Integrability in AdS_3/CFT_2

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Nov 16, 2015

Based on work done together with A. Babichenko, R. Borsato, A. Dekel, T. Lloyd, A. Sfondrini, B. Stefański and A. Torrielli

NORDITA

AdS_3/CFT_2

- AdS_3 backgrounds preserving 16 supersymmetries:

$$AdS_3 \times S^3 \times T^4 \qquad\qquad AdS_3 \times S^3 \times S^3 \times S^1$$

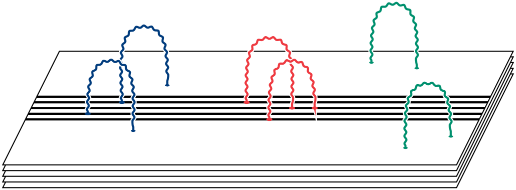
- String backgrounds supported by RR+NSNSN three-form flux
- Dual conformal field theories:

$D = 2$ CFT with small $\mathcal{N} = (4, 4)$ symmetry	$D = 2$ CFT with large $\mathcal{N} = (4, 4)$ symmetry

- Non-linear sigma models are classically integrable
- Goal: use integrability to solve spectral problem of AdS_3/CFT_2

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$

	$\mathbb{R}^{1,5}$						T^4			
	0	1	2	3	4	5	6	7	8	9
Q_{D5} D5:	•	•					•	•	•	•
Q_{D1} D1:	•	•								



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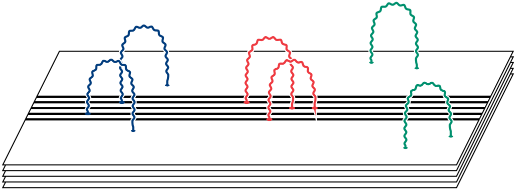
Near horizon geometry:	$\mathrm{AdS}_3 \times \mathrm{S}^3$						T^4			



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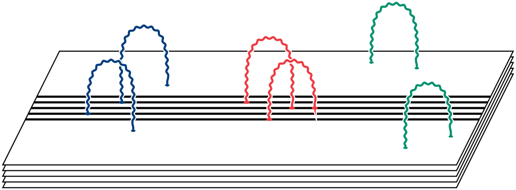
Near horizon geometry: $AdS_3 \times S^3$ T^4

Radius: $L^2 = g_s Q_{D5}$



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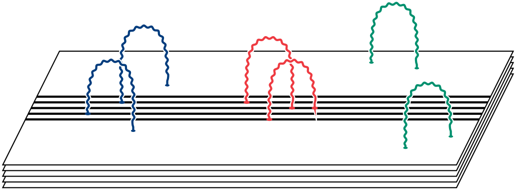
Q_{F1} F1 + Q_{D1} D1:	•	•								
Near horizon geometry:	$\mathrm{AdS}_3 \times \mathrm{S}^3$						T^4			

Radius: $$L^2 = \sqrt{Q_{\mathrm{NS5}}^2 + g_s^2 Q_{\mathrm{D5}}^2} = \alpha' \sqrt{\lambda}$$

World-sheet coupling



Integrability for the AdS_3/CFT_2 spectral problem

- Spectral problem for AdS_3/CFT_2:

Energy of free closed strings $\longleftrightarrow$ Dimensions of gauge invariant local operators

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- Work in the free/planar ('t Hooft) limit

$$g_s \to 0 \qquad \sqrt{\lambda} \text{ fixed}$$

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- CFT_2: spin-chain picture of local operators See also Bogdan's talk

$$\mathbf{D}\mathcal{O} = \Delta\mathcal{O} \qquad\qquad \mathcal{O} = \sum \mathrm{tr}\big(ZZZZXZZXZZZ\big)$$

Outline

Integrability in AdS_3/CFT_2

Integrability

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- Classical mechanics: N d.o.f, N conserved charges

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- Classical field theory: an infinite number of d.o.f
- Local conserved quantities

  Sine-Gordon: $\qquad H_{2k+1} = p^{2k+1} \qquad H_{2k+2} = p^{2k}\sqrt{p^2 + m^2}$

- Momentum-dependent translations

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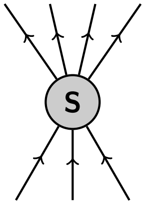
- Quantum field theory: factorised scattering

Factorised scattering

- N-particle scattering: $|p_1, p_2, \ldots, p_N\rangle \to |p'_1, p'_2, \ldots, p'_M\rangle$



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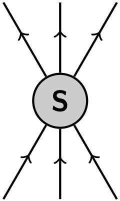

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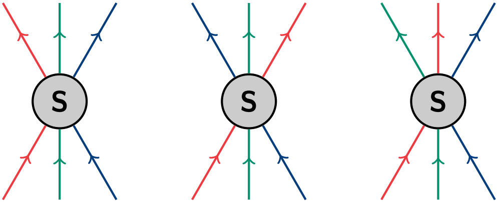

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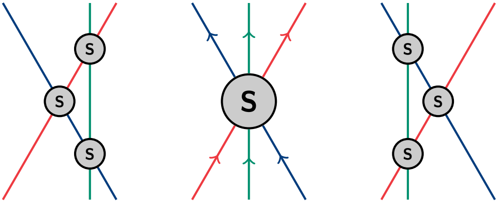

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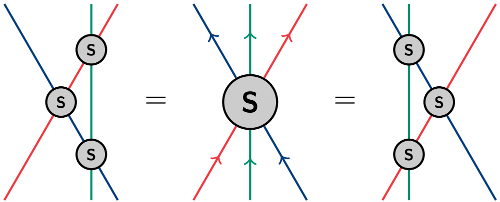

Coset sigma models

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Background geometries as cosets:

$$\frac{SO(2,2)}{SO(1,2)} \times \frac{SO(4)}{SO(3)} \qquad\qquad \frac{SO(2,2)}{SO(1,2)} \times \frac{SO(4)}{SO(3)} \times \frac{SO(4)}{SO(3)}$$

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Super-coset sigma models:

$$\frac{PSU(1,1|2) \times PSU(1,1|2)}{SL(2) \times SU(2)} \times U(1)^4 \qquad\qquad \frac{D(2,1;\alpha) \times D(2,1;\alpha)}{SL(2) \times SU(2) \times SU(2)} \times U(1)$$

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Super-cosets with
$\mathbb{Z}_4$ automorphism

Classical integrability

Coset sigma models with $\mathbb{Z}_4$ grading

- Super-coset space G/H_0
- $\mathbb{Z}_4$ grading of super-Lie algebra

$$g = h_0 \oplus h_1 \oplus h_2 \oplus h_3$$

- Compatibility with (anti-)commutation relations

$$[h_n, h_m\} \subset h_{(n+m) \bmod 4}$$

- In our case:
  - h_0 and h_2 are bosonic
  - h_1 and h_3 are fermionic

Coset sigma models with $\mathbb{Z}_4$ grading

- Graded currents ($g(x) \in G$)

$$J = g^{-1}dg = J_0 + J_1 + J_2 + J_3$$

- Sigma model action

$$\mathcal{S} = \int d^2\sigma \, \mathrm{Str}\big(J_2 \wedge *J_2 + J_1 \wedge J_3\big)$$

- Introduce the Lax connection

$$L(x) = J_0 + \tfrac{x^2+1}{x^2-1} J_2 - \tfrac{2x}{x^2-1} *J_2 + \sqrt{\tfrac{x+1}{x-1}} J_1 + \sqrt{\tfrac{x-1}{x+1}} J_3$$

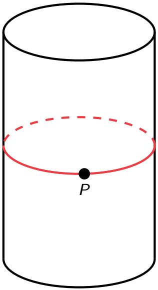
- Equations of motion $\quad \longleftrightarrow \quad$ flatness of L

$$dL + L \wedge L = 0, \qquad \forall x$$

The monodromy matrix and integrability

- Construct the monodromy matrix

$$\mathcal{M}_P(x) = \mathcal{P}\exp\int_P^P L(x)$$



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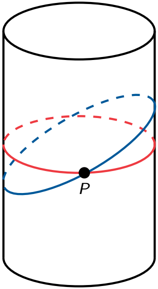

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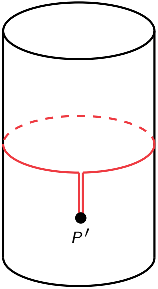

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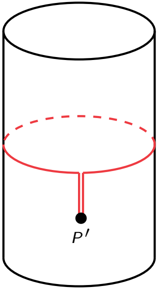

- Expanding $\operatorname{tr}\mathcal{M}(x)$ gives an infinite set of conserved charges

Coset sigma models and Green-Schwarz strings

Green-Schwarz string		Coset sigma model
IIB on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3$	$\leftrightarrow$	$\dfrac{D(2,1;\alpha) \times D(2,1;\alpha)}{SL(2) \times SU(2) \times SU(2)}$
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Fully fix kappa symmetry		Add free scalars

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- Coset backgrounds supported by pure RR flux

[Babichenko, Stefański, Zarembo '09]

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- Coset backgrounds supported by pure RR flux

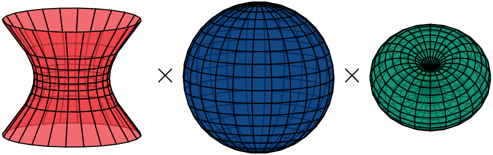
  [Babichenko, Stefański, Zarembo '09]
- Include NSNS flux by adding WZ term

$$k \int \mathrm{Str}\left(\tfrac{2}{3} J_2 \wedge J_2 \wedge J_2 + J_1 \wedge J_3 \wedge J_2 + J_3 \wedge J_1 \wedge J_2\right)$$

- WZ term breaks $\mathbb{Z}_4$ symmetry but a Lax connection can still be constructed [Cagnazzo, Zarembo '12]

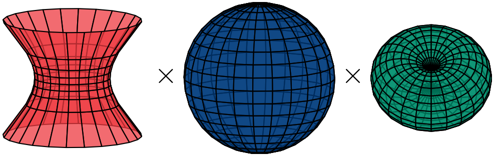
String theory in uniform light-cone gauge

String theory on $AdS_3 \times S^3 \times T^4$



- Consider strings in $AdS_3 \times S^3 \times T^4$ supported by pure RR flux
- Fix light-cone gauge
- $8 + 8$ physical world-sheet excitation
- World-sheet integrability:
  - Dispersion relation for fundamental excitations
  - Two-particle S matrix
- S matrix defined on a non-compact world-sheet

String theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$



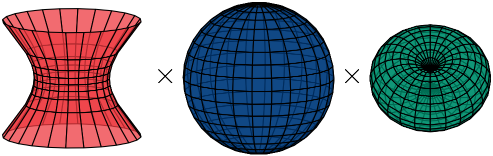
- Isometries:

$$PSU(1,1|2) \times PSU(1,1|2) \times U(1)^4 \quad \underbrace{\qquad\qquad}_{SU(2)_\bullet \times SU(2)_\circ}$$

- Bosonic subgroup

$$SO(2,2) \times SO(4) \times U(1)^4$$

String theory on $AdS_3 \times S^3 \times T^4$



- Isometries in the decompactification limit:

$$PSU(1,1|2) \times PSU(1,1|2) \times U(1) \times \underbrace{SO(4)}_{SU(2)_\bullet \times SU(2)_\circ}$$

- Bosonic subgroup

$$SO(2,2) \times SO(4) \times U(1) \times SO(4)$$

Light-cone gauge

$$\underbrace{}_{}$$

- Fix light-cone gauge: $X^+ = \overset{\text{Equator of } S^3}{\phi} + \overset{\text{AdS}_3 \text{ time}}{t} = \tau$

Light-cone gauge

Equator of S^3 AdS$_3$ time

- Fix light-cone gauge: $X^+ = \phi + t = \tau$
- World-sheet Hamiltonian: $\mathbf{H} = E - J$

AdS$_3$ energy Angular momentum on S^3

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- 8+8 fluctuations:

$$m_{\mathrm{B}} = 4 \times \{0, 1\} \qquad m_{\mathrm{F}} = 4 \times \{0, 1\}$$

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Note massless modes

"Off-shell" symmetries

- Physical states satisfy level matching:

$$\mathbf{P}\,|p_1, \ldots, p_n\rangle = (p_1 + \cdots + p_n)\,|p_1, \ldots, p_n\rangle = 0$$

- "Off-shell" states have:

$$\mathbf{P}\,|p_1, \ldots, p_n\rangle \neq 0$$

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- Not all isometries are manifest in light-cone gauge
- Construct off-shell symmetry algebra $\mathcal{A}$ of generators $\mathbf{J}$ that
  1. Commute with the gauge-fixed Hamiltonian $[\mathbf{H},\mathbf{J}]=0$
  2. Act on generic off-shell states
- World-sheet supercurrents constructed to quartic order

[Borsato, OOS, Sfondrini, Stefański, Torrielli '14]
[Lloyd, OOS, Sfondrini, Stefański '14]

- For on-shell states $\mathcal{A} \subset psu(1,1|2)^2 \times so(4)$

Light-cone gauge symmetry algebra

Light-cone gauge breaks isometries to $psu(1|1)^4_{\text{c.e.}} \times so(4)$

8 supercharges

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Light-cone gauge breaks isometries to $psu(1|1)^4_{\text{c.e.}} \times so(4)$

The on-shell algebra $\mathbf{P} = 0$

$$\{\mathbf{Q}_{\mathrm{L}\,a}, \overline{\mathbf{Q}}_{\mathrm{L}}{}^{b}\} = \tfrac{1}{2}\delta_a{}^b(\mathbf{H} + \mathbf{M}) \qquad \{\mathbf{Q}_{\mathrm{L}\,a}, \mathbf{Q}_{\mathrm{R}}{}^{b}\} = 0$$

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$su(2)_\bullet \subset so(4)$ indices

Light-cone gauge symmetry algebra

Light-cone gauge breaks isometries to $psu(1|1)^4_{\text{c.e.}} \times so(4)$

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Two additional central charges

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Central charge

$$\mathbf{C} = \tfrac{i}{2}h(\lambda)\left(e^{i\mathbf{P}} - 1\right)$$

Coupling constant

$$h(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + \mathcal{O}(1/\sqrt{\lambda})$$

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Non-trivial coproduct

$$\mathbf{C}\,|p_1p_2\rangle = \begin{cases} \big(\#\,\mathbf{C}\otimes\mathbf{1} + \#\,\mathbf{1}\otimes\mathbf{C}\big)\,|p_1p_2\rangle \\ \frac{ih}{2}\big(e^{i(p_1+p_2)}-1\big)\,|p_1p_2\rangle \end{cases}$$

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Representations

Particles transform in short representations

$$\mathbf{H}^2 = \mathbf{M}^2 + 4\mathbf{C}\overline{\mathbf{C}}$$

Central charge $\mathbf{C} = \frac{ih}{2}\left(e^{i\mathbf{P}} - 1\right)$ gives the dispersion relation

$$E_p = \sqrt{m^2 + 4h^2 \sin^2 \frac{p}{2}}$$

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Two massive + two massless $psu(1|1)^4_{\text{c.e.}}$ multiplets

$$\begin{tikzcd}[column sep=small]
 & {|Y^{\mathrm{L}}\rangle} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
{|\eta^{\mathrm{L}1}\rangle} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {|\eta^{\mathrm{L}2}\rangle} \arrow[dl, leftrightarrow] \\
 & {|Z^{\mathrm{L}}\rangle} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & {|Z^{\mathrm{R}}\rangle} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
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 & {|Y^{\mathrm{R}}\rangle} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & {|\chi^{1}\rangle} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
{|T^{11}\rangle} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {|T^{21}\rangle} \arrow[dl, leftrightarrow] \\
 & {|\tilde{\chi}^{1}\rangle} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & {|\chi^{2}\rangle} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
{|T^{12}\rangle} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & {|T^{22}\rangle} \arrow[dl, leftrightarrow] \\
 & {|\tilde{\chi}^{2}\rangle} &
\end{tikzcd}$$

$m{=}{+}1 \qquad m{=}{-}1 \qquad m{=}0 \qquad m{=}0$

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 & {|Z^{\mathrm{L}}\rangle} \arrow[ur, leftrightarrow] &
\end{tikzcd}
\qquad
\begin{tikzcd}
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 & {|Y^{\mathrm{R}}\rangle} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

$m{=}{+}1$ $\qquad$ $m{=}{-}1$

$$\underbrace{
\begin{tikzcd}
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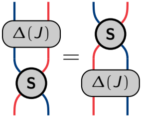
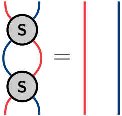
 & {|\tilde{\chi}^{2}\rangle} \arrow[ur, leftrightarrow] &
\end{tikzcd}
}_{\text{Doublet under } su(2)_\circ \subset so(4)}$$

Properties of the S matrix

- Symmetry invariance

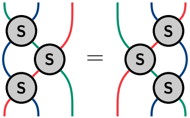


- Unitarity



$$\mathbf{S}^\dagger \mathbf{S} = 1$$

- Yang-Baxter equation



The two-particle S matrix

- Find S matrix by imposing off-shell symmetry

$$[\Delta(\mathbf{Q}), \mathbf{S}] = 0$$

Non-trivial coproduct

The two-particle S matrix

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Non-trivial coproduct

- Unique matrix for each pair of representations
- Four undetermined coefficients – "dressing phases"

$$\sigma^2 \qquad \tilde{\sigma}^2 \qquad \sigma^2_{\bullet\circ} \qquad \sigma^2_{\circ\circ}$$

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Scattering of excitations with $m = +1$ and $m = +1$

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Scattering of excitations with $m = +1$ and $m = -1$

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Scattering of excitations with $m = +1$ and $m = 0$

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Scattering of excitations with $m = 0$ and $m = 0$

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Non-trivial coproduct

- Unique matrix for each pair of representations
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$$\sigma^2 \qquad \tilde{\sigma}^2 \qquad \sigma^2_{\bullet\circ} \qquad \sigma^2_{\circ\circ}$$

- Phases satisfy crossing equations
- S matrix exact to all orders in $h(\lambda)$

Massless S matrix

- In a relativistic theory scattering of massless modes is problematic
$$v = \frac{\partial E}{\partial p} = \pm 1$$
- Here there is no Lorentz invariance and the “massless” modes have a non-linear dispersion relation

$$v = \frac{\partial E}{\partial p} = \pm h \cos \frac{p}{2}$$

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- Massless modes form doublet under $su(2)_\circ$
    - extra $su(2)$ S matrix

$$\mathbf{S}_{su(2)} = 1 + i(w_p - w_q)\Pi$$

Unknown function of momentum

Mixed flux background

- $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ supported by RR+NSNS three-form flux

$$F = \tilde{q}\big(\mathrm{Vol}_{\mathrm{AdS}_3} + \mathrm{Vol}_{\mathrm{S}^3}\big) \qquad H = q\big(\mathrm{Vol}_{\mathrm{AdS}_3} + \mathrm{Vol}_{\mathrm{S}^3}\big)$$

- Coefficients related by $\tilde{q}^2 + q^2 = 1$
- Quantised WZW level

$$Q_{\mathrm{NS5}} = 2\pi \not{k} = q\sqrt{\lambda} \in \mathbb{Z}$$

- Dispersion relation

$$E_p = \sqrt{(m + \not{k} p)^2 + 4\tilde{q}^2 h^2 \sin^2 \frac{p}{2}}$$

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Momentum-dependent "mass"

$\not{k} \sim Q_{\mathrm{NS5}}$

Rescaled coupling

$\tilde{q} h \sim g_s Q_{\mathrm{D5}} + \cdots$

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- S matrix takes the same functional form of p and E_p for any q

[Hoare, Tseytlin '13]

[Lloyd, OOS, Stefański, Sfondrini '14]

Bethe ansatz and the spin-chain picture

Bethe ansatz equations

- Impose periodic boundary conditions

$$e^{ip_k L} = \prod_{j \neq k} S(p_k, p_j)$$

- Non-diagonal S matrix $\longrightarrow$ nested Bethe equations
- 3 types of momentum-carrying roots
- 3 types of auxiliary roots
- Simplifies in the weak coupling limit $h(\lambda) \to 0$

Massive sector

At weak coupling

- Two decoupled $PSU(1,1|2)$ spin-chains
- The two spin-chains couple through level matching

$$e^{ip_{\text{total}}} = 1$$

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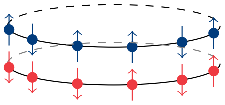
Higher orders

- Sites in the $(\frac{1}{2};\frac{1}{2})_{\mathrm{L}} \otimes (\frac{1}{2};\frac{1}{2})_{\mathrm{R}}$ representation



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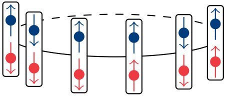

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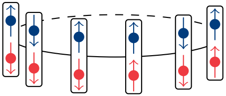
- Dynamic supersymmetries



Spin-chain representation $(\frac{1}{2}; \frac{1}{2})$

Doublet under $su(2) \subset psu(1,1|2)$

- Two bosons $\qquad \phi^{\pm} \qquad$ Dimension $\frac{1}{2}$

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- Two fermions $\psi^{\pm}$ Dimension 1

Doublet under $su(2)_{\bullet}$ automorphism

Spin-chain representation $(\frac{1}{2};\frac{1}{2})$

Doublet under $su(2) \subset psu(1,1|2)$

- Two bosons $\partial^n \phi^\pm$ Dimension $\frac{1}{2}+n$
- Two fermions $\partial^n \psi^\pm$ Dimension $1+n$

Doublet under $su(2)_\bullet$ automorphism

- Derivatives generate $sl(2)$ descendants

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Doublet under $su(2)_\bullet$ automorphism

- Derivatives generate $sl(2)$ descendants
- 1/2-BPS representation

In the full $psu(1,1|2) \times psu(1,1|2)$ (massive) spin-chain:

- Sites make up 8+8 primary fields

$$
\begin{tikzcd}
 & \phi^{a\dot{a}} \arrow[dl] \arrow[dr] & \\
\psi_L^{\alpha\dot{a}} \arrow[dr] & & \psi_R^{a\alpha} \arrow[dl] \\
 & D^{\alpha\beta} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \left(\frac{1}{2}, \frac{1}{2}\right) \arrow[dl] \arrow[dr] & \\
\left(1, \frac{1}{2}\right) \arrow[dr] & & \left(\frac{1}{2}, 1\right) \arrow[dl] \\
 & (1,1) &
\end{tikzcd}
$$

Massless modes in the spin-chain

When we include the massless modes additional chiral representations appear as sites in the spin-chain

- Four free scalars $\quad T^{\alpha\dot{\beta}} \quad (0,0)$
- Two $(\frac{1}{2};\frac{1}{2})\otimes 1$ multiplets

$$\chi_L^{a\dot{\alpha}} \swarrow \partial_L T^{\alpha\dot{\alpha}} \qquad (\tfrac{1}{2},0) \swarrow (1,0)$$

- Two $1\otimes(\frac{1}{2};\frac{1}{2})$ multiplets

$$\chi_R^{\dot{a}\dot{\alpha}} \searrow \partial_R T^{\alpha\dot{\alpha}} \qquad (0,\tfrac{1}{2}) \searrow (0,1)$$

Massless modes in the spin-chain

With massive + massless modes

- Sites in different representations – "reducible spin-chain"

[OOS, Stefański, Torrielli '12]

At weak coupling

- Two decoupled $psu(1,1|2)$ spin-chains of different length
- Extra equations describing scattering between massless modes
- Level matching condition

$$\exp\left(ip_{\mathsf{L}} + ip_{\mathsf{R}} + ip_{\text{massless}}\right) = 1$$

BPS states

From $psu(1,1|2)^2$ representation theory

- Primaries of three types of 1/2-BPS sites

$$\begin{array}{rl} \phi & \text{massive scalar} \\ \chi_L^{\pm} & \text{massless chiral fermion} \\ \chi_R^{\pm} & \text{massless anti-chiral fermion} \end{array}$$

- Expect 1/2-BPS states of the form

$$\big(\phi\big)^{J_M}\big(\chi_L\big)^{J_L}\big(\chi_R\big)^{J_R} \qquad \big(\tfrac{1}{2}(J_M+J_L),\, \tfrac{1}{2}(J_M+J_R)\big)$$

- Only completely symmetric states protected when interactions are included

BPS states

From $psu(1,1|2)^2$ representation theory + interactions

- J massive bosons

$$\left(\frac{J}{2}, \frac{J}{2}\right)$$

BPS states

From $psu(1,1|2)^2$ representation theory + interactions

- J massive bosons
- Two + two massless fermions, each appearing maximally once

$$\left(\tfrac{J}{2}, \tfrac{J}{2}\right)$$

$$\left(\tfrac{J}{2}+\tfrac{1}{2}, \tfrac{J}{2}\right)^2 \qquad \left(\tfrac{J}{2}, \tfrac{J}{2}+\tfrac{1}{2}\right)^2$$

$$\left(\tfrac{J}{2}+1, \tfrac{J}{2}\right) \qquad \left(\tfrac{J}{2}+\tfrac{1}{2}, \tfrac{J}{2}+\tfrac{1}{2}\right)^4 \qquad \left(\tfrac{J}{2}, \tfrac{J}{2}+1\right)$$

$$\left(\tfrac{J}{2}+1, \tfrac{J}{2}+\tfrac{1}{2}\right)^2 \qquad \left(\tfrac{J}{2}+\tfrac{1}{2}, \tfrac{J}{2}+1\right)^2$$

$$\left(\tfrac{J}{2}+1, \tfrac{J}{2}+1\right)$$

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- Two + two massless fermions, each appearing maximally once

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$$\left(\tfrac{J}{2}+\tfrac{1}{2},\tfrac{J}{2}\right)^2 \qquad \left(\tfrac{J}{2},\tfrac{J}{2}+\tfrac{1}{2}\right)^2$$

$$\left(\tfrac{J}{2}+1,\tfrac{J}{2}\right) \qquad \left(\tfrac{J}{2}+\tfrac{1}{2},\tfrac{J}{2}+\tfrac{1}{2}\right)^4 \qquad \left(\tfrac{J}{2},\tfrac{J}{2}+1\right)$$

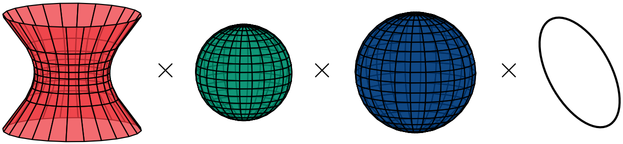
$$\left(\tfrac{J}{2}+1,\tfrac{J}{2}+\tfrac{1}{2}\right)^2 \qquad \left(\tfrac{J}{2}+\tfrac{1}{2},\tfrac{J}{2}+1\right)^2$$

$$\left(\tfrac{J}{2}+1,\tfrac{J}{2}+1\right)$$

- Matches supergravity spectrum [de Boer '98]

String theory on $AdS_3 \times S^3 \times S^3 \times S^1$

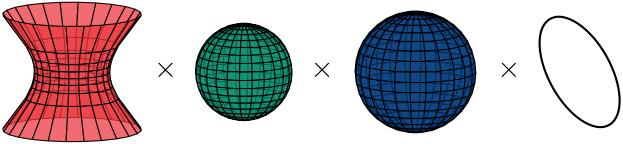
$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$



- Supersymmetry realtes the radii:

$$\frac{1}{L^2} = \frac{1}{R_+^2} + \frac{1}{R_-^2} \qquad \frac{1}{R_+^2} = \frac{\alpha}{L^2} \qquad \frac{1}{R_-^2} = \frac{1-\alpha}{L^2}$$

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$



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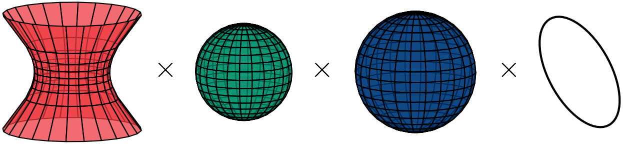
One parameter
family of backgrounds
$0 < \alpha < 1$

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$



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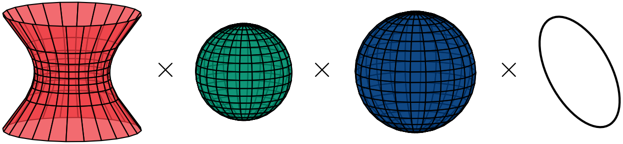
- Isometries:

$$D(2,1;\alpha)\times D(2,1;\alpha)\times U(1) \supset SO(2,2)\times SO(4)\times SO(4)\times U(1)$$

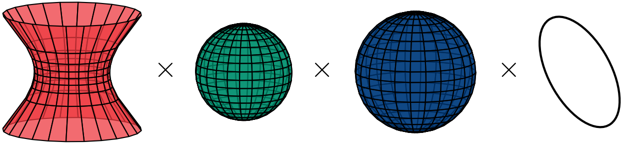
- In the $\alpha \to 0$ and $\alpha \to 1$ limits one of the sphere blows up
  $\longrightarrow$ obtain the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background

$AdS_3 \times S^3 \times S^3 \times S^1$



- Unique supersymmetric geodesic on $AdS_3 \times S^3 \times S^3$
- Preserves 4 supersymmetries

$AdS_3 \times S^3 \times S^3 \times S^1$



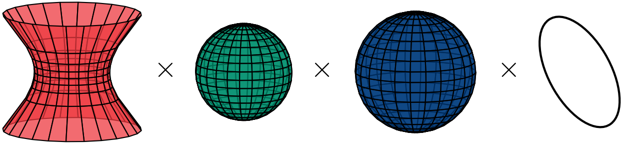
- Unique supersymmetric geodesic on $AdS_3 \times S^3 \times S^3$
- Preserves 4 supersymmetries
- Light-cone gauge "off-shell" symmetry algebra

$$psu(1|1)^2_{\text{c.e.}} \quad \text{with four central elements}$$

- Fundamental excitations

$$m_{\text{B}} = 2 \times \{0, \alpha, 1 - \alpha, \ 1\} \qquad m_{\text{F}} = 2 \times \{0, \alpha, 1 - \alpha, \ 1\}$$

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$



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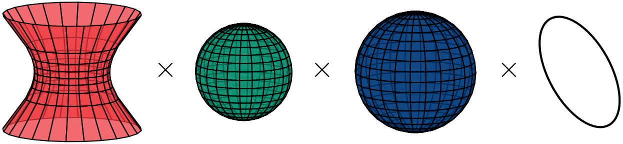
Composite?

$AdS_3 \times S^3 \times S^3 \times S^1$



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- Preserves 4 supersymmetries
- Light-cone gauge "off-shell" symmetry algebra

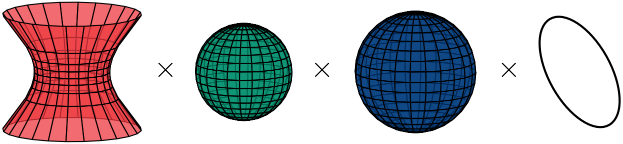
$$psu(1|1)^2_{\text{c.e.}} \quad \text{with four central elements}$$

- Fundamental excitations

$$m_{\text{B}} = 2 \times \{0, \alpha, 1-\alpha,\ 1\} \qquad m_{\text{F}} = 2 \times \{0, \alpha, 1-\alpha,\ 1\}$$

- Form $1+1$ dimensional representations of $psu(1|1)^2_{\text{c.e.}}$

$AdS_3 \times S^3 \times S^3 \times S^1$



Off-shell symmetry algebra gives

- Dispersion relation

$$E_p = \sqrt{(m + \not{k} p)^2 + 4\tilde{q}^2 h^2 \sin^2 \frac{p}{2}}$$

- Matrix form of S matrix
- 9 dressing phases

Summary

Summary

Integrability in AdS_3/CFT_2

Discussed string theory on $AdS_3 \times S^3 \times T^4$

- Supported by RR+NSNS flux
- Classical theory is integrable
- Quantum theory: light-cone gauge
- Constructed "off-shell" symmetry algebra
  - Exact dispersion relation
  - All-loop S matrix – satisfies Yang-Baxter equation
- Spin-chain picture from Bethe equations

Results generalise to $AdS_3 \times S^3 \times S^3 \times S^1$

Outlook

Open string theory questions

- Dressing phases – solve crossing equations [Work in progress]
- Match with perturbation theory [Sundin, Wulff '12–'15] [Engelund, McKeown, Roiban '13] [Bianchi, Hoare '14]
  - S matrix matches with perturbative results
  - Two-loop missmatch for massless dispersion relation

$$E_p^{\text{Exact}} = p - \frac{p^3}{24h^2} + \cdots \qquad E_p^{\text{Pert}} = p - \frac{p^3}{4\pi^2 h^2} + \cdots$$

- Massless $su(2)_\circ$ S matrix
- Winding modes on T^4

Outlook

Bigger questions

- Full spectrum from integrability – TBA
- Spin-chain from CFT_2? See Bogdan's talk
- Virasoro? Full $\mathcal{N} = (4, 4)$ superconformal symmetry?
- Relation with symmetric product orbifold?
  [Pakman, Rastelli, Razamat '10]
- Black holes in AdS_3 and integrability [David, Sadhukhan '11]
- Relation with higher spin theories in AdS_3?
  [Gaberdiel, Gopakumar '14]

Thank you!

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