Wess-Zumino-Witten Models

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Abstract

We introduce the principal chiral model in two dimensions and its extension by the Wess-Zumino term. We discuss the symmetries, corresponding currents and their quantum algebra. We explain the Sugawara construction to demonstrate that the Wess-Zumino-Witten model defines a conformal field theory. We then move on and discuss representations, characters, fusion and modular invariants by following the example of $\mathfrak{su}(2)$. Finally, we briefly discuss the coset construction.

Contents

1 Introduction

In these lectures, we will introduce Wess-Zumino-Witten (WZW)-models. They are a prime example of rational conformal field theories and are completely solvable. They feature an extended chiral algebra: The Virasoro algebra gets extended to an affine Lie algebra, due to the existence of additional symmetries in the theory.

We start by the classical formulation of the theory. Wess-Zumino-Witten models can be viewed as non-linear sigma models, where the target space is a group manifold. To make them conformal, one introduces an additional term in the action. We discuss the action and derive from it the quantum symmetry algebra. It is given by an affine generalization of finite-dimensional Lie algebras:

$$[J^a_m, J^a_n] = k\delta^{ab}\delta_{m+n,0} + if^{ab}{}_c J^c_{m+n} \ . \tag{1.1}$$

This algebra forms the chiral algebra of the model. We explain how to construct out of it the holomorphic component of the energy-momentum tensor, thereby demonstrating conformal invariance of the theory. A study of the representations will show that they are severely restricted, in particular there are only finitely many possible representations. This is very much in contrast to finite-dimensional Lie algebras.

We then discuss the structure of the complete theory. A study of the fusion rules gives us information about possible three-point functions of the theory. We will not explore three- and four-point functions any further. They can be computed via the Knizhnik-Zamolodchikov (KZ)-equations. We discuss the constraint of modular invariance, which is a consistency condition arising when putting the theory on the torus. We explain the classification of modular invariants of the SU(2) WZW-model.

Finally, we introduce the coset construction, which corresponds to gauged WZW-models. It accounts for a wealth of known rational conformal field theories. In particular, we show how to retrieve the unitary minimal models as a coset.

The material presented here has become standard. Good general references are [1, 2]. The 'big yellow book' [3] is a very complete treatment of all the topics discussed in these lectures. The material is however stretched over several hundred pages. For these lectures, we require only a basic knowledge of CFT in two dimensions.

There are exercises scattered throughout the text. The reader is very much urged to solve those exercises.

2 The classical theory

2.1 Principal chiral model

We consider a quantum field theory on the Riemann sphere, viewed as the one-point compactification of the two-dimensional plane. Throughout this lecture, we will work in Euclidean signature. This will provide us with the tools of complex analysis. The model we will discuss in this lecture is a principal chiral model on a Lie-group G.[1] We will use a matrix notation for group and Lie algebra elements. In particular, for two Lie algebra elements X and Y, we can define a non-degenerate invariant product as

$$\langle X, Y \rangle = \mathrm{tr}\big(XY\big) \ . \tag{2.1}$$

[1]Technically, we need to assume this Lie group to be semi-simple, i.e. (locally) a product of simple Lie groups. This condition is equivalent to the existence of a non-degenerate invariant trace on the corresponding Lie algebra $\mathfrak{g} = \mathrm{Lie(G)}$.

We will normalise the trace such that $\mathrm{tr}\,(t^a t^b) = \frac{1}{2}\delta^{ab}$ for two generators of the Lie algebra in the fundamental representation.[2]

The field of the model is denoted by $g(z,\bar{z})$. We think of them as matrices, i.e. we choose some (faithful) representation of G. $g(z,\bar{z})$ is hence a map

$$g : \mathrm{S}^2 \longrightarrow \mathrm{G} \ . \tag{2.2}$$

Consider the action

$$\mathcal{S}_0 = \frac{1}{4\lambda^2} \int_{\mathrm{S}^2} \mathrm{d}^2 z \ \mathrm{tr}\left(g^{-1}\partial_\mu g\, g^{-1}\partial^\mu g\right) \ . \tag{2.3}$$

Since g is an element in the Lie group, $g^{-1}\partial_\mu g$ defines an element of the Lie algebra.[3]

The action has a global $\mathrm{G}\times\mathrm{G}$-symmetry, given by

$$g(z,\bar{z}) \longmapsto g_{\mathrm{L}}\, g(z,\bar{z})\, g_{\mathrm{R}}^{-1} \tag{2.4}$$

for some constant group elements g_{L}, $g_{\mathrm{R}} \in \mathrm{G}$.

This theory is classically conformally invariant. Computing the β-function shows that this is no longer true at the quantum level. One finds a negative β-function. Hence conformal invariance is broken at the quantum level and the theory is asymptotically free. This is why we will modify the action (2.3) to obtain the action of a conformal field theory, which is referred to as Wess-Zumino-Witten (WZW) model.

Let us compute the classical equations of motion by varying the action w.r.t. g:

$$\delta\mathcal{S}_0 = \frac{1}{2\lambda^2} \int_{\mathrm{S}^2} \mathrm{d}^2 z \ \mathrm{tr}\left((-g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\delta\partial_\mu g) g^{-1}\partial^\mu g\right) \tag{2.5}$$

$$= \frac{1}{2\lambda^2} \int_{\mathrm{S}^2} \mathrm{d}^2 z \ \mathrm{tr}\left(\partial_\mu(g^{-1}\delta g)\, g^{-1}\partial^\mu g\right) \tag{2.6}$$

$$= -\frac{1}{2\lambda^2} \int_{\mathrm{S}^2} \mathrm{d}^2 z \ \mathrm{tr}\left(g^{-1}\delta g \partial_\mu(g^{-1}\partial^\mu g)\right) \ . \tag{2.7}$$

Here, we used the cyclicity of the trace and integrated by parts. Since this holds for every variation, we deduce the classical equations of motions:

$$\partial_\mu(g^{-1}\partial^\mu g) = 0 \ , \tag{2.8}$$

hence the Lie algebra valued current $J^\mu = g^{-1}\partial^\mu g$ is conserved. This current is in fact the associated current to the right multiplication symmetry $g \mapsto g g_{\mathrm{R}}^{-1}$. The current for the left-multiplication symmetry $g \mapsto g_{\mathrm{L}} g$ is given by $\tilde{J}^\mu = \partial^\mu g g^{-1}$ and is also conserved. Indeed, we can rewrite the equations of motion (2.8) as

$$\partial_\mu(g^{-1}\partial^\mu g) = 0 \quad \Longleftrightarrow \quad g\partial_\mu(g^{-1}\partial^\mu g)g^{-1} = \partial_\mu(\partial^\mu g g^{-1}) = 0 \ . \tag{2.9}$$

In complex coordinates, current conservation reads

$$\partial J_{\bar{z}} + \bar{\partial} J_z = 0 \ , \tag{2.10}$$

[2]A fundamental representation does not exist in some cases, but the following analysis still goes through.

[3]Mathematically, $g^{-1}\mathrm{d}g$ is the pullback of the Maurer-Cartan form to S^2 under the map defined by g.

where here and in the following $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. In conformal field theory, we would like to have (anti)holomorphic quantities, in other words both terms in this equation should vanish separately. Indeed, the situation is entirely analogous to the conservation of the stress energy tensor in a CFT. We have

$$\partial T_{\bar{z}\bar{z}} + \bar{\partial} T_{z\bar{z}} = 0 \ , \qquad \partial T_{\bar{z}z} + \bar{\partial} T_{zz} = 0 \ . \tag{2.11}$$

Tracelessness of the energy-momentum tensor in a CFT reads $T_{z\bar{z}} = T_{\bar{z}z} = 0$ in complex coordinates. Hence $T \equiv T_{zz}$ and $\bar{T} \equiv T_{\bar{z}\bar{z}}$ are holomorphic (antiholomorphic) quantities. In the WZW-case, one of the components of the currents vanishes identically and hence the other component will be (anti)holomorphic. This is precisely achieved by adding the WZ-term to the action, as we shall see below.

But for the principal chiral model, not both terms can vanish separately. Indeed, J^μ can also be seen to be a gauge potential for the gauge group G. By definition, it is of pure gauge and hence its field strength vanishes, so

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \, . \tag{2.12}$$

If in (2.10) both terms would vanish separately, then this would imply that the dual current $\epsilon_{\mu\nu} J^\nu$ would also be conserved. Here $\epsilon_{\mu\nu}$ is the totally antisymmetric tensor. However, we have

$$\partial_\mu(\epsilon^{\mu\nu} J_\nu) = \frac{1}{2}\epsilon^{\mu\nu}(\partial_\mu J_\nu - \partial_\nu J_\mu) = -\frac{1}{2}\epsilon^{\mu\nu}[J_\mu, J_\nu] = -\epsilon^{\mu\nu} J_\mu J_\nu \, . \tag{2.13}$$

This expression vanishes only for an abelian Lie algebra,[4] so the currents are not separately conserved. Thus, the theory will not contain (anti)holomorphic currents. As a consequence, we do not expect this to be a two-dimensional conformal field theory at the quantum level.

Note that our task would be complete for an abelian Lie algebra. It turns out that in this case, this indeed defines a conformally invariant theory and the Wess-Zumino term we define below can be seen to vanish. In fact, the WZW-model for the abelian Lie algebra $\mathbb{R}^n$ gives the theory of n free bosons. In the following, we want to focus on the non-abelian case.

2.2 WZW action

To rectify the issue of non-holomorphicity in the non-abelian case, we will modify the action and will add to it the Wess-Zumino term, which is given by

$$\Gamma[g] \equiv -\frac{i}{12\pi} \int_B \mathrm{d}^3 y \, \epsilon_{\alpha\beta\gamma} \, \mathrm{tr} \left(g^{-1}\partial^\alpha g g^{-1} \partial^\beta g g^{-1} \partial^\gamma g \right) \tag{2.14}$$

$$= -\frac{i}{2\pi} \int_B \mathrm{tr} \, \left(g^{-1} \, \mathrm{d}g \wedge g^{-1} \, \mathrm{d}g \wedge g^{-1} \, \mathrm{d}g \right) \, . \tag{2.15}$$

This needs some explanation. The field g is defined on the Riemann sphere S^2, which can be thought of being the boundary of a three-dimensional ball B. The fields are smooth maps

$$g : \mathrm{S}^2 \longrightarrow \mathrm{G} \, . \tag{2.16}$$

[4] An abelian Lie algebra is a Lie algebra, where all commutators vanish.

Such maps are classified up to homotopy by the second homotopy group $\pi_2(\mathrm{G})$. It is a mathematical fact that the second fundamental group of every Lie group vanishes, i.e. $\pi_2(\mathrm{G}) = 0$. This implies that every map as in (2.16) is homotopic to the constant map. Clearly the constant map can be continued to the interior of S^2, and hence so can any map g. We denote this extension of g also by g, i.e. g maps now from B to G and on $\partial B = \mathrm{S}^2$ it is equal to the original map. Of course this extension is not unique. Let us analyse how changing the relevant extension will modify the value of Γ. If we have two different extensions, we can glue them together along the common boundary (where they agree by assumption). This provides a map

$$g : (B \sqcup B)/\partial B \approx \mathrm{S}^3 \longrightarrow \mathrm{G}\,, \tag{2.17}$$

where $\approx$ denotes homeomorphism. Such maps are classified up to homotopy by the third homotopy group $\pi_3(\mathrm{G})$. It is also a mathematical fact that $\pi_3(\mathrm{G}) \cong \mathbb{Z}$ for every compact simple Lie group. In fact, every continuous mapping of $\mathrm{S}^3 \mapsto \mathrm{G}$ is homotopic to a mapping into a SU(2)-subgroup.[5] So, we may for the moment assume that $\mathrm{G} = \mathrm{SU}(2) \cong \mathrm{S}^3$. Thus, gluing the two different extensions together provides us with a map $\mathrm{S}^3 \to \mathrm{S}^3 \subset \mathrm{G}$. Then the integer $\mathbb{Z}$ simply classifies how many times S^3 wraps around itself.

The WZ-term (2.15) features the extension of the field g. In order for this action to be well-defined, we have to check, that the result is independent of the extension we choose. First, let us check that the action is invariant under small perturbation of the extension g (technically under homotopies of g relative $\partial B = \mathrm{S}^2$). Under this variation, the integrand will be exact, so we can use Stokes' Theorem:

$$\begin{aligned}
\delta\Gamma &= -\frac{i}{4\pi}\int_B \mathrm{tr}\,\big((-g^{-1}\delta g g^{-1}\,\mathrm{d}g + g^{-1}\,\mathrm{d}\delta g)\wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\big) && (2.18)\\
&= -\frac{i}{4\pi}\int_B \mathrm{d}\,\mathrm{tr}\,\big(g^{-1}\delta g g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\big) && (2.19)\\
&= -\frac{i}{4\pi}\int_{\mathrm{S}^2} \mathrm{tr}\,\big(g^{-1}\delta g g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\big) && (2.20)\\
&= -\frac{i}{4\pi}\int_{\mathrm{S}^2} \mathrm{tr}\,\big(g^{-1}\delta g\,\mathrm{d}(g^{-1}\,\mathrm{d}g)\big)\ . && (2.21)
\end{aligned}$$

Hence, if the variation of g vanishes on S^2, then Γ is invariant.

It remains to check the behaviour of Γ under taking topologically different extensions. In fact, for the classical theory to be well-defined, the action does not have to be invariant under topologically inequivalent extensions. After all, we can still get equations of motion just by varying the action. However, this has an important consequence for the quantum theory. As seen above, Γ is invariant under homotopies and g maps without loss of generality into an SU(2) subgroup, so to analyse the behaviour of Γ under change of homotopy class, we can just take g to be the identity, i. e.

$$g(y) = y^0 - iy^k\sigma_k\ , \quad y \in \mathrm{S}^3 \subset \mathbb{R}^4\ . \tag{2.22}$$

Then S^3 wraps the target S^3 exactly once, so we compute the difference of Γ for two extensions with neighbouring homotopy classes. Then $g^{-1}\partial^k g = -i\sigma^k$ and using the form (2.14) of the action, we obtain

$$\Delta\Gamma = -\frac{i}{12\pi}\int_B \mathrm{d}^3y\,\epsilon_{\alpha\beta\gamma}\,\mathrm{tr}\,\big(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\big) \tag{2.23}$$

[5] This is assuming that the group as an SU(2)-subgroup. This may fail for global reasons as for SO(3).

$$= -\frac{i(-i)^3}{12\pi} 2\pi^2 \sum_{i,j,k} \epsilon_{ijk} \mathrm{tr}\,(\sigma^i \sigma^j \sigma^k) \tag{2.24}$$

$$= \frac{\pi}{12} \sum_{i,j,k} \epsilon_{ijk} \mathrm{tr}\,([\sigma^i, \sigma^j]\sigma^k) \tag{2.25}$$

$$= \frac{\pi}{12} 2i \sum_{i,j,k,\ell} \epsilon_{ijk} \mathrm{tr}\,(\epsilon^{ij\ell} \sigma_\ell \sigma_k) \tag{2.26}$$

$$= \frac{i\pi}{6} \sum_{k,\ell} 2\mathrm{tr}\,(\sigma^k \sigma_k) = \frac{i\pi}{6} 12 = 2\pi i \ . \tag{2.27}$$

We used that the integrand is a constant and inserted the volume of the unit 3-sphere.

Let us now set

$$\mathcal{S}[g] \equiv \mathcal{S}_0[g] + k\Gamma[g] \tag{2.28}$$

for some real number k. As we have discussed, this is a perfectly well-defined classical theory for any value of k. However, in the quantum theory, we will have an additional constraint on k. Indeed, let us compute path integrals of the form

$$\langle \mathcal{O} \rangle = \int \mathcal{D}g \, \mathcal{O}[g] \, \mathrm{e}^{-\mathcal{S}[g]} \tag{2.29}$$

for some operators $\mathcal{O}[g]$. In order for this procedure to be well-defined, the exponential $\mathrm{e}^{-\mathcal{S}[g]}$ has to be single-valued. This means that $\mathcal{S}[g]$ has to be well-defined up to the addition of $2\pi i$ times an integer. Since $\Delta\Gamma = 2\pi i$, we conclude that this is the case if

$$k \in \mathbb{Z} \ . \tag{2.30}$$

Thus, for compact Lie groups, the number k has to be an integer for topological reasons.[6] It is referred to as the *level* of the model. Notice that for non-compact Lie groups, there is no such quantization condition.

Let us again consider the classical model.

Exercise 1 Derive the equations of motion for the complete action. Verify that the equations of motion still reflect the conservation of the global $\mathrm{G} \times \mathrm{G}$-symmetry. Express the result in complex coordinates. You should obtain

$$\left(1 + \frac{\lambda^2 k}{\pi}\right) \partial(g^{-1}\bar{\partial} g) + \left(1 - \frac{\lambda^2 k}{\pi}\right) \bar{\partial}(g^{-1}\partial g) = 0 \ . \tag{2.31}$$

Solution 1 We already derived the variation of the two pieces of the action $\mathcal{S}_0 + k\Gamma$, see eqs. (2.7) and(2.21). So our task consists in translating the terms into complex coordinates. We have

$$z = x + iy \ , \qquad \bar{z} = x - iy \ , \qquad \partial = \tfrac{1}{2}(\partial_x - i\partial_y) \ , \qquad \bar{\partial} = \tfrac{1}{2}(\partial_x + i\partial_y) \ . \tag{2.32}$$

[6] This analysis relied on the fact that we have a SU(2)-subgroup inside G. For SO(3), no such subgroup exists and the quantization condition reads instead $k \in 2\mathbb{Z}$.

The measure reads

$$\mathrm{d}^2 z = \mathrm{d}x \wedge \mathrm{d}y = \tfrac{1}{4i}(\mathrm{d}z + \mathrm{d}\bar{z}) \wedge (\mathrm{d}z - \mathrm{d}\bar{z}) = \tfrac{i}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\ . \tag{2.33}$$

Thus, the variation (2.7) becomes

$$\delta \mathcal{S}_0 = -\frac{i}{4\lambda^2}\int_{\mathrm{S}^2} \mathrm{d}z\,\mathrm{d}\bar{z}\ \mathrm{tr}\ \left(g^{-1}\delta g\big(\partial(g^{-1}\bar{\partial}g) + \bar{\partial}(g^{-1}\partial g)\big)\right)\ . \tag{2.34}$$

For the term (2.21), we compute

$$\partial\Gamma = -\frac{i}{4\pi}\int_{\mathrm{S}^2} \mathrm{tr}\ \left(g^{-1}\delta g\,\mathrm{d}(g^{-1}\partial g\,\mathrm{d}z + g^{-1}\bar{\partial}g\,\mathrm{d}\bar{z})\right) \tag{2.35}$$

$$= -\frac{i}{4\pi}\int_{\mathrm{S}^2} \mathrm{tr}\ \left(g^{-1}\delta g\big(\bar{\partial}(g^{-1}\partial g)\,\mathrm{d}\bar{z}\wedge\mathrm{d}z + \partial(g^{-1}\bar{\partial}g)\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\big)\right) \tag{2.36}$$

$$= -\frac{i}{4\pi}\int_{\mathrm{S}^2} \mathrm{d}z\,\mathrm{d}\bar{z}\ \mathrm{tr}\ \left(g^{-1}\delta g\big(-\bar{\partial}(g^{-1}\partial g) + \partial(g^{-1}\bar{\partial}g)\big)\right)\ . \tag{2.37}$$

Adding the two contributions yields (2.31).

By a suitable choice of λ^2 and k, we can achieve for one of the two brackets to vanish. We consider $k \in \mathbb{Z}_{>0}$ and $\lambda^2 = \pi/k$.[7] This choice defines the WZW-model action. We obtain an antiholormophic current $\bar{J} \equiv kg^{-1}\bar{\partial}g$:

$$\partial(g^{-1}\bar{\partial}g) = 0\ , \tag{2.38}$$

This implies also that the current $J \equiv -k\partial g g^{-1}$ is holomorphic:

$$\bar{\partial}(\partial g g^{-1}) = g\partial(g^{-1}\bar{\partial}g)g^{-1} = 0\,. \tag{2.39}$$

Thus, the theory possesses now the desired holomorphic and antiholomorphic currents. Notice however the asymmetry in the definitions of the holomorphic and the antiholomorphic current.

(Anti)holomorphicity arises from an enhanced symmetry of the action. The global $\mathrm{G}\times\mathrm{G}$ symmetry extends to a *local* $\mathrm{G}(z)\times\mathrm{G}(\bar{z})$ symmetry acting as

$$g(z,\bar{z}) \mapsto g_{\mathrm{L}}(z)g(z,\bar{z})g_{\mathrm{R}}^{-1}(\bar{z})\ . \tag{2.40}$$

Here $g_{\mathrm{L}}(z)$ is an arbitrary holomorphic map $\mathrm{S}^2 \mapsto \mathrm{G}$ and $g_{\mathrm{R}}(\bar{z})$ is an arbitrary antiholomorphic map.

3 The quantum theory

3.1 Current algebras

So far, the analysis was mostly classical. In particular, we have not yet shown that the WZW-model is conformal on the quantum level. The usual way to move to the quantum level is to compute the Poisson brackets and then canonically quantize by replacing them

[7] For $k < 0$, the action would not be positive definite and path-integrals do not converge.

with commutators. This also fixes the OPE's of the currents. Instead of taking this route, we will take a shortcut. In fact, we will see that possible OPE's are highly constrained.

So, let us start with the holomorphic current $J^a(z)$ (or rather its components). Here and in the following, indices $a, b, c, \ldots$ always denote adjoint indices. By dimensional analysis, this current has conformal weight one. Thus, $J(z) = -k\partial g g^{-1}$ has conformal dimension one (at this point, we should rather only say dimension). Holomorphicity protects the operator from acquiring an anomalous dimension in the quantum theory. Similarly, the right-moving conformal weight is zero, and so the spin is $h - \bar{h} = 1$, as expected. In fact, this is true in general — any conserved current of a CFT has conformal weight $(1, 0)$ or $(0, 1)$. Hence, when writing out the OPE in the form

$$J^a(z)J^b(w) \sim \sum_p \frac{X_p(w)}{(z-w)^p} \ , \tag{3.1}$$

the holomorphic field $X_p(w)$ has conformal weight $2 - p$. Unitarity imposes severe restrictions on the possible field content of a CFT. In particular, there are no operators of negative conformal dimensions. Thus, the highest possible pole order is a second order pole. Furthermore, there is a unique field of conformal dimension zero, namely the identity. The field $X_1(w)$ is of conformal dimension one and is hence a current itself! Thus, the OPE between $J^a(z)$ and $J^b(w)$ has to take the form

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w} \tag{3.2}$$

for some constants κ^{ab} and $f^{ab}{}_c$.

Symmetry under exchange of $J^a(z)$ and $J^b(w)$ implies $\kappa^{ab} = \kappa^{ba}$ and $f^{ab}{}_c = -f^{ba}{}_c$. Finally, associativity of the OPE gives the following additional conditions:

$$\kappa^{cd} f^{ab}{}_d = \kappa^{bd} f^{ca}{}_d = \kappa^{ad} f^{bc}{}_d \ , \qquad f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0 \ . \tag{3.3}$$

In other words, $f^{ab}{}_c$ are the structure constants of a Lie algebra and κ^{ab} is a symmetric invariant tensor. Since the currents reflect the symmetry of the model, the relevant Lie algebra is $\mathfrak{g} \equiv \mathrm{Lie(G)}$. For simple Lie algebras, there is a unique invariant 2-tensor — the Killing form. We may choose the basis of the Lie algebra such that it becomes proportional to the identity. In this case, we have $\kappa^{ab} = k\delta^{ab}$ for some constant k. Thus, we finally have the OPE structure

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w} \ , \tag{3.4}$$

where $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$. The parameter k is in fact identified with the level k appearing in the action.

The OPE structure (3.4) is called *current algebra*. We have similarly an antiholomorphic copy satisfying

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{i f^{ab}{}_c J^c(\bar{w})}{\bar{z}-\bar{w}} \ . \tag{3.5}$$

This algebraic structure is the main organizing principle of WZW-models.

3.2 Affine Kac-Moody algebras

In this subsection, we reformulate the OPEs in terms of modes. Their equal-time commutator gives an equivalent way of characterizing the algebraic structure. Since $J^a(z)$ is holomorphic (or in fact *meromorphic*), we can consider its Laurent expansion around the origin:

$$J^a(z) \equiv \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \ . \tag{3.6}$$

where the -1 in the exponent is motivated by the fact that $J^a(z)$ is a field of conformal weight 1. Note in particular that with this definition, J^a_0 are the conserved charges of the symmetry current, since

$$J^a_0 = \oint \mathrm{d}z\ J^a(z) \tag{3.7}$$

and the contour integral goes over an equal-time surface. We can compute the commutation relations by contour deformation:

$$\begin{aligned}
[J^a_m, J^b_n] &= \frac{1}{(2\pi i)^2}\left(\oint_{|z|>|w|} \mathrm{d}z \oint \mathrm{d}w - \oint_{|z|<|w|} \mathrm{d}z \oint \mathrm{d}w\right) z^m w^n J^a(z) J^b(w) && (3.8)\\
&= \frac{1}{(2\pi i)^2}\oint_0 \mathrm{d}w \oint_w \mathrm{d}z\ z^m w^n \left(\frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(z)}{z-w}\right) && (3.9)\\
&= \frac{1}{2\pi i}\oint_0 \mathrm{d}w\left(km\delta^{ab} w^{m+n-1} + i f^{ab}{}_c J^c(w) w^{m+n}\right) && (3.10)\\
&= km\delta^{ab}\delta_{m+n,0} + i f^{ab}{}_c J^c_{m+n} \ . && (3.11)
\end{aligned}$$

This Lie algebra is called an *affine Kac-Moody algebra*. We note in particular that the zero modes (the conserved charges) satisfy the algebra $\mathfrak{g}$. The additional term $km\delta^{ab}\delta_{m+n,0}$ is a central term, in a similar way that the Virasoro algebra is a central extension of the Witt algebra. In fact, to any simple Lie algebra $\mathfrak{g}$, we can associate the corresponding affine Kac-Moody algebra, which is often denoted by $\widehat{\mathfrak{g}}_k$ or simply $\mathfrak{g}_k$. We stress that this algebra is *not* a symmetry algebra of the theory. In fact, we shall see below that all modes J^a_m with $m \neq 0$ do not commute with the Hamiltonian. Only the zero-modes corresponding to the conserved charges form a symmetry algebra. For this reason, the affine Kac-Moody algebra is sometimes referred to as *spectrum-generating algebra*.

3.3 The Sugawara construction

To show that the theory is conformal, we have to show that the energy-momentum tensor satisfies the Virasoro algebra. In terms of OPE's,

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \ . \tag{3.12}$$

This is the statement that the conformal algebra acts on the Hilbert space. To understand this, we first have to identify the energy-momentum tensor of the model.

Classically, it is given by

$$T(z) = \frac{1}{2k}\delta_{ab} J^a(z) J^b(z) = \frac{1}{2k} J^a(z) J^a(z) \ . \tag{3.13}$$

A corresponding quantum version will need to implement normal ordering, in order for the vacuum expectation value to be finite. It turns out that the prefactor is corrected in the quantum theory and hence we will denote it for now by γ. Thus, we define

$$T(z) \equiv \gamma (J^a J^a)(z) \ , \tag{3.14}$$

where normal ordering is defined as the constant part of the $J^a(z)J^b(w)$ OPE. Hence all short-distance singularities are subtracted. It is convenient to extract the constant part via a contour integral:

$$(J^a J^a)(z) \equiv \frac{1}{2\pi i} \oint_z \frac{\mathrm{d}x}{x-z} J^a(x) J^a(z) \ . \tag{3.15}$$

As it should be, our energy-momentum tensor defined in this way is indeed holomorphic, even in the quantum theory.

As a first step, we compute the OPE of $T(z)$ with $J^a(w)$. We are only interested in the singular part of the OPE, which we denote by a contraction.

$$\overbracket{J^a(z)T(w)} = \frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \overbracket{J^a(z) \big(J^b(x)} J^b(w)\big) \tag{3.16}$$

$$= \frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \big(\overbracket{J^a(z) J^b(x)} J^b(w) + J^b(x) \overbracket{J^a(z) J^b(w)} \big) \tag{3.17}$$

$$\begin{aligned} &= \frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \left(\left[\frac{k\delta^{ab}}{(z-x)^2} + \frac{i f^{ab}{}_c J^c(x)}{z-x} \right] J^b(w) \right. \\ &\qquad \left. + J^b(x) \left[\frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w} \right] \right) \ . \end{aligned} \tag{3.18}$$

Now, we have to evaluate the remaining JJ-OPE's using (3.4). Here the singular and the regular terms appear. However, since the structure constants are totally antisymmetric in this basis and since the Kronecker symbol is symmetric, the second order pole does not contribute.

$$\begin{aligned} \overbracket{J^a(z)T(w)} &= \frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \left(\frac{k\delta^{ab} J^b(w)}{(z-x)^2} + \frac{k\delta^{ab} J^b(x)}{(z-w)^2} \right. \\ &\qquad + \frac{i f^{ab}{}_c}{z-x} \left[\frac{i f^{cb}{}_d J^d(w)}{x-w} + (J^c J^b)(w) \right] \\ &\qquad \left. + \frac{i f^{ab}{}_c}{z-w} \left[\frac{i f^{bc}{}_d J^d(w)}{x-w} + (J^b J^c)(w) \right] \right) \end{aligned} \tag{3.19}$$

$$= \gamma \left(\frac{2k\delta^{ab} J^b(w)}{(w-z)^2} - \frac{f^{ab}{}_c f^{cb}{}_d J^d(w)}{(w-z)^2} \right) \ . \tag{3.20}$$

We used that the normal ordered expressions vanish after integration, they cancel out by (anti)symmetry in b and c.

We recall the definition of the dual Coxeter number $h^\vee$, the quadratic Casimir of the adjoint representation:

$$f^{ab}{}_c f^{bc}{}_d = 2h^\vee \delta^{ad} \, . \tag{3.21}$$

Indeed, since $f^{ab}{}_c f^{bc}{}_d$ is an invariant 2-tensor, it has to be proportional to the Killing form, which we chose to be the identity. With this definition, we finally obtain

$$T(z)J^a(w) = J^a(w)T(z) \tag{3.22}$$
$$= 2\gamma(k+h^\vee)\frac{J^a(z)}{(w-z)^2} \tag{3.23}$$
$$= 2\gamma(k+h^\vee)\left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right) . \tag{3.24}$$

In order for $J^a(z)$ to have conformal weight one, we define

$$\gamma \equiv \frac{1}{2(k+h^\vee)} . \tag{3.25}$$

Then we obtain

$$T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} . \tag{3.26}$$

This is indeed the expected OPE of the stress-energy tensor with a primary field of dimension one.

Now we still need to demonstrate that $T(z)T(w)$ obeys the Virasoro algebra. We are particularly interested in the central charge of the model. We compute

$$T(z)T(w) = \frac{1}{4\pi i(k+h^\vee)}\oint \frac{\mathrm{d}x}{x-w}\left(T(z)J^a(x)J^a(w) + T(z)J^a(x)J^a(w)\right) \tag{3.27}$$
$$= \frac{1}{4\pi i(k+h^\vee)}\oint \frac{\mathrm{d}x}{x-w}\Bigg(\left[\frac{J^a(x)}{(z-x)^2} + \frac{\partial J^a(x)}{z-x}\right]J^a(w) \tag{3.28}$$
$$+ J^a(x)\left[\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right]\Bigg) \tag{3.29}$$
$$= \frac{1}{4\pi i(k+h^\vee)}\oint \frac{\mathrm{d}x}{x-w}\Bigg(\frac{k\dim\mathfrak{g}}{(z-x)^2(x-w)^2} - \frac{2k\dim\mathfrak{g}}{(z-x)(x-w)^3}$$
$$+ \frac{k\dim\mathfrak{g}}{(x-w)^2(z-w)^2} + \frac{2k\dim\mathfrak{g}}{(z-w)(x-w)^3}\Bigg)$$
$$+ \frac{1}{4\pi i(k+h^\vee)}\oint \frac{\mathrm{d}x}{x-w}\Bigg(\frac{(J^aJ^a)(w)}{(z-x)^2} + \frac{(\partial J^aJ^a)(w)}{z-x}$$
$$+ \frac{(J^aJ^a)(w)}{(z-w)^2} + \frac{(J^a\partial J^a)(w)}{z-w}\Bigg) \tag{3.30}$$
$$= \frac{(3-2+0+0)k\dim\mathfrak{g}}{2(k+h^\vee)(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} . \tag{3.31}$$

Thus, the OPE gives indeed the Virasoro algebra with central charge

$$c \equiv \frac{k\dim\mathfrak{g}}{k+h^\vee} . \tag{3.32}$$

This finally demonstrates that the conformal algebra is also realized on the quantum level and hence WZW-models indeed define CFTs. This construction of the energy-momentum tensor is called *Sugawara construction* [4].

Exercise 2 Rederive the same result using the commutation relations (3.11). In terms of modes, the normal ordering reads

$$L_m = \gamma \sum_{n\in\mathbb{Z}} : J^a_n J^a_{m-n} := \gamma \left(\sum_{n\leq -1} J^a_n J^a_{m-n} + \sum_{n\geq 0} J^a_{m-n} J^a_n \right) . \tag{3.33}$$

You should obtain

$$[L_m, J^a_n] = -n J^a_{m+n} \ , \tag{3.34}$$

$$[L_m, L_n] = \frac{c}{12} m(m^2-1)\delta_{m+n,0} + (m-n)L_{m+n} \ , \tag{3.35}$$

where the central charge is again given by (3.32).

Solution 2 We start with $[L_m, J^a_n]$. We have

$$[L_m, J^a_n] = \gamma \left[\sum_{r\leq -1} J^b_r J^b_{m-r} + \sum_{r\geq 0} J^b_{m-r} J^b_r ,\ J^a_n \right] \tag{3.36}$$

$$\begin{aligned} &= \gamma \Bigg\{ \sum_{r\leq -1} \Big(J^b_r \big(k(m-r)\delta^{ab}\delta_{m+n-r,0} + i f^{ba}{}_c J^c_{m+n-r} \big) \\ &\qquad\qquad + \big(kr\delta^{ab}\delta_{r+n,0} + i f^{ba}{}_c J^c_{r+n} \big) J^b_{m-r} \Big) \\ &\quad + \sum_{r\geq 0} \Big(J^b_{m-r} \big(kr\delta^{ab}\delta_{r+n,0} + i f^{ba}{}_c J^c_{r+n} \big) \\ &\qquad\qquad + \big(k(m-r)\delta^{ab}\delta_{m+n-r,0} + i f^{ba}{}_c J^c_{m+n-r} \big) J^b_r \Big) \Bigg\} . \end{aligned} \tag{3.37}$$

Let us first simply the central terms. This yields

$$[L_m, J^a_n]_{\text{central}} = \gamma \left\{ \sum_{r\in\mathbb{Z}} k(m-r)\delta^{ab} J^b_r \delta_{m+n-r,0} + \sum_{r\in\mathbb{Z}} kr\delta^{ab} J^b_{m-r}\delta_{r+n,0} \right\} \tag{3.38}$$

$$= -2\gamma k n J^a_{m+n} \ . \tag{3.39}$$

The other terms are a little more complicated. The trick is to try to bring the terms back to normal ordering. Two of the terms are already correctly normal ordered, the others need correction terms. This yields

$$\begin{aligned} [L_m, J^a_n]_{\text{non-central}} = \gamma i \Bigg\{ &\sum_{r\leq -1} f^{ba}{}_c J^b_r J^c_{m+n-r} + \sum_{r\geq 0} f^{ba}{}_c J^c_{m+n-r} J^b_r \\ &+ \sum_{r\leq n-1} f^{ba}{}_c J^c_r J^b_{m+n-r} + \sum_{r\geq n} f^{ba}{}_c J^b_{m+n-r} J^c_r \Bigg\} \end{aligned} \tag{3.40}$$

$$= if^{ba}{}_c (J^b J^c)_{m+n} + if^{ba}{}_c (J^c J^b)_{m+n} + \sum_{r=0}^{n-1} if^{ba}{}_c [J^c_r, J^b_{m+n-r}] \quad (3.41)$$

$$= \gamma \sum_{r=0}^{n-1} if^{ba}{}_c \left(kr\delta^{bc}\delta_{m+n,0} + if^{cb}{}_d J^d_{m+n} \right) \quad (3.42)$$

$$= \gamma n f^{ab}{}_c f^{cb}{}_d J^d_{m+n} \quad (3.43)$$

$$= -2\gamma h^\vee n J^a_{m+n} \; . \quad (3.44)$$

Here, we used the definition of the dual Coxeter number. The two normal ordered terms cancel by renaming $b \leftrightarrow c$ in one of them and by the antisymmetry of the structure constants. In total, we hence obtain

$$[L_m, J^a_n] = -2\gamma(k + h^\vee) n J^a_{m+n} = -n J^a_{m+n} \quad (3.45)$$

by definition of γ.

Next, we compute the Virasoro algebra:

$$[L_m, L_n] = \gamma \left[L_m, \sum_{r\leq -1} J^a_r J^a_{n-r} + \sum_{r\geq 0} J^a_{n-r} J^a_r \right] \quad (3.46)$$

$$= \gamma \Bigg\{ \sum_{r\leq -1} \big(-rJ^a_{m+r} J^a_{n-r} - (n-r) J^a_r J^a_{m+n-r} \big)$$

$$+ \sum_{r\geq 0} \big(-rJ^a_{n-r} J^a_{m+r} - (n-r) J^a_{m+n-r} J^a_r \big) \Bigg\} \quad (3.47)$$

$$= \gamma \Bigg\{ \sum_{r\leq m-1} (m-r) J^a_r J^a_{m+n-r} - \sum_{r\leq -1} (n-r) J^a_r J^a_{m+n-r} \big)$$

$$+ \sum_{r\geq m} (m-r) J^a_{m+n-r} J^a_r - \sum_{r\geq 0} (n-r) J^a_{m+n-r} J^a_r \big) \quad (3.48)$$

$$= \gamma(m-n) \left(\sum_{r\leq -1} J^a_r J^a_{m+n-r} + \sum_{r\geq 0} J^a_{m+n-r} J^a_r \right) + \gamma \sum_{r=0}^{m-1} (m-r) [J^a_r, J^a_{m+n-r}] \quad (3.49)$$

$$= (m-n) L_{m+n} + \gamma \sum_{r=0}^{m-1} (m-r) rk\delta^{aa}\delta_{m+n,0} \quad (3.50)$$

$$= (m-n) L_{m+n} + \frac{k}{6}\gamma \dim(\mathfrak{g})\, m(m^2-1)\delta_{m+n,0} \; . \quad (3.51)$$

Thus, the Virasoro algebra with the correct central charge

$$c = 2k\gamma \dim(\mathfrak{g}) = \frac{k \dim(\mathfrak{g})}{k + h^\vee} \quad (3.52)$$

is satisfied.

The Sugawara construction shows that each affine Kac-Moody algebra has naturally a Virasoro algebra contained in its universal enveloping algebra. The combined algebra consisting of the affine Kac-Moody modes and the Virasoro modes forms a semidirect product, as (3.34) shows that the affine Kac-Moody algebra is a Lie ideal inside the combined algebra.

When $\mathfrak{g}$ is only semisimple, i.e.

$$\mathfrak{g} = \bigoplus_{i=1}^{j} \mathfrak{g}_j \,, \tag{3.53}$$

we can simply define

$$T_{\mathfrak{g}}(z) = \sum_{i=1}^{j} T_{\mathfrak{g}_i}(z) \,. \tag{3.54}$$

This again satisfies the Virasoro algebra with central charge

$$c_{\mathfrak{g}} = \sum_{i=1}^{j} c_{\mathfrak{g}_i} \ . \tag{3.55}$$

In the semisimple case, we can choose k independently for all the individual factors.

3.4 The central charge

We now assume again that $\mathfrak{g}$ is simple and consider the expression (3.32) in some detail. We first note that the central charge is in general not a (half)integer, hence we are dealing with a truly interacting theory.

The values of the dual Coxeter numbers for the simple Lie algebras and their dimensions are listed in table 1.

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$h^\vee(\mathfrak{g})$	$n+1$	$2n-1$	$n+1$	$2n-2$	12	18	30	9	4
$\dim(\mathfrak{g})$	n^2+2n	$2n^2+n$	$2n^2+n$	$2n^2-n$	78	133	248	52	14

Table 1: The dual Coxeter number for the simple Lie algebras. We are always using the compact real form of these complex Lie algebras. The subscript always refers to the rank of the Lie algebra. To physicists, the regular series are better known as $\mathrm{A}_n = \mathfrak{su}(n+1)$, $\mathrm{B}_n = \mathfrak{so}(2n+1)$, $\mathrm{C}_n = \mathfrak{sp}(2n)$ and $\mathrm{D}_n = \mathfrak{so}(2n)$. We also tabulated the dimensions.

One can show that for a positive integer k, the central charge satisfies the bound

$$\text{rank } \mathfrak{g} \le c_{\mathfrak{g}} < \dim \mathfrak{g} \ . \tag{3.56}$$

The lower bound is saturated precisely if $\mathfrak{g}$ is *simply-laced* (i.e. of type A, D or E) and $k = 1$. In particular, for these values, the central charge becomes an integer suggesting that these theories are free-field theories in disguise. This turns out to be true and is referred to as the Frenkel-Kac-Segal construction [5, 6]. The free theory consists of rank $\mathfrak{g}$ bosons on the Cartan torus of the group. For the algebras $\mathfrak{su}(n)$ (actually $\mathfrak{u}(n)$) and $\mathfrak{so}(2n)$, there is alternatively a construction in terms of free fermions.

We also notice that there is also another canonical value, where the central charge attains a half-integer value, namely $k = h^\vee$:

$$c(h^\vee) = \frac{h^\vee \dim \mathfrak{g}}{h^\vee + h^\vee} = \frac{1}{2}\dim \mathfrak{g} \ . \tag{3.57}$$

This suggests that we can represent this algebra in terms of $\dim \mathfrak{g}$ fermions. We shall see this below.

3.5 Free field constructions

Let us now work out the free-field representations we mentioned above. We start with $\mathfrak{so}(n)_1$, whose central charge is $c = \frac{1}{2}n$. Thus, we suspect that this theory might be equivalent to n free (Majorana-Weyl) fermions. Indeed, the theory has Lagrangian

$$\mathcal{S} = \int_{\mathrm{S}^2} \mathrm{d}^2 z \ \bar{\psi}^i \gamma^\mu \partial_\mu \psi^i \ , \tag{3.58}$$

where ψ^i, $i = 1, \dots, n$ are the fermion fields. This action clearly has an $\mathrm{SO}(n)$-symmetry acting by rotating the fermions. The corresponding currents are

$$J^a(z) = \frac{1}{2} T^a_{ij} (\psi^i \psi^j)(z) \ , \tag{3.59}$$

where T^a_{ij} are generators of $\mathfrak{so}(n)$ in the fundamental representation. By our general argument, these currents have to satisfy an affine Kac-Moody algebra, which turns out to have level one.

Similarly, we can start with $\dim \mathfrak{g}$ fermions and construct a current algebra via

$$J^a(z) = \frac{i}{2} f^a{}_{bc} (\psi^b \psi^c)(z) \ . \tag{3.60}$$

The level turns out to be $h^\vee$.

Exercise 3 Based on the basic OPE

$$\psi^i(z)\psi^j(w) \sim \frac{\delta^{ij}}{z-w} \ , \tag{3.61}$$

show that the current $\frac{1}{2}T^a_{ij}(\psi^i\psi^j)(z)$ indeed satisfies the $\mathfrak{so}(n)_1$ current algebra.

Similarly show that starting from the OPE

$$\psi^a(z)\psi^b(w) \sim \frac{\delta^{ab}}{z-w} \ , \tag{3.62}$$

the current $J^a(z) = \frac{i}{2} f^a{}_{bc}(\psi^b\psi^c)(z)$ satisfies the algebra $\mathfrak{g}_{h^\vee}$.

Solution 3 We will show the following more general statement: For fermions transforming in any representation labelled by a highest weight state λ (see also next section), the bilinears $J^a = \frac{1}{2}T^a_{ij}(\psi^i\psi^j)$ give a Kac-Moody algebra of $\mathfrak{g}$ at level

$$k = \frac{\mathcal{C}(\lambda)\dim(\lambda)}{2\dim(\mathfrak{g})} \ . \tag{3.63}$$

Here, $\mathcal{C}(\lambda)$ is the Casimir of the representation and $\dim(\lambda)$ its dimension. The exercise asks then about the cases where λ describes the fundamental representation and the adjoint representation respectively. (For general representations, it is however not true that the Sugawara tensor of the current algebra coincides with the Sugawara tensor of the free fermions.)

The computation can be either done in modes or OPEs, whatever one prefers. Let us show it in the OPE language. The normal ordered product is again defined by

$$J^a(z) = \frac{1}{2} T^a_{ij} \oint_z \frac{\mathrm{d}x}{2\pi i(x-z)} \psi^i(x)\psi^j(z) \ . \tag{3.64}$$

We then compute first the OPE between $J^a(z)$ and $\psi^\ell(w)$:

$$\begin{aligned} J^a(z)\psi^\ell(w) &\sim \frac{1}{2} T^a_{ij} \oint_z \frac{\mathrm{d}x}{2\pi i(x-z)} \psi^i(x)\psi^j(z)\psi^\ell(w) &(3.65)\\ &\sim \frac{1}{2} T^a_{ij} \oint_z \frac{\mathrm{d}x}{2\pi i(x-z)} \left(-\frac{\delta^{i\ell}\psi^j(z)}{x-w} + \frac{\delta^{j\ell}\psi^i(x)}{z-w}\right) &(3.66)\\ &\sim \frac{1}{2(z-w)}\Big(-T^a_{\ell j}\psi^j(z) + T^a_{i\ell}\psi^i(w)\Big) &(3.67)\\ &\sim \frac{1}{2(z-w)}\Big(-T^a_{\ell j}\psi^j(w) + T^a_{i\ell}\psi^i(w)\Big) &(3.68)\\ &\sim -\frac{T^a_{\ell i}\psi^i(w)}{z-w} \ . &(3.69) \end{aligned}$$

Here, one has to be attentive to the minus sign due to the fermionic statistics. We also used that $\mathrm{SO}(n)$ representation are antisymmetric matrices. This means that the fermions will transform in the respective representation of the current, see eq. (3.83) below. Next, we compute the OPE between the currents:

$$\begin{aligned} J^a(z)J^b(w) &\sim \frac{1}{2} T^b_{ij} \oint_w \frac{\mathrm{d}x}{2\pi i(x-w)} J^a(z)\psi^i(x)\psi^j(w) &(3.70)\\ &\sim \frac{1}{2} T^b_{ij} \oint_w \frac{\mathrm{d}x}{2\pi i(x-w)} \left(\frac{-T^a_{i\ell}\psi^\ell(x)\psi^j(w)}{z-x} + \frac{-T^a_{j\ell}\psi^i(x)\psi^\ell(w)}{z-w}\right) &(3.71)\\ &\sim \frac{1}{2} T^b_{ij} \oint_w \frac{\mathrm{d}x}{2\pi i(x-w)} \Bigg(-\frac{T^a_{i\ell}\delta^{\ell j}}{(z-x)(x-w)} - \frac{T^a_{j\ell}\delta^{i\ell}}{(z-w)(x-w)} \\ &\qquad - \frac{T^a_{i\ell}(\psi^\ell\psi^j)(w)}{z-x} - \frac{T^a_{j\ell}(\psi^i\psi^\ell)(w)}{z-w}\Bigg) &(3.72)\\ &\sim -\frac{T^b_{ij}T^a_{ij}}{2(z-w)^2} + \frac{T^b_{ij}T^a_{i\ell}(\psi^\ell\psi^j)(w)}{2(z-w)} - \frac{T^b_{ij}T^a_{j\ell}(\psi^i\psi^\ell)(w)}{2(z-w)} &(3.73)\\ &\sim \frac{\mathrm{tr}\,(T^aT^b)}{2(z-w)^2} + \frac{[T^a,T^b]_{j\ell}(\psi^j\psi^\ell)(w)}{2(z-w)} &(3.74) \end{aligned}$$

Now let us evaluate the representation theoretic quantities. In the first order pole, we have

$$[T^a, T^b] = i f^{ab}{}_c T^c \ , \tag{3.75}$$

this is the characterising property of a representation. For the second order pole, we recall the definition of the quadratic Casimir:

$$(T^a T^a)_{ij} = \mathcal{C}(\lambda)\delta_{ij} \ . \tag{3.76}$$

By taking the trace of this relation, we learn

$$\mathcal{C}(\lambda)\mathrm{dim}(\lambda) = \mathrm{tr}\left(T^a T^a\right) \ . \tag{3.77}$$

On the other hand, $\mathrm{tr}\left(T^a T^b\right)$ has to be an invariant tensor of $\mathfrak{g}$ and hence proportional to δ^{ab}. By taking the trace we hence conclude that

$$\mathrm{tr}\left(T^a T^b\right) = \frac{\mathcal{C}(\lambda)\,\mathrm{dim}(\lambda)}{\mathrm{dim}(\mathfrak{g})}\delta^{ab} \ . \tag{3.78}$$

The constant is sometimes called the index of the representation. Thus, we conclude that the current algebra is satisfied with level

$$k = \frac{\mathcal{C}(\lambda)\mathrm{dim}(\lambda)}{2\ \mathrm{dim}(\mathfrak{g})} \ . \tag{3.79}$$

In particular, for $\mathfrak{so}(n)$, we have $\mathrm{dim}(\mathfrak{so}(n)) = \frac{1}{2}n(n-1)$. The Casimir of the fundamental representation is $n-1$ and hence we have

$$k_{\mathbf{fund}} = \frac{(n-1)n}{n(n-1)} = 1 \ . \tag{3.80}$$

For the adjoint representation we have instead

$$k_{\mathbf{adj}} = \frac{\mathcal{C}(\mathbf{adj})}{2} = h^\vee \ . \tag{3.81}$$

We learn that seemingly different conformal field theories may actually become equivalent at the quantum level. The actions (2.28) and (3.58) look certainly very different, but turn out to describe the same quantum theory! In particular, one action is bosonic and the other is fermionic. The simplest incarnation of this fact is the observation that one free boson (on a circle with a particular radius) is equivalent to two fermions. This phenomenon is called *non-abelian bosonization* [7].

In general, for any CFT with a holomorphic conserved current, the theory will contain a subsector with a Kac-Moody symmetry.

3.6 Representations

We have established that a Kac-Moody algebra acts on the Hilbert space of the theory. Hence all states of the CFT will transform in representations of the algebra $\mathfrak{g}_k$. Thus, we will now discuss representations of these algebras. For this, we again restrict to simple compact Lie groups. The non-compact case is much more complicated and much less understood.

As for the Virasoro algebra, only highest weight representations of the Kac-Moody algebra are physically relevant, since these have bounded energy spectrum from below.

Thus, there is a highest weight state $|\lambda\rangle$ satisfying

$$J_n^a |\lambda\rangle = 0 \quad \text{for} \quad n > 0 \ . \tag{3.82}$$

Furthermore, the zero-modes J_0^a satisfy the original Lie algebra $\mathfrak{g}$. In particular, we require $|\lambda\rangle$ to be a highest weight state of a highest weight representation of $\mathfrak{g}$, which is labelled by λ.[8] This then characterizes the representation completely.

Exercise 4 The Sugawara construction may be used to derive the analog of the BPZ-equations for WZW-models. Show first that the primary condition (3.82) translates in OPE language to

$$J^a(z)\Phi^\lambda(w) \sim \frac{t^a \Phi^\lambda(w)}{z-w} \ , \tag{3.83}$$

where t^a are the matrices of the representation of the zero-modes and Φ^λ is the field corresponding to the primary state $|\lambda\rangle$. By inserting the identity

$$L_{-1} - \frac{1}{k+h^\vee}(J^a J^a)_{-1} = 0 \tag{3.84}$$

in a correlator of affine primary fields, show that the correlator obeys the Knizhnik-Zamolodchikov equation [8]:

$$\left(\partial_{z_i} - \frac{1}{k+h^\vee} \sum_{i \neq j} \frac{t_i^a \otimes t_j^a}{z_i - z_j} \right) \langle \Phi^{\lambda_1}(z_1) \cdots \Phi^{\lambda_n}(z_n) \rangle = 0 \ . \tag{3.85}$$

Here, the subscripts in the generators t_i^a indicate on which field they are acting. Thus, the existence of the affine algebra constrains the n-point functions severely. Solving this differential equation is challenging, but the four-point functions may be computed in this fashion.

Solution 4 We have in general for an OPE:

$$J^a(z)\Phi^\lambda(w) \sim \sum_{p=1}^{\infty} \frac{V(J_{p-1}^a |\lambda\rangle, w)}{(z-w)^p} \ , \tag{3.86}$$

where $V(J_{p-1}^a |\lambda\rangle, w)$ is the field associated to the state $J_{p-1}^a |\lambda\rangle$ via the operator-state correspondence. Since $|\lambda\rangle$ is primary, this is only non-vanishing for $p = 1$, in which case $V(J_0^a |\lambda\rangle, w) = t^a V(|\lambda\rangle, w) = t^a \Phi^\lambda(w)$, hence (3.83) is the corresponding statement for the OPEs.

Let us now consider the correlator

$$0 = \left\langle \Phi^{\lambda_1}(z_1) \cdots \left(L_{-1} - \frac{1}{k+h^\vee}(J^a J^a)_{-1} \right) \Phi^{\lambda_i}(z_i) \cdots \Phi^{\lambda_n}(z_n) \right\rangle \ . \tag{3.87}$$

We have $L_{-1}\Phi^{\lambda_i}(z_i) = \partial_{z_i}\Phi^{\lambda_i}(z_i)$. Let us now compute the bilinear current term. By definition, $(J^a J^a)_{-1}\Phi^{\lambda_i}(z_i)$ is the first order pole of the OPE between $(J^a J^a)(z)$

[8] λ can be understood as the Dynkin labels of the representation.

and $\Phi^{\lambda_i}(z_i)$. Thus, we have

$$(J^a J^a)_{-1}\Phi^{\lambda_i}(z_i) = \oint_{z_i} \frac{\mathrm{d}z}{2\pi i}(J^a J^a)(z)\Phi^{\lambda_i}(z_i) \tag{3.88}$$

$$= \oint_{z_i} \frac{\mathrm{d}z}{2\pi i} \oint_{z} \frac{\mathrm{d}x}{2\pi i(x-z)} J^a(x)J^a(z)\Phi^{\lambda_i}(z_i)\ . \tag{3.89}$$

Now note that we are computing the correlators on the Riemann sphere. Instead of encircling z, we can let x encircle the complement of z. Possible singularities come then from all z_i's. Thus, we have

$$\left\langle \Phi^{\lambda_1}(z_1)\cdots(J^aJ^a)_{-1}\Phi^{\lambda_i}(z_i)\cdots\Phi^{\lambda_n}(z_n)\right\rangle \tag{3.90}$$

$$= -\sum_{j=1}^{n}\oint_{z_i}\frac{\mathrm{d}z}{2\pi i}\oint_{z_j}\frac{\mathrm{d}x}{2\pi i(x-z)}\left\langle J^a(x)J^a(z)\Phi^{\lambda_1}(z_1)\cdots\Phi^{\lambda_n}(z_n)\right\rangle \tag{3.91}$$

$$= -\sum_{j=1}^{n}\oint_{z_i}\frac{\mathrm{d}z}{2\pi i}\oint_{z_j}\frac{\mathrm{d}x}{2\pi i(x-z)}\frac{t_j^a}{x-z_j}\left\langle J^a(z)\Phi^{\lambda_1}(z_1)\cdots\Phi^{\lambda_n}(z_n)\right\rangle \tag{3.92}$$

$$= -\sum_{j=1}^{n}\oint_{z_i}\frac{\mathrm{d}z}{2\pi i}\frac{t_i^a\otimes t_j^a}{(z_j-z)(z-z_i)}\left\langle \Phi^{\lambda_1}(z_1)\cdots\Phi^{\lambda_n}(z_n)\right\rangle \tag{3.93}$$

$$= \sum_{j=1}^{n}\frac{t_i^a\otimes t_j^a}{z_i-z_j}\left\langle \Phi^{\lambda_1}(z_1)\cdots\Phi^{\lambda_n}(z_n)\right\rangle\ . \tag{3.94}$$

Let us exemplify the meaning of primary fields at the simplest example of $\mathfrak{g} = \mathfrak{su}(2)$. In a suitable basis, the affine Kac-Moody algebra reads[9]

$$[J^3_m, J^3_n] = \frac{k}{2} m\delta_{m+n,0}\ , \tag{3.95}$$

$$[J^3_m, J^\pm_n] = \pm J^\pm_{m+n}\ , \tag{3.96}$$

$$[J^+_m, J^-_n] = km\delta_{m+n,0} + 2J^3_{m+n}\ . \tag{3.97}$$

A highest weight state can now be denoted by $|\ell\rangle$ and satisfies

$$J^a_n\,|\ell\rangle = 0 \quad \text{for} \quad n > 0\ , \tag{3.98}$$

$$J^+_0\,|\ell\rangle = 0\ , \tag{3.99}$$

$$J^3_0\,|\ell\rangle = \ell\,|\ell\rangle\ . \tag{3.100}$$

Hence, the ground state transforms in the spin ℓ representation. We can now build up the representation by acting with as many of the other oscillators as desired. More precisely, this defines a representation

$$\mathcal{V} = \left\{\cdots(J^3_{-2})^{n^3_2}(J^+_{-2})^{n^+_2}(J^-_{-1})^{n^-_1}(J^3_{-1})^{n^3_1}(J^+_{-1})^{n^+_1}(J^-_0)^{n^0_-}\,|\ell\rangle\,\middle|\,n^+_i, n^-_i, n^3_i \geq 0\right\}\ . \tag{3.101}$$

This space is called the *Verma-module* of the representation. However, this representation is typically not an irreducible representation, because of the existence of null-vectors.

[9]This is *not* the basis we have been using so far in which the Killing form is proportional to the identity. However, this basis is much more useful for understanding the representation theory.

To talk about such null-vectors, we first have to introduce the canonical norm on this vectorspace. It is induced from the following hermitian properties:

$$(J_n^+)^\dagger = J_{-n}^- \ , \qquad (J_n^3)^\dagger = J_{-n}^3 \ . \tag{3.102}$$

This specifies a real form of the Kac-Moody algebra and implies that we are indeed considering the compact form $\mathfrak{su}(2)$.

A simple such null-vector arises already from the ground states, $(J_0^-)^{2\ell+1} |\ell\rangle = 0$. But there is a second null-vector taking the form

$$|\mathcal{N}\rangle = (J_{-1}^+)^{k+1-2\ell} |\ell\rangle = 0 \ . \tag{3.103}$$

Since this fact is of central importance in what follows, we shall demonstrate it. We even show the stronger property of being singular, meaning that this vector is again a highest weight state of an affine representation. This implies in particular that it is null. To start, J_0^+ annihilates this singular vector, since $[J_0^+, J_{-1}^+] = 0$. We now show that the vector is also annihilated by all modes J_1^a. This is obvious for J_1^+, since $[J_1^+, J_{-1}^+] = 0$ and hence we can simply commute the oscillator J_1^+ through, where it hits the highest weight state $|\ell\rangle$. Next, we consider J_1^3. We again commute J_1^3 through all oscillators:

$$J_1^3 (J_{-1}^+)^{k+1-2\ell} |\ell\rangle = [J_1^3, (J_{-1}^+)^{k+1-2\ell}] |\ell\rangle = \sum_{m=0}^{k-2\ell} (J_{-1}^+)^m J_0^+ (J_{-1}^+)^{k-2\ell-m} |\ell\rangle = 0 \ , \tag{3.104}$$

since J_0^+ can be commuted through, where it hits the highest weight state. Thus, we only have to show that J_1^- also annihilates the state. This will complete the proof, since we can obtain any other positive mode as a repeated commutator of J_1^a's. We calculate

$$\begin{aligned} J_1^- (J_{-1}^+)^{k+1-2\ell} |\ell\rangle &= [J_1^-, (J_{-1}^+)^{k+1-2\ell}] |\ell\rangle && (3.105) \\ &= \sum_{m=0}^{k-2\ell} (J_{-1}^+)^m (k - 2J_0^3)(J_{-1}^+)^{k-2\ell-m}] |\ell\rangle && (3.106) \\ &= \sum_{m=0}^{k-2\ell} (J_{-1}^+)^m (k - 2(k-\ell-m))(J_{-1}^+)^{k-2\ell-m}] |\ell\rangle && (3.107) \\ &= \sum_{m=0}^{k-2\ell} (2\ell + 2m - k)(J_{-1}^+)^{k+1-2\ell} |\ell\rangle = 0 \ . && (3.108) \end{aligned}$$

Exercise 5 Show the following generalization of this result. The vector $(J_{-1}^+)^N |\ell\rangle$ has norm

$$\langle \ell| (J_1^-)^N (J_{-1}^+)^N |\ell\rangle = \prod_{n=1}^{N} n(k+1-n-2\ell) \ , \tag{3.109}$$

where we assume that $\langle \ell|\ell\rangle = 1$. In particular, this norm is positive for $N < k+1-2\ell$ and zero for $N \geq k+1-2\ell$, provided that $k \in \mathbb{Z}$.

Solution 5 Let us proceed by induction over N. The case $N=0$ is trivial. To show $N \to N+1$, we compute

$$\langle \ell | (J_1^-)^N (J_{-1}^+)^N |\ell\rangle = \langle \ell | (J_1^-)^{N-1} [J_1^-, (J_{-1}^+)^N] |\ell\rangle \tag{3.110}$$

$$= \sum_{m=0}^{N-1} \langle \ell | (J_1^-)^{N-1} (J_{-1}^+)^m \big(k - 2J_0^3\big) (J_{-1}^+)^{N-1-m} |\ell\rangle \tag{3.111}$$

$$= \sum_{m=0}^{N-1} \langle \ell | (J_1^-)^{N-1} (J_{-1}^+)^m \big(k - 2(\ell + N - 1 - m)\big) (J_{-1}^+)^{N-1-m} |\ell\rangle \tag{3.112}$$

$$= \sum_{m=0}^{N-1} \big(k - 2(\ell + N - 1 - m)\big) \langle \ell | (J_1^-)^{N-1} (J_{-1}^+)^{N-1} |\ell\rangle \tag{3.113}$$

$$= N(k+1-N-2\ell) \prod_{n=1}^{N-1} n(k+1-n-2\ell) \tag{3.114}$$

$$= \prod_{n=1}^{N} n(k+1-n-2\ell) \ , \tag{3.115}$$

where we used the induction hypothesis in the penultimate line.

It turns out that these two vectors together with their descendants are the only null-vectors in the representation. Thus, we can obtain the corresponding irreducible representation by dividing the *Verma-module* by the null-vector relations. We denote the resulting space by $\mathcal{M}_\ell$.

The existence of this null-vector constrains the possible representations severely. Since WZW-models (on compact Lie groups) are unitary, it is vital that no negative-norm states are part of the representation. For this to be the case, the exercise shows that two conditions have to be met. First, we see algebraically that $k \in \mathbb{Z}_{>0}$. Indeed, if k is not a positive integer, we cannot have $k+1-2\ell \in \mathbb{Z}_{>0}$ for any value of $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Hence these theories could not have any (unitary) representations. Second, this null-vector has to occur at a positive level and hence $k+1-2\ell \geq 1$. So we conclude that only representations with

$$\ell \leq \frac{k}{2} \tag{3.116}$$

define consistent unitary representations of the affine Kac-Moody algebra. In particular there are only finitely many representations. A CFT with only finitely many representations is called *rational* and WZW-models are prime examples of rational CFTs. A similar statement is true for any WZW-model based on a compact Lie group. Only finitely many representations of $\mathfrak{g}$ lift to unitary representations of $\mathfrak{g}_k$. We will denote the modules in general by $\mathcal{M}_\lambda$, where $\lambda \in \mathcal{R}$ labels the allowed representations of the model.

Let us compute the conformal weight of the Kac-Moody highest weight state for an arbitrary Lie group:

$$L_0 |\lambda\rangle = \frac{1}{2(k+h^\vee)} \Bigg(\sum_{n \leq -1} J_n^a J_{-n}^a + \sum_{n \geq 0} J_{-n}^a J_n^a \Bigg) |\lambda\rangle \tag{3.117}$$

$$= \frac{1}{2(k+h^\vee)} J_0^a J_0^a |\lambda\rangle \tag{3.118}$$

$$= \frac{\mathcal{C}(\lambda)}{2(k+h^\vee)} |\lambda\rangle \ . \tag{3.119}$$

Here, $\mathcal{C}(\lambda)$ denotes the value of the quadratic Casimir of the zero-mode representation specified by λ. In particular for $\mathfrak{su}(2)$, we have

$$h(|\ell\rangle) = \frac{\ell(\ell+1)}{k+2} \ . \tag{3.120}$$

Knowing the conformal weight of the highest weight state fixes the conformal weights of all states in the representation. Indeed, from (3.34), we conclude that

$$h\Big(J^{a_1}_{-n_1} \cdots J^{a_m}_{-n_m} |\lambda\rangle\Big) = \frac{\mathcal{C}(\lambda)}{2(k+h^\vee)} + \sum_{p=1}^m n_p \ . \tag{3.121}$$

In other words, the conformal weight of a state equals the conformal weight of the highest weight state plus the total mode number of oscillators applied to the ground state.

3.7 Characters

Characters are a convenient way of summarizing all states in a given representation. Let us recall the character of a conformal representation. We define $q = \mathrm{e}^{2\pi i \tau}$, where $\tau \in \mathbb{H}$ is in the upper half plane. Then the character of a given representation measures the contribution of this representation to the partition function of the theory. We define

$$\chi_\ell(\tau) = \mathrm{tr}_{\mathcal{M}_\ell}\left(q^{L_0 - \frac{c}{24}}\right) \ . \tag{3.122}$$

Thus the character keeps track of all states in the representation and counts them with their corresponding multiplicity.

Recall that the collection $(\chi_\lambda)_{\lambda\in\mathcal{R}}$ of characters of a rational conformal field theory with allowed representations $\mathcal{R}$ transform into each other under modular transformations as follows:

$$\chi_\lambda(\tau+1) = \sum_{\mu\in\mathcal{R}} T_{\lambda\mu}\chi_\mu(\tau) \ , \tag{3.123}$$

$$\chi_\lambda(-\tfrac{1}{\tau}) = \sum_{\mu\in\mathcal{R}} S_{\lambda\mu}\chi_\mu(\tau) \ . \tag{3.124}$$

Here T and S are finite-dimensional constant matrices.

In the context of the $\mathfrak{su}(2)_k$ WZW-model, it is convenient to refine the notion of characters slightly by keeping track of the $\mathfrak{su}(2)$-charge as well. Thus, we define

$$\chi_\ell(z;\tau) = \mathrm{tr}_{\mathcal{M}_\ell}\left(q^{L_0 - \frac{c}{24}} y^{J_0^3}\right) \ , \tag{3.125}$$

where $y = \mathrm{e}^{2\pi i z}$.[10] z is usually called 'chemical potential', since it represents the chemical potential associated to the U(1)-symmetry. The characters defined in this way still transforms covariantly under modular transformations:

$$\chi_\lambda(z;\tau+1) = \sum_{\mu\in\mathcal{R}} T_{\lambda\mu}\chi_\mu(z;\tau) \ , \tag{3.126}$$

[10]Geometrically, z should be understood as a coordinate on the torus with modular parameter τ.

$$\chi_\lambda(\tfrac{z}{\tau};-\tfrac{1}{\tau})=\mathrm{e}^{\frac{kz^2}{2\tau}}\sum_{\mu\in\mathcal{R}}S_{\lambda\mu}\chi_\mu(z;\tau)\ . \tag{3.127}$$

For the $\mathfrak{su}(2)_k$ WZW-model, the characters take the following form. Let us first define the level-k theta functions

$$\Theta^{(k)}_m(z;\tau)\equiv\sum_{n\in\mathbb{Z}+\frac{m}{2k}}q^{kn^2}y^{kn}\ . \tag{3.128}$$

Then $\mathfrak{su}(2)_k$-characters are given by

$$\chi_\ell(z;\tau)=\frac{\Theta^{(k+2)}_{2\ell+1}(z;\tau)-\Theta^{(k+2)}_{-2\ell-1}(z;\tau)}{\Theta^{(2)}_{1}(z;\tau)-\Theta^{(2)}_{-1}(z;\tau)} \tag{3.129}$$

$$=\frac{\Theta^{(k+2)}_{2\ell+1}(z;\tau)-\Theta^{(k+2)}_{-2\ell-1}(z;\tau)}{q^{\frac{1}{8}}(y^{\frac{1}{2}}-y^{-\frac{1}{2}})\prod_{n=1}^\infty(1-q^n)(1-yq^n)(1-y^{-1}q^n)}\ . \tag{3.130}$$

To go from the first to the second line, we used Jacobi's triple product identity

$$\sum_{n\in\mathbb{Z}}(-1)^n y^{n+\frac{1}{2}}q^{\frac{1}{2}(n+\frac{1}{2})^2}=q^{\frac{1}{8}}(y^{\frac{1}{2}}-y^{-\frac{1}{2}})\prod_{n=1}^\infty(1-q^n)(1-yq^n)(1-y^{-1}q^n)\ . \tag{3.131}$$

Such an identity exists for any simple Kac-Moody algebra, where it is called the Kac-Weyl denominator formula.

Let us unpack the character. For this, we use the version (3.130). The infinite product in the denominator comes about as follows. We have three oscillators J^3_{-n} and $J^\pm_{-n}$ of which we can apply arbitrary many on the ground states to generate new states. Since these oscillators are bosonic, they contribute

$$1+q^n+q^{2n}+q^{3n}+\cdots=(1-q^n)^{-1} \tag{3.132}$$

to the character. Two of the oscillators also have a charge, which explains the appearance of the additional factors of y and y^{-1}. The rest of the expression accounts for the fact that there is more than one ground state and null vectors in the module. Let us look at the lowest order (in q) terms of the numerator. We have

$$\Theta^{(k+2)}_{2\ell+1}(z;\tau)-\Theta^{(k+2)}_{-2\ell-1}(z;\tau)=q^{\frac{(\ell+\frac{1}{2})^2}{k+2}}\left(y^{\ell+\frac{1}{2}}-y^{-\ell-\frac{1}{2}}\right)-q^{\frac{(k-\ell+\frac{3}{2})^2}{k+2}}\left(y^{k-\ell+\frac{3}{2}}-y^{-k+\ell-\frac{3}{2}}\right)+\cdots \tag{3.133}$$

Let us also recall the finite dimensional $\mathfrak{su}(2)$-character. It is given by

$$\chi^{\mathfrak{su}(2)}_\ell(z)=\sum_{m=-\ell,\,m+\ell\in\mathbb{Z}}^{\ell}y^m=\frac{y^{\ell+\frac{1}{2}}-y^{-\ell-\frac{1}{2}}}{y^{\frac{1}{2}}-y^{-\frac{1}{2}}}\ . \tag{3.134}$$

Thus, the oscillator contribution is multiplied by

$$q^{\frac{(\ell+\frac{1}{2})^2}{k+2}-\frac{1}{8}}\chi^{\mathfrak{su}(2)}_\ell(z)-q^{\frac{(k-\ell+\frac{3}{2})^2}{k+2}-\frac{1}{8}}\chi^{\mathfrak{su}(2)}_{k+1-\ell}(z)+\cdots$$
$$=q^{-\frac{k}{8(k+2)}+\frac{\ell(\ell+1)}{k+2}}\left(\chi^{\mathfrak{su}(2)}_\ell(z)-q^{k+1-2\ell}\chi^{\mathfrak{su}(2)}_{k+1-\ell}(z)+\cdots\right)\ . \tag{3.135}$$

We recognize the prefactor as $q^{h-\frac{c}{24}}$, where the central charge is given in (3.32) and the conformal weight of the ground state in (3.120). We also understand the second term in the series as subtracting out the null-vector $(J^+_{-1})^{k+1-2\ell}\,|\ell\rangle$, which we have discussed before at length. The next terms in the series correspond to the fact that we have also subtracted all the descendants of the null-vector. However, some descendants are actually not there and have to be put in again into the character. This pattern continues and yields an alternating sum.

We now investigate the behaviour under modular transformations. Under the T-modular transformation, $q \to q\mathrm{e}^{2\pi i}$ and the character picks up a phase. We conclude

$$T_{\ell\ell'} = \mathrm{e}^{2\pi i\left(\frac{\ell(\ell+1)}{k+2}-\frac{k}{8(k+2)}\right)}\delta_{\ell,\ell'}\ . \tag{3.136}$$

The behaviour under the S-modular transformation is much more interesting.

Exercise 6 By using the Poisson ressumation formula

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi a n^2+bn} = \frac{1}{\sqrt{a}}\sum_{m\in\mathbb{Z}} \mathrm{e}^{-\frac{\pi}{a}(m-\frac{b}{2\pi i})^2}\ , \tag{3.137}$$

show first that

$$\Theta^{(k)}_m\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right) = \sqrt{\frac{-i\tau}{2k}}\mathrm{e}^{\frac{2\pi i k z^2}{4\tau}}\sum_{m'=-k+1}^{k} \mathrm{e}^{-\frac{2\pi i m m'}{2k}}\Theta^{(k)}_{m'}(z;\tau)\ . \tag{3.138}$$

Deduce then that the S-matrix in (3.124) is given by

$$S_{\ell\ell'} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(2\ell+1)(2\ell'+1)}{k+2}\right)\ . \tag{3.139}$$

Check that the S-matrix is unitary and symmetric.

Solution 6 For completeness, let us first prove the Poisson ressumation formula. Let $f(x)$ be a (reasonable nice) function on the real line. The consider

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n)\ . \tag{3.140}$$

By construction, this is a 1-periodic function. Hence we can compute its Fourier series. The Fourier coefficients are given by

$$a_m = \int_0^1 \mathrm{d}x\ F(x)\mathrm{e}^{-2\pi i m x} \tag{3.141}$$
$$= \sum_{n\in\mathbb{Z}}\int_0^1 \mathrm{d}x\ f(x+n)\mathrm{e}^{-2\pi i m(x+n)} \tag{3.142}$$
$$= \int_{-\infty}^{\infty} \mathrm{d}y\ f(y)\mathrm{e}^{-2\pi i m y} \tag{3.143}$$
$$= \hat{f}(m)\ . \tag{3.144}$$

Here, $\hat{f}(p)$ denotes the Fourier transform of $f(x)$. Thus, we have

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\mathrm{e}^{2\pi imx} \ . \tag{3.145}$$

Specialising to $x = 0$, we obtain the general form of the ressumation formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m) \ . \tag{3.146}$$

The given formula is simply the special case of

$$f(x) = \mathrm{e}^{-\pi ax^2+bx} \ . \tag{3.147}$$

Its Fourier transform is

$$\hat{f}(p) = \int_{-\infty}^{\infty} \mathrm{d}x \ \mathrm{e}^{-\pi ax^2+bx-2\pi ipx} = \frac{1}{\sqrt{a}}\mathrm{e}^{-\frac{\pi}{a}(p-\frac{b}{2\pi i})^2} \ . \tag{3.148}$$

Thus, the result follows.

Now, we apply this to the Theta-function. We have

$$\Theta_m^{(k)}\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{2\pi ik}{\tau}(n+\frac{m}{2k})^2+\frac{2\pi izk}{\tau}(n+\frac{m}{2k})} \tag{3.149}$$

$$= \sqrt{\frac{-i\tau}{2k}}\mathrm{e}^{\frac{2\pi ikz^2}{4\tau}}\sqrt{\frac{2k}{-i\tau}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{2\pi ik}{\tau}(n+\frac{m}{2k}-\frac{z}{2})^2} \ . \tag{3.150}$$

We already have the desired prefactor. The rest is precisely the right hand side of the Poisson ressumation formula with $a = \frac{-i\tau}{2k}$ and $b = \pi i(z - \frac{m}{k})$. Thus, we have

$$\sqrt{\frac{2k}{-i\tau}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{2\pi ik}{\tau}(n+\frac{m}{2k}-\frac{z}{2})^2} = \sum_{r\in\mathbb{Z}} \mathrm{e}^{\frac{\pi i\tau r^2}{2k}+\pi i(z-\frac{m}{k})r} \ . \tag{3.151}$$

We now write

$$r = 2kn \tag{3.152}$$

where $n \in \mathbb{Z} + \frac{m'}{2k}$ and $m' \in \{-k+1,\dots,k\}$. This change of variables clearly still covers every integer value of r precisely once. Thus:

$$\sum_{r\in\mathbb{Z}} \mathrm{e}^{\frac{\pi i\tau r^2}{2k}+\pi i(z-\frac{m}{k})r} = \sum_{m'=-k+1}^{k}\ \sum_{n\in\mathbb{Z}+\frac{m'}{2k}} \mathrm{e}^{2\pi i\tau n^2+2\pi i(kz-m)n} \tag{3.153}$$

$$= \sum_{m'=-k+1}^{k} \mathrm{e}^{-\frac{2\pi imm'}{2k}}\Theta_{m'}^{(k)}(z;\tau) \ . \tag{3.154}$$

This shows the first part.

Now we work out the transformation behaviour of the $\mathfrak{su}(2)_k$ characters. The denominator becomes:

$$\Theta_1^{(2)}\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right)-\Theta_{-1}^{(2)}\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right)=i\sqrt{-i\tau}\,\mathrm{e}^{\frac{\pi i z^2}{\tau}}\sum_{m'=-1}^{2}\sin\left(-\frac{\pi i m'}{2}\right)\Theta_{m'}^{(2)}(z;\tau) \tag{3.155}$$

$$=-i\sqrt{-i\tau}\,\mathrm{e}^{\frac{\pi i z^2}{\tau}}\left(\Theta_1^{(2)}(z;\tau)-\Theta_{-1}^{(2)}(z;\tau)\right)\,. \tag{3.156}$$

Thus, the denominator only receives a prefactor under the S-modular transformation. Now, let us look at the numerator:

$$\Theta_{2\ell+1}^{(k+2)}\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right)-\Theta_{-2\ell-1}^{(k+2)}\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right)=-i\sqrt{\frac{-2i\tau}{k+2}}\,\mathrm{e}^{\frac{2\pi i(k+2)z^2}{4\tau}}\sum_{m'=-k-1}^{k+2}\sin\left(\frac{\pi i(2\ell+1)m'}{k+2}\right)$$
$$\times\,\Theta_{m'}^{(k+2)}(z;\tau)\,. \tag{3.157}$$

Next, we observe that the right hand side is identically equal to zero if $m'=0$ or $m'=k+2$. Furthermore, we can pair up m' with $-m'$ and rename $m'=2\ell'+1$, ℓ' then runs over half-integer from 0 to $\frac{k}{2}$. Thus,

$$\Theta_{2\ell+1}^{(k+2)}\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right)-\Theta_{-2\ell-1}^{(k+2)}\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right)=-i\sqrt{\frac{-2i\tau}{k+2}}\,\mathrm{e}^{\frac{2\pi i(k+2)z^2}{4\tau}}\sum_{\ell'=0}^{\frac{k}{2}}\sin\left(\frac{\pi i(2\ell+1)(2\ell'+1)}{k+2}\right)$$
$$\times\left(\Theta_{2\ell'+1}^{(k+2)}(z;\tau)-\Theta_{-2\ell'-1}^{(k+2)}(z;\tau)\right)\,. \tag{3.158}$$

Finally, we can combine numerator and denominator and conclude

$$\chi_\ell\left(\tfrac{z}{\tau};-\tfrac{1}{\tau}\right)=\sqrt{\frac{2}{k+2}}\,\mathrm{e}^{\frac{2\pi i k z^2}{4\tau}}\sum_{\ell'=0}^{\frac{k}{2}}\sin\left(\frac{\pi i(2\ell+1)(2\ell'+1)}{k+2}\right)\chi_{\ell'}(z;\tau)\,. \tag{3.159}$$

This proves that the S-matrix has indeed the form (3.139).

The S-matrix is obviously symmetric, so let us show that it is unitary. It is convenient to set $m=2\ell+1$, which then runs from 1 to $k+1$. We have

$$\sum_{\ell=0}^{\frac{k}{2}}S^\dagger_{\ell\ell''}S_{\ell\ell'}=\frac{2}{k+2}\sum_{m=1}^{k+1}\sin\left(\frac{\pi m m''}{k+2}\right)\sin\left(\frac{\pi m m'}{k+2}\right) \tag{3.160}$$

$$=\frac{1}{k+2}\sum_{m=-k-1}^{k+2}\sin\left(\frac{\pi m m''}{k+2}\right)\sin\left(\frac{\pi m m'}{k+2}\right) \tag{3.161}$$

$$=-\frac{1}{4(k+2)}\sum_{m=-k-1}^{k+2}\left(\mathrm{e}^{\frac{\pi m(m'+m'')}{k+2}}+\mathrm{e}^{\frac{\pi m(-m'-m'')}{k+2}}\right.$$
$$\left.-\,\mathrm{e}^{\frac{\pi m(m'-m'')}{k+2}}-\mathrm{e}^{\frac{\pi m(-m'+m'')}{k+2}}\right)\,. \tag{3.162}$$

We extended the range of summation, which does not change the result. The remaining terms are now simple geometric series, which impose $m' = m''$ or $m' = -m''$. The latter cannot be by assumption and hence we conclude

$$\sum_{\ell=0}^{\frac{k}{2}} S^\dagger_{\ell\ell''} S_{\ell\ell'} = -\frac{1}{4(k+2)} \times 2 \times (-2(k+2))\delta_{m',m''} = \delta_{\ell',\ell''} \ . \tag{3.163}$$

Thus, the S-matrix is unitary.

3.8 Fusion rules

A conformal field theory has an important invariant: the fusion ring. If we bring two fields $\phi_\lambda(z)$ and $\phi_\mu(w)$ close together, their product again transforms in a certain representation. In the context of WZW-models, the fusion rules express simply the Clebsch-Gordan coefficients of affine Lie algebras. One may think of the fusion product as some sort of tensor product. We denoted it as $\times$. We thus have a commutative, associative ring structure on the set of allowed modules $(\mathcal{M}_\lambda)_{\lambda\in\mathcal{R}}$ of the theory:

$$\mathcal{M}_\lambda \times \mathcal{M}_\mu = \bigoplus_{\nu\in\mathcal{R}} \mathcal{N}_{\lambda\mu}{}^\nu \, \mathcal{M}_\nu \ . \tag{3.164}$$

The coefficients $\mathcal{N}_{\lambda\mu}{}^\nu$ encode the multiplicity of the representation $\mathcal{M}_\nu$ appearing in the fusion product. The *Verlinde-formula* establishes a surprising relation between the fusion rules of the theory and its modular properties [9]. The S-matrix diagonalizes the fusion rules and one can deduce

$$\mathcal{N}_{\lambda\mu}{}^\nu = \sum_{\sigma\in\mathcal{R}} \frac{S_{\lambda\sigma}S_{\mu\sigma}S^*_{\nu\sigma}}{S_{0\sigma}} \ . \tag{3.165}$$

Let us apply this again to the case of $\mathfrak{su}(2)_k$. By using the explicit form of the S-matrix (3.139), we can derive the fusion rules. One obtains

$$\mathcal{M}_{\ell_1} \times \mathcal{M}_{\ell_2} = \bigoplus_{\ell=|\ell_1-\ell_2|,\, \ell+\ell_1+\ell_2\in\mathbb{Z}}^{\min(\ell_1+\ell_2,k-\ell_1-\ell_2)} \mathcal{M}_\ell \ . \tag{3.166}$$

As a comparison, we recall the Clebsch-Gordan coefficients of two $\mathfrak{su}(2)$-representations. They read

$$(\ell_1) \otimes (\ell_2) = \bigoplus_{\ell=|\ell_1-\ell_2|,\, \ell+\ell_1+\ell_2\in\mathbb{Z}}^{\ell_1+\ell_2} (\ell) \ , \tag{3.167}$$

where (ℓ) denotes the spin-ℓ representation of $\mathfrak{su}(2)$. The fact that the affine representations are cut off at $\ell = \frac{k}{2}$ leads to a folding-back of the representations appearing in the tensor product.

Notice in particular

$$\mathcal{M}_\ell \times \mathcal{M}_{\frac{k}{2}} = \mathcal{M}_{\frac{k}{2}-\ell} \ . \tag{3.168}$$

A representation with the property that the fusion with any other representation yields only a single representation is called a *simple current*. Simple currents are useful in constructing modular invariants (see next section). The simple current $\mathcal{M}_{k/2}$ has conformal weight $h = \frac{k}{4}$ and will lead to an extended modular invariant if k is divisible by four.

3.9 Modular invariants

Given that we know all possible irreducible modules of the $\mathfrak{su}(2)_k$ WZW-model, we now ask how to consistently combine them to a full-fledged theory. This gives also the opportunity to discuss conformal embeddings.

We declare the full Hilbert space of the theory to have the form

$$\mathcal{H} = \bigoplus_{\lambda \in \mathcal{R}} M_{\lambda\mu}\, \mathcal{M}_\lambda \otimes \mathcal{M}_\mu \ . \tag{3.169}$$

Here, we have been general for a rational conformal field theory. λ runs over all allowed representations, we denote the modules by $\mathcal{M}_\lambda$. We should not forget that we actually have *two* copies of affine Kac-Moody algebras in the model, one left-moving and one right-moving. Correspondingly, the full Hilbert space is a linear combination of representations of the form $\mathcal{M}_\lambda \otimes \mathcal{M}_\mu$. The matrix $M_{\lambda\mu}$ encodes the multiplicity of the respective representation.

What consistency condition should $M_{\lambda\mu}$ obey? We want to impose the following conditions:

(a) Integrality. $M_{\lambda\nu}$ should be a matrix of non-negative integers.

(b) Uniqueness of the vacuum. The vacuum representation should appear exactly once in the model. Hence $M_{00} = 1$, where $\lambda = 0$ labels the vacuum representation.

(c) Modular invariance. As we have discussed before, the partition function should be invariant under modular transformations. This amounts to the conditions

$$S^\dagger M S = M \quad \Leftrightarrow \quad [S, M] = 0 \ , \tag{3.170}$$
$$T^\dagger M T = M \quad \Leftrightarrow \quad [T, M] = 0 \ . \tag{3.171}$$

Every conformal field theory has one obvious solution to these constraints, the so-called *diagonal modular invariant*. It consists of taking M to be the identity, i.e.

$$M_{\lambda\mu} = \delta_{\lambda\mu} \ . \tag{3.172}$$

For the $\mathfrak{su}(2)_k$ WZW-model, there exists a complete classification of modular invariants due to Cappelli, Itzykson and Zuber [10, 11]. There are three types of modular invariants:

(a) A-type. The A-type modular invariant corresponds to the diagonal modular invariant. It exists for every level $k \in \mathbb{Z}_{\geq 1}$.

(b) D-type. This modular invariant exists for even level and takes the form

$$k = 4n : \qquad \mathcal{H} = 2\,\mathcal{M}_n \otimes \mathcal{M}_n \oplus \bigoplus_{\ell=0,\, \ell \in \mathbb{Z}}^{n-1} (\mathcal{M}_\ell \oplus \mathcal{M}_{2n-\ell}) \otimes (\mathcal{M}_\ell \oplus \mathcal{M}_{2n-\ell}) \ , \tag{3.173}$$

$$\begin{aligned} k = 4n-2 \quad \mathcal{H} &= \mathcal{M}_{n-\frac{1}{2}} \otimes \mathcal{M}_{n-\frac{1}{2}} \oplus \bigoplus_{\ell=0,\, \ell\in\mathbb{Z}}^{2n-1} \mathcal{M}_\ell \otimes \mathcal{M}_\ell \\ &\oplus \bigoplus_{\ell=\frac{1}{2},\, \ell \in \mathbb{Z}+\frac{1}{2}}^{n-\frac{3}{2}} \big(\mathcal{M}_\ell \otimes \mathcal{M}_{2n-1-\ell} \oplus \mathcal{M}_{2n-1-\ell} \otimes \mathcal{M}_\ell\big) \ . \end{aligned} \tag{3.174}$$

(c) E-type. These are three exceptional invariants existing at the levels $k=10$, $k=16$ and $k=28$. They take the form

$$\begin{aligned}
k=10: \quad \mathcal{H} &= (\mathcal{M}_0 \oplus \mathcal{M}_3) \otimes (\mathcal{M}_0 \oplus \mathcal{M}_3) \oplus (\mathcal{M}_{\frac{3}{2}} \oplus \mathcal{M}_{\frac{7}{2}}) \otimes (\mathcal{M}_{\frac{3}{2}} \oplus \mathcal{M}_{\frac{7}{2}}) \\
&\quad \oplus (\mathcal{M}_2 \oplus \mathcal{M}_5) \otimes (\mathcal{M}_2 \oplus \mathcal{M}_5) \ , \qquad (3.175) \\
k=16: \quad \mathcal{H} &= (\mathcal{M}_0 \oplus \mathcal{M}_8) \otimes (\mathcal{M}_0 \oplus \mathcal{M}_8) \oplus (\mathcal{M}_2 \oplus \mathcal{M}_6) \otimes (\mathcal{M}_2 \oplus \mathcal{M}_6) \\
&\quad \oplus (\mathcal{M}_3 \oplus \mathcal{M}_5) \otimes (\mathcal{M}_3 \oplus \mathcal{M}_5) \oplus \mathcal{M}_4 \otimes \mathcal{M}_4 \\
&\quad \oplus \mathcal{M}_4 \otimes (\mathcal{M}_1 \oplus \mathcal{M}_7) \oplus (\mathcal{M}_1 \oplus \mathcal{M}_7) \otimes \mathcal{M}_4 \ , \qquad (3.176) \\
k=28: \quad \mathcal{H} &= (\mathcal{M}_0 \oplus \mathcal{M}_5 \oplus \mathcal{M}_9 \oplus \mathcal{M}_{14}) \otimes (\mathcal{M}_0 \oplus \mathcal{M}_5 \oplus \mathcal{M}_9 \oplus \mathcal{M}_{14}) \\
&\quad \oplus (\mathcal{M}_3 \oplus \mathcal{M}_6 \oplus \mathcal{M}_8 \oplus \mathcal{M}_{11}) \otimes (\mathcal{M}_3 \oplus \mathcal{M}_6 \oplus \mathcal{M}_8 \oplus \mathcal{M}_{11}) \ . \qquad (3.177)
\end{aligned}$$

The terminology ADE comes from the fact that the problem can be mapped to the classification of simply-laced Lie algebras.

The modular invariants have the following physical interpretation. The A-type modular invariant defines the SU(2) WZW-model at level k. It contains every representation exactly once. The D-type modular invariant corresponds to the SO(3) WZW-model. Indeed, as we have remarked in footnote 6, this is exactly the quantization condition on the level. We will comment on the meaning of the three exceptional theories below.

Let us have a closer look at the $k=4$ D-type modular invariant. Explicitly, we have by (3.173)

$$\mathcal{H} = (\mathcal{M}_0 \oplus \mathcal{M}_2) \otimes (\mathcal{M}_0 \oplus \mathcal{M}_2) \oplus 2\,\mathcal{M}_1 \otimes \mathcal{M}_1 \ . \qquad (3.178)$$

Thus the vacuum $\mathcal{M}_0$ is combined with the spin-2 representation $\mathcal{M}_2$. The conformal weight of the spin-2 representation is

$$h(|2\rangle) = \frac{2(2+1)}{4+2} = 1 \ . \qquad (3.179)$$

Thus, there are in fact 5 more holomorphic spin-1 fields in the model, which by our previous discussion constitute five more conserved currents. Thus, we have in total 8 conserved currents, which turn out to generate $\mathfrak{su}(3)$. The D-type level-4 modular invariant has hence in fact $\mathfrak{su}(3)$-symmetry! We thus conclude that

$$\text{D-type } \mathfrak{su}(2)_4 = \mathfrak{su}(3)_k \ . \qquad (3.180)$$

What should be the level k of the $\mathfrak{su}(3)$-theory? This can be determined by comparing central charges and yields $k=1$. In the $\mathfrak{su}(3)_1$-language, the modular invariant (3.178) becomes either one of

$$\mathcal{H} = \mathcal{M}_{\mathbf{1}} \otimes \mathcal{M}_{\mathbf{1}} \oplus \mathcal{M}_{\mathbf{3}} \otimes \mathcal{M}_{\mathbf{3}} \oplus \mathcal{M}_{\bar{\mathbf{3}}} \otimes \mathcal{M}_{\bar{\mathbf{3}}} \ , \qquad (3.181)$$

$$\mathcal{H} = \mathcal{M}_{\mathbf{1}} \otimes \mathcal{M}_{\mathbf{1}} \oplus \mathcal{M}_{\mathbf{3}} \otimes \mathcal{M}_{\bar{\mathbf{3}}} \oplus \mathcal{M}_{\bar{\mathbf{3}}} \otimes \mathcal{M}_{\mathbf{3}} \ . \qquad (3.182)$$

The three allowed modules of the $\mathfrak{su}(3)_1$ theory are the vacuum module $\mathcal{M}_{\mathbf{1}}$ and the module based on the fundamental (antifundamental) representation $\mathcal{M}_{\mathbf{3}}$ ($\mathcal{M}_{\bar{\mathbf{3}}}$). Both Hilbert spaces are possible, the first corresponds to the diagonal modular invariant, the second to the *charge conjugated modular invariant*.

This phenomenon is called *modular extension*. For a modular extension, we choose a modular invariant in which the vacuum gets combined with some other representations and we obtain a bigger chiral algebra. This happens for all D-type invariants of level

$k = 4n$ and the three exceptional invariants. If the new fields have spin-1, these modular extensions are associated to *conformal embeddings*. A conformal embedding of two Kac-Moody algebras is an embedding $\mathfrak{h}_{k'} \subset \mathfrak{g}_k$ such that the two Sugawara-tensors become identified. A necessary and sufficient condition for this to happen [12] is the requirement of equal central charges

$$c(\mathfrak{h}_{k'}) = c(\mathfrak{g}_k) \ . \tag{3.183}$$

We have seen above the example $\mathfrak{su}(2)_4 \subset \mathfrak{su}(3)_1$, where $c(\mathfrak{su}(2)_4) = c(\mathfrak{su}(3)_1) = 2$.

Exercise 7 Show that the existence of the E-type modular invariants at level $k = 10$ and $k = 28$ are explainable by the conformal embeddings

$$\mathfrak{su}(2)_{10} \subset \mathfrak{so}(5)_1 \ , \qquad \mathfrak{su}(2)_{28} \subset (\mathfrak{g}_2)_1 \ . \tag{3.184}$$

The data of table 1 might be useful. Recast the structure of the Hilbert space in terms of $\mathfrak{so}(5)_1$ and $(\mathfrak{g}_2)_1$-modules. $\mathfrak{so}(5)_1$ has the following allowed modules:

$$\mathcal{M}_{\mathbf{1}} \ , \qquad \mathcal{M}_{\mathbf{4}} \ , \qquad \mathcal{M}_{\mathbf{5}} \ , \tag{3.185}$$

corresponding to the modules based on the trivial, the spinor and the vector representation. $(\mathfrak{g}_2)_1$ has only two allowed modules

$$\mathcal{M}_{\mathbf{1}} \ , \qquad \mathcal{M}_{\mathbf{7}} \ , \tag{3.186}$$

based on the trivial and the vector representation.

Solution 7 We are given the two conformal embeddings (3.184), whose existence we assume. Let us first discuss $\mathfrak{so}(5)_1$, which has the three modules $\mathcal{M}_{\mathbf{1}}$, $\mathcal{M}_{\mathbf{4}}$ and $\mathcal{M}_{\mathbf{5}}$. These three modules have to decompose into $\mathfrak{su}(2)_{10}$ modules in some way. We know that $\mathfrak{su}(2)_{10}$ has the following modules $\mathcal{M}_\ell$ for $\ell = 0, \frac{1}{2}, \ldots, 5$. Their conformal weights are

$$h(\mathcal{M}_0) = 0 \ , \qquad h(\mathcal{M}_{\frac{1}{2}}) = \tfrac{1}{16} \ , \qquad h(\mathcal{M}_1) = \tfrac{1}{6} \ , \qquad h(\mathcal{M}_{\frac{3}{2}}) = \tfrac{5}{16} \ , \tag{3.187}$$

$$h(\mathcal{M}_2) = \tfrac{1}{2} \ , \qquad h(\mathcal{M}_{\frac{5}{2}}) = \tfrac{35}{48} \ , \qquad h(\mathcal{M}_3) = 1 \ , \qquad h(\mathcal{M}_{\frac{7}{2}}) = \tfrac{21}{16} \ , \tag{3.188}$$

$$h(\mathcal{M}_4) = \tfrac{5}{3} \ , \qquad h(\mathcal{M}_{\frac{9}{2}}) = \tfrac{33}{16} \ , \qquad h(\mathcal{M}_5) = \tfrac{5}{2} \ . \tag{3.189}$$

Let us assume the decomposition

$$\mathcal{M}_{\mathbf{1}} = \bigoplus_{\ell=0}^{5} N_\ell \, \mathcal{M}_\ell \tag{3.190}$$

of the vacuum module of the $\mathfrak{so}(5)_1$ theory, where N_ℓ are non-negative integers. Since the vacuum representation of $\mathfrak{so}(5)_1$ contains only integer conformal weights by (3.121), the only modules which can appear on the right hand side are actually $\mathcal{M}_0$ and $\mathcal{M}_3$. How many times do they appear? Since the vacuum appears precisely once, the $\mathfrak{su}(2)_{10}$ vacuum representation has to appear precisely once. Thus

$$\mathcal{M}_{\mathbf{1}} = \mathcal{M}_0 \oplus N_3 \, \mathcal{M}_3 \tag{3.191}$$

for some integer N_3. N_3 is fixable by counting the $h = 1$ states. These correspond by definition the Kac-Moody fields of $\mathfrak{so}(5)_1$, so $\mathcal{M}_{\mathbf{1}}$ contains $\frac{1}{2}5 \times 4 = 10$ $h = 1$ states, whereas $\mathfrak{su}(2)_{10}$ contains only 3. Now we observe that $\mathcal{M}_3$ contains precisely 7 $h = 1$ states (since the spin 3 representation is 7-dimensional). Thus we have to set $N_3 = 1$ in order to get a matching. We conclude

$$\mathcal{M}_{\mathbf{1}} = \mathcal{M}_0 \oplus \mathcal{M}_3 \ , \tag{3.192}$$

and this is indeed the combination which appears in the modular invariant. For the other factors, one can argue similarly and one concludes

$$\mathcal{M}_{\mathbf{4}} = \mathcal{M}_{\frac{3}{2}} \oplus \mathcal{M}_{\frac{7}{2}} \ , \tag{3.193}$$

$$\mathcal{M}_{\mathbf{5}} = \mathcal{M}_2 \oplus \mathcal{M}_5 \ . \tag{3.194}$$

Thus, the E_6 modular invariant takes in $\mathfrak{so}(5)_1$ language the diagonal form

$$\mathcal{H} = \mathcal{M}_{\mathbf{1}} \otimes \mathcal{M}_{\mathbf{1}} \oplus \mathcal{M}_{\mathbf{4}} \otimes \mathcal{M}_{\mathbf{4}} \oplus \mathcal{M}_{\mathbf{5}} \otimes \mathcal{M}_{\mathbf{5}} \ . \tag{3.195}$$

Similarly, one can argue in the $(\mathfrak{g}_2)_1$ case to obtain simply

$$\mathcal{H} = \mathcal{M}_{\mathbf{1}} \otimes \mathcal{M}_{\mathbf{1}} \oplus \mathcal{M}_{\mathbf{7}} \otimes \mathcal{M}_{\mathbf{7}} \ . \tag{3.196}$$

This explains at least the physical origin of the $k = 10$ and $k = 28$ E-type invariants. The $k = 16$ E-type invariant is closely related to the $k = 16$ D-type invariant and can be obtained by a permutation of fields.

A similar classification exists for $\mathfrak{su}(3)_k$ WZW-models. In addition to the A-type and D-type series and their charge conjugated versions, there are five exceptional invariants. Some of them can be explained by the conformal embeddings

$$\mathfrak{su}(3)_5 \subset \mathfrak{su}(6)_1 \ , \qquad \mathfrak{su}(3)_9 \subset (\mathrm{E}_6)_1 \ , \qquad \mathfrak{su}(3)_{21} \subset (\mathrm{E}_7)_1 \ . \tag{3.197}$$

A general classification for arbitrary Lie algebras is not known and is an important open problem.

4 Cosets

In this final chapter, we will briefly introduce the coset construction. Cosets are one of the main examples of rational conformal field theories.

4.1 The coset construction

A coset is a *gauged* WZW-model, where a subgroup of G is gauged. While this can be described on the level of the action, we will focus on the algebraic construction. We will illustrate the concepts with the following two theories:

$$\text{Parafermions}: \quad \frac{\mathfrak{su}(2)_k}{\mathfrak{u}(1)} \ , \tag{4.1}$$

$$\text{Minimal models}: \quad \frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1}{\mathfrak{su}(2)_{k+1}} \ . \tag{4.2}$$

As one can see, we usually denote coset theories as quotients of two affine Kac-Moody algebras. For the parafermion theory, the denominator is embedded as the Cartan-generator of the numerator theory. In the minimal model series, the denominator is embedded as the diagonal subalgebra of $\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1$. The names of the two theories are traditional and will become clear in the following.

The coset construction starts with a WZW-model $\mathfrak{g}_k$ and the subalgebra $\mathfrak{h}_{k'} \subset \mathfrak{g}_k$ to be gauged. The chiral algebra (i.e. the holomorphic fields of the theory) consists of all holomorphic fields in the $\mathfrak{g}_k$ WZW model which have regular OPE with the Kac-Moody fields of $\mathfrak{h}_{k'}$.

This is indeed an algebra for the following reason. Let $X(z)$ and $Y(z)$ be two fields with trivial OPE with the Kac-Moody fields of $\mathfrak{h}_{k'}$, which we denote by $K^a(z)$. Then we have to check that the normal-ordered product of X with Y has also trivial OPE with $K^a(z)$. We compute

$$K^a(z)(XY)(w) = \frac{1}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} K^a(z)X(x)Y(w) \ . \tag{4.3}$$

But the OPE of $K^a(z)$ with both $X(x)$ and $Y(w)$ is regular and hence the complete OPE is regular.

We illustrate this by computing the vacuum module of the parafermion theory. Let us assume that the level k is large to avoid the complication of null-vectors. The chiral algebra consists of all fields having trivial OPE with $J^3(z)$. By definition, neither $J^-(z)$, $J^+(z)$, nor $J^3(z)$ have this property. So, we have to look at composite fields. There is exactly one bilinear field with this property and two trilinear fields. They are given by

$$T = \frac{(J^+J^-) - \partial J^3}{k+2} - \frac{2(J^3J^3)}{k(k+2)} \ , \tag{4.4}$$

$$\partial T = \frac{(\partial J^+J^-) + (J^+\partial J^-) - \partial^2 J^3}{k+2} - \frac{4(\partial J^3 J^3)}{k(k+2)} \ , \tag{4.5}$$

$$W^{(3)} = \frac{4(J^3(J^3J^3))}{3k} - (J^3(J^+J^-)) + \frac{k}{4}\big((\partial J^+J^-) - (J^+\partial J^-)\big) + (\partial J^3J^3) + \frac{k\partial^2 J^3}{12} \ . \tag{4.6}$$

We gave the spin-2 field the suggestive name T, since it will be the energy-momentum tensor of the coset theory. One spin-3 field is the derivative of the energy-momentum tensor, whereas the other one is a primary (with respect to the coset energy-momentum tensor T) spin-3 field, which we denoted by $W^{(3)}$. Continuing like this, we obtain one primary field at every spin. For finite k, the k-th field turns out to be a null field. Thus the chiral algebra is generated by $k-1$ fields of spin $2, \dots, k$. It is also known as $\mathcal{W}_k$-algebra. The fact that this model is constructed via a coset is somewhat obscured, in particular, there is no trace of an affine Kac-Moody symmetry left in the chiral algebra.

Repeating the same calculation for the theory (4.2) shows that there is no spin-3 primary field. It turns out that the chiral algebra of the coset is given only by the Virasoro algebra. For this reason, it gives a concrete realisation of the minimal-model series.

Let us show that there is always an energy-momentum tensor in the coset. For this, we denote by $T^{\mathfrak{g}}$ the energy-momentum tensor in the numerator theory and by $T^{\mathfrak{h}}$ the energy-momentum tensor in the denominator theory. Then we claim that

$$T^{\mathfrak{g}/\mathfrak{h}} = T^{\mathfrak{g}} - T^{\mathfrak{h}} \tag{4.7}$$

is the energy-momentum tensor of the coset. For this, we first have to check that its OPE with any Kac-Moody generator $K^a(z)$ of $\mathfrak{h}$ is non-singular. By definition, $K^a(z)$ is a primary field of spin 1 with respect to the energy-momentum tensor $T^{\mathfrak{h}}$. But since $K^a(z)$ is in particular also in the numerator algebra $\mathfrak{g}_k$, it is also a primary field of spin 1 with respect to the energy-momentum tensor $T^{\mathfrak{g}}$. Hence, $T^{\mathfrak{h}}$ and $T^{\mathfrak{g}}$ have identical OPEs with $K^a(z)$, so the OPE of $T^{\mathfrak{g}/\mathfrak{h}}$ with K^a is non-singular. Hence, $T^{\mathfrak{g}/\mathfrak{h}}$ defines indeed a field in the coset. Next, we have to check that (4.7) satisfies the Virasoro algebra. For this, we note that by the same argument as for the currents themselves, the OPE between $T^{\mathfrak{g}/\mathfrak{h}}$ and $T^{\mathfrak{h}}$ is non-singular. Thus, we obtain

$$\begin{aligned} T^{\mathfrak{g}/\mathfrak{h}}(z)T^{\mathfrak{g}/\mathfrak{h}}(w) &= (T^{\mathfrak{g}}(z) - T^{\mathfrak{h}}(z))T^{\mathfrak{g}}(w) && (4.8)\\ &= T^{\mathfrak{g}}(z)T^{\mathfrak{g}}(w) - T^{\mathfrak{h}}(z)T^{\mathfrak{h}}(w) - T^{\mathfrak{h}}(z)T^{\mathfrak{g}/\mathfrak{h}}(w) && (4.9)\\ &\sim \frac{c_{\mathfrak{g}} - c_{\mathfrak{h}}}{2(z-w)^4} + \frac{2T^{\mathfrak{g}/\mathfrak{h}}(w)}{(z-w)^2} + \frac{\partial T^{\mathfrak{g}/\mathfrak{h}}(w)}{z-w}\ . && (4.10) \end{aligned}$$

This shows that the coset theory again defines a CFT with central charge $c = c_{\mathfrak{g}} - c_{\mathfrak{h}}$. For the central charges of the theories (4.1) and (4.2), we conclude in particular

$$c\left(\frac{\mathfrak{su}(2)_k}{\mathfrak{u}(1)}\right) = \frac{3k}{k+2} - 1 = \frac{2(k-1)}{k+2}\ , \tag{4.11}$$

$$c\left(\frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1}{\mathfrak{su}(2)_{k+1}}\right) = \frac{3k}{k+2} + \frac{3\cdot 1}{1+2} - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}\ . \tag{4.12}$$

Observe that in the special case of $k = 2$, the parafermion theory has central charge $c = \frac{1}{2}$ and turns out to be equivalent to the free-fermion theory (with a suitable modular invariant). This is where the name parafermion stems from. On the other hand, the central charge of the minimal model coset is always smaller than one. The minimal model coset gives a concrete realization of the minimal Virasoro models.

4.2 Representations and characters

After having determined the chiral algebra of the theory, we study its representations. Constructing them is quite easy. For the chiral algebra, we considered fields of the numerator algebra in the vacuum representation which transformed in the trivial representation of the numerator algebra. We can expand this to fields in any representation of the numerator algebra with some fixed representation of the denominator algebra. Concretely, let us consider some module $\mathcal{M}^{\mathfrak{g}}_\lambda$ of the parent theory. Then we decompose it into primary representations of the denominator algebra:

$$\mathcal{M}^{\mathfrak{g}}_\lambda = \bigoplus_{\mu \in \mathcal{R}^{\mathfrak{h}}} \mathcal{M}^{\mathfrak{g}/\mathfrak{h}}_{(\lambda,\mu)} \otimes \mathcal{M}^{\mathfrak{h}}_\mu\ . \tag{4.13}$$

$\mathcal{M}^{\mathfrak{g}/\mathfrak{h}}_{(\lambda,\mu)}$ forms then a module of the coset algebra. In fact, all modules are constructed in this way. Thus, representations are labelled by a pair of indices $(\lambda,\mu) \in \mathcal{R}^{\mathfrak{g}} \times \mathcal{R}^{\mathfrak{h}}$.

Not all pairs (λ,μ) might be allowed. This is formalised by the so-called *selection rules*. They express the fact that not all representations of $\mathfrak{h}_{k'}$ might appear in a given representation of $\mathfrak{g}_k$. Indeed, an example of this is given by the parafermion theory. Integer-spin representations of $\mathfrak{su}(2)_k$ contain only integer $\mathfrak{u}(1)$-charges, whereas half-integer-spin representations contain only half-integer $\mathfrak{u}(1)$-charges.

There is a further subtlety, which is in some sense complementary. While not all pairs (λ, μ) might be allowed, some might be equivalent. This is referred to as *field identifications*. The reason for this is the existence of outer automorphisms of the parent theory, which act trivially on the coset algebra. In the parafermion theory, the corresponding field identification is $(\ell, m) \cong (\frac{k}{2} - \ell, \frac{k}{2} - m)$.[11]

We can apply characters to (4.13). This implies that we have the character identity

$$\chi^{\mathfrak{g}}_{\lambda}(\tau) = \sum_{\mu \in \mathcal{R}^{\mathfrak{h}}} \chi^{\mathfrak{g}/\mathfrak{h}}_{(\lambda,\mu)}(\tau) \chi^{\mathfrak{h}}_{\mu}(\tau) \ . \tag{4.14}$$

Thus, given the characters of the numerator and denominator theory, one can work out the characters of the coset theory. Similarly, the modular transformation behaviour can be deduced. We claim that the S-matrix of the coset theory is given by

$$S^{\mathfrak{g}/\mathfrak{h}}_{(\lambda,\mu)(\lambda',\mu')} = S^{\mathfrak{g}}_{\lambda\lambda'} \big(S^{\mathfrak{h}}\big)^{-1}_{\mu\mu'} \ . \tag{4.15}$$

To show this, we simply have to apply the S-modular transformation to both sides of (4.14) and check that they indeed agree. This then also implies that modulo field identifications, modular invariants of the coset theory are given by products of modular invariants of the numerator theory and the denominator theory. Similarly, we conclude via the Verlinde formula that the fusion rules of the coset theory are the product of the fusion rules of numerator and denominator theory.

For illustration, we work this out for the minimal model theory. Representations are in principle labelled by three spins $(\ell_1, \ell_2; j)$, where $0 \le \ell_1 \le \frac{k}{2}$, $0 \le \ell_2 \le \frac{1}{2}$ and $0 \le j \le \frac{k+1}{2}$. There is again a selection rule, similar to the parafermion theory. We have $\ell_1 + \ell_2 + j \in \mathbb{Z}$ and hence ℓ_2 is determined in terms of the two other labels. For this reason, it is customary to suppress ℓ_2 and use only two labels $(\ell = \frac{r-1}{2}, j = \frac{s-1}{2})$. Using (4.7), the representation (r, s) has conformal weight

$$\Delta_{(r,s)} = \frac{\frac{r-1}{2}\big(\frac{r-1}{2}+1\big)}{k+2} - \frac{\frac{s-1}{2}\big(\frac{s-1}{2}+1\big)}{k+3} + \begin{cases} n & r-s \in 2\mathbb{Z} \ , \\ n+\frac{1}{4} & r-s \in 2\mathbb{Z}+1 \end{cases} \ , \tag{4.16}$$

where n is a non-negative integer. It can appear, since the representation $j = \frac{s-1}{2}$ does not necessarily appear in the ground state of the representation $\ell = \frac{r-1}{2}$, but higher up in the Verma module. For an appropriate choice of n, this becomes

$$\Delta_{(r,s)} = \frac{\big(r(k+3) - s(k+2)\big)^2 - 1}{4(k+2)(k+3)} \ , \tag{4.17}$$

which matches the conformal dimensions of primary fields in the unitary minimal models. We deduce the minimal model fusion rules:

$$\mathcal{M}_{r_1,s_1} \times \mathcal{M}_{r_2,s_2} = \bigoplus_{\substack{r_3=|r_1-r_2|+1, \\ r_1+r_2+r_3+1 \in 2\mathbb{Z}}}^{\min(r_1+r_2-1, 2k+3-r_1-r_2)} \ \bigoplus_{\substack{s_3=|s_1-s_2|+1, \\ s_1+s_2+s_3+1 \in 2\mathbb{Z}}}^{\min(s_1+s_2-1, 2k+5-s_1-s_2)} \mathcal{M}_{r_3,s_3} \ . \tag{4.18}$$

Similarly, we can reproduce the existence of the D-series minimal models by the D-series modular invariant of the $\mathfrak{su}(2)_k$ WZW-model.

[11] We mean the extended $\mathfrak{u}(1)_k$ algebra here, so there is an identification $m \sim m + k$ of $\mathfrak{u}(1)_k$-representations.

Exercise 8 Reproduce the spectrum of the D-series minimal models you learned about it Sylvain's lectures from the D-series $\mathfrak{su}(2)_k$ modular invariants. Find then all modular invariants of minimal models, including the exceptional modular invariants.

Solution 8 We have $p = k+2$ and $q = k+3$ (or vice versa). The D-series minimal models only exist for p even, which means also k even. So we use the D-modular invariant for the numerator theory and the A-modular invariant for the denominator theory. There are two cases, corresponding to the two cases (3.174) and (3.173).

$k = 4n$: The modular invariant reads by definition

$$2\mathcal{H} = \bigoplus_{s=1}^{q-1} \left(2\,\mathcal{M}_{2n+1,s} \otimes \mathcal{M}_{2n+1,s} \oplus \bigoplus_{r \stackrel{2}{=} 1}^{2n-1} (\mathcal{M}_{r,s} \oplus \mathcal{M}_{4n-r+2,s}) \otimes (\mathcal{M}_{r,s} \oplus \mathcal{M}_{4n-r+2,s}) \right) \tag{4.19}$$

$$= \bigoplus_{s=1}^{q-1} \left(\bigoplus_{r \stackrel{2}{=} 1}^{4n+1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{r,s} \oplus \bigoplus_{r \stackrel{2}{=} 1}^{4n+1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{4n-r+2,s} \right) \tag{4.20}$$

$$= \bigoplus_{r \stackrel{2}{=} 1}^{p-1} \bigoplus_{s=1}^{q-1} \left(\mathcal{M}_{r,s} \otimes \mathcal{M}_{r,s} \oplus \mathcal{M}_{r,s} \otimes \mathcal{M}_{p-r,s} \right) . \tag{4.21}$$

Here, $r \stackrel{2}{=} 1$ to $4n-1$ means that r runs over all odd numbers.

$k = 4n-2$: The modular invariant reads in this case

$$2\mathcal{H} = \bigoplus_{s=1}^{q-1} \left(\mathcal{M}_{2n,s} \otimes \mathcal{M}_{2n,s} \oplus \bigoplus_{r \stackrel{2}{=} 1}^{4n-1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{r,s} \right.$$
$$\left. \oplus \bigoplus_{r \stackrel{2}{=} 2}^{2n-2} \left(\mathcal{M}_{r,s} \otimes \mathcal{M}_{4n-r,s} \oplus \mathcal{M}_{4n-r,s} \otimes \mathcal{M}_{r,s} \right) \right) \tag{4.22}$$

$$= \bigoplus_{s=1}^{q-1} \left(\bigoplus_{r \stackrel{2}{=} 1}^{4n-1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{r,s} \oplus \bigoplus_{r \stackrel{2}{=} 2}^{4n-2} \mathcal{M}_{r,s} \otimes \mathcal{M}_{4n-r,s} \right) \tag{4.23}$$

$$= \bigoplus_{r \stackrel{2}{=} 1}^{p-1} \bigoplus_{s=1}^{q-1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{r,s} \oplus \bigoplus_{r \stackrel{2}{=} 2}^{p-2} \bigoplus_{s=1}^{q-1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{4n-r,s} . \tag{4.24}$$

Thus, in total we obtain

$$\mathcal{H} = \frac{1}{2} \bigoplus_{r \stackrel{2}{=} 1}^{p-1} \bigoplus_{s=1}^{q-1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{r,s} \oplus \frac{1}{2} \bigoplus_{\substack{r \equiv \frac{p}{2} \bmod 2 \\ 1 \le r \le p-1}} \bigoplus_{s=1}^{q-1} \mathcal{M}_{r,s} \otimes \mathcal{M}_{4n-r,s} . \tag{4.25}$$

This reproduces equation (4.10) in Sylvain's lecture notes.

The modular invariants are in general of type (A,A), (D,A), (A,D), (E,A) or (A,E), where the first entry denotes the numerator theory and the second entry the denominator theory. At least one entry has to be of A-type since not both the levels of the numerator as well as of the denominator can be even.

We have seen that there are surprising identifications of CFTs at the quantum level. We mention here the somewhat surprising *level-rank* duality. It states the equivalence of the conformal field theories

$$\frac{\mathfrak{su}(M+N)_k}{\mathfrak{su}(M)_k \oplus \mathfrak{su}(N)_k \oplus \mathfrak{u}(1)} \cong \frac{\mathfrak{su}(k)_M \oplus \mathfrak{su}(k)_N}{\mathfrak{su}(k)_{M+N}} . \tag{4.26}$$

We have seen the simplest instance of this equivalence, namely for $k = 2$ and $M = N = 1$, where both sides describe the Ising model. Let us check at least the equality of central charges:

$$c\left(\frac{\mathfrak{su}(k)_M \oplus \mathfrak{su}(k)_N}{\mathfrak{su}(k)_{M+N}}\right) = \frac{M(k^2-1)}{k+M} + \frac{N(k^2-1)}{k+N} - \frac{(M+N)(k^2-1)}{k+M+N} \tag{4.27}$$

$$= \frac{k((M+N)^2-1)}{k+M+N} - \frac{k(M^2-1)}{k+M} - \frac{k(N^2-1)}{k+N} - 1 \tag{4.28}$$

$$= c\left(\frac{\mathfrak{su}(M+N)_k}{\mathfrak{su}(M)_k \oplus \mathfrak{su}(N)_k \oplus \mathfrak{u}(1)}\right) . \tag{4.29}$$

5 Outlook

Let us mention a few topics we did not have time to mention or elaborate on.

(i) Affine Kac-Moody algebras. We have introduced affine Kac-Moody algebras in passing. There is a mathematical theory of these algebras, developing their representations, characters, etc. from a purely mathematical point of view. This is described in detail in chapter 14 of [3] or chapter 3 of [1].

(ii) Correlation functions. We have mentioned correlation functions only in passing. Their form is fixed by the Knizhnik-Zamlodochikov equations [8].

(iii) Non-compact WZW-models. We have almost exclusively discussed the compact case. For a non-compact Lie group, the story is much more complicated: The level is no longer quantised, infinite-dimensional ground state representations appear and spectrally flowed representations appear. Moreover, they typically contain indecomposable modules and fall into the realm of *logarithmic conformal field theory*. Good general overviews of logarithmic conformal field theories are [13, 14]. Arguably the simplest and most important non-compact WZW-model is the $\mathrm{SL}(2,\mathbb{R})$ WZW-model. An introduction can be found in the concluding chapter of [15], a more extensive review in [16].

(iv) Supersymmetry. We have not explained how to make WZW-models supersymmetric. Every WZW-model can be made $\mathcal{N} = 1$ supersymmetric by adding fermions transforming in the adjoint representation. Some cosets are moreover $\mathcal{N} = 2$ (or even large $\mathcal{N} = 4$) supersymmetric. The better known case of $\mathcal{N} = 2$ is referred to as Kazama-Suzuki models [17].

(v) Relation to string theory. While WZW-models provide interesting CFTs in their own right, they have important applications to string theory. The addition of the WZ-term to the action corresponds to adding NS-NS flux to the background. Thus, all pure NS-NS background string theories on group manifolds can be described by WZW-models. This includes in particular the important case of AdS_3-backgrounds. For instance, the maximally supersymmetric background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ can be described by the worldsheet theory $\mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k-2} \oplus \mathfrak{u}(1)^4 \oplus 10$ free fermions. The level k has then the interpretation of the amount of B-field flux.

(vi) $\mathcal{W}$-algebras. Many $\mathcal{W}$-algebras can be constructed via cosets and *Casimir algebras*, thereby providing an explicit definition. A general overview can be found in [18].

(vii) Admissible levels. While we have defined WZW models at positive integer levels, they can actually be defined also at other levels (in which case they define a non-unitary theory). A particularly interesting class of levels is given by the so-called admissible levels, which have the property that they have still null vectors in their Verma modules. A good introduction can be found in the yellow book [3].

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