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Superstrings on $\text{AdS}_4 \times \text{CP}^3$ as a Coset Sigma Model

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Introduction

- Multiple M2 branes $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ \implies $\mathcal{N} = 6$ Chern-Simons Theory
 $\text{SU}(N) \times \text{SU}(N)$ at level k
Aharony, Bergman, Jafferis and Maldacena, hep-th/0806.1218

- Parameters of Chern-Simons theory N and k or

$$N \quad \text{and} \quad \lambda = 2\pi^2 N/k \quad \leftarrow \quad \text{'t Hooft coupling}$$

- 't Hooft limit $N \rightarrow \infty$, λ finite

$$\begin{array}{ccc} \text{Type IIA Strings} & \implies & \mathcal{N} = 6 \text{ Chern-Simons Theory} \\ \text{AdS}_4 \times \mathbb{CP}^3 & & \text{planar, perturbative in } \lambda \end{array}$$

The goal is to understand the
dynamics of

Type IIA Strings on

$$\text{AdS}_4 \times \mathbb{CP}^3$$

Plan

1. Coset Sigma Model
2. Brief Intro into $osp(2, 2|6)$
3. Automorphism of Order Four
4. The Lagrangian and Eoms
5. Local Fermionic Symmetry
6. Integrability of the Coset Model
7. Plane-wave Limit
8. Conclusions

Sigma Model on the Coset Space

$$\frac{\text{OSP}(2, 2|6)}{\text{SO}(3, 1) \times \text{U}(3)}$$

$\text{OSP}(2, 2|6)$ has a bosonic subgroup $\text{USP}(2, 2) \times \text{SO}(6)$

$$\frac{\text{USP}(2, 2)}{\text{SO}(3, 1)} \times \frac{\text{SO}(6)}{\text{U}(3)} = \text{AdS}_4 \times \mathbb{CP}^3$$

The coset superspace contains 24 fermions – too little for Type IIA!

Superalgebra $osp(2, 2|6)$

$osp(2, 2|6)$ can be realized by 10×10 supermatrices

$$A = \begin{pmatrix} X_{4 \times 4} & \theta_{4 \times 6} \\ \eta_{6 \times 4} & Y_{6 \times 6} \end{pmatrix}$$

The matrix A must satisfy two conditions

$$A^{st} \begin{pmatrix} C_4 & 0 \\ 0 & \mathbb{I}_{6 \times 6} \end{pmatrix} + \begin{pmatrix} C_4 & 0 \\ 0 & \mathbb{I}_{6 \times 6} \end{pmatrix} A = 0 \Rightarrow A^{st} = -\check{C} A \check{C}^{-1}$$

$$A^\dagger \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{I}_{6 \times 6} \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{I}_{6 \times 6} \end{pmatrix} A = 0 \Rightarrow A^\dagger = -\check{\Gamma} A \check{\Gamma}^{-1}$$

- ✓ C_4 is real skew-symmetric matrix, $C_4^2 = -\mathbb{I}$
- ✓ Γ^μ represent the Clifford algebra for $SO(3, 1)$
- ✓ C_4 is charge conjugation matrix: $(\Gamma^\mu)^t = -C_4 \Gamma^\mu C_4^{-1}$

Automorphism of order 4

\mathbb{Z}_4 -automorphism with a stationary algebra $\text{SO}(3, 1) \times \text{U}(3)$?

Introduce

$$K_4 = -\Gamma^1 \Gamma^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad K_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

These matrices obey $K_4^2 = -\mathbb{I}$ and $K_6^2 = -\mathbb{I}$ and also

$$(\Gamma^\mu)^t = K_4 \Gamma^\mu K_4^{-1}$$

for all gamma-matrices

Automorphism of order 4

$$\Omega(A) = \begin{pmatrix} K_4 X^t K_4 & K_4 \eta^t K_6 \\ -K_6 \theta^t K_4 & K_6 Y^t K_6 \end{pmatrix}$$

For any two supermatrices A and B

$$\Omega(AB) = -\Omega(B)\Omega(A)$$

i.e. it is an automorphism of $\mathfrak{osp}(2, 2|6)$

$$\Omega([A, B]) = -[\Omega(B), \Omega(A)] = [\Omega(A), \Omega(B)].$$

Automorphism of order 4

The algebra relations imply

$$\Omega(A) = \begin{pmatrix} K_4 C_4 & 0 \\ 0 & -K_6 \end{pmatrix} \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix} \begin{pmatrix} K_4 C_4 & 0 \\ 0 & -K_6 \end{pmatrix}^{-1} \equiv \Upsilon A \Upsilon^{-1}$$

- Since $(K_4 C_4)^2 = \mathbb{I}$ and $K_6^2 = -\mathbb{I}$ one finds $\Upsilon^4 = \mathbb{I}$
- $K_4 C_4$ coincides with Γ^5 given by $\Gamma^5 = -i\Gamma^0\Gamma^1\Gamma^2\Gamma^3$
- Υ takes values in the complexified $\text{OSP}(2, 2|6)$,
i.e. it is orthosymplectic but not unitary: $\Upsilon^\dagger \check{\Upsilon} \Upsilon \check{\Upsilon}^{-1} = -\mathbb{I}$

\mathbb{Z}_4 -grading of $\mathfrak{osp}(2, 2|6)$

As the vector space $\mathbf{A} = \mathfrak{osp}(2, 2|6)$ can be decomposed as

$$\mathbf{A} = \mathbf{A}^{(0)} \oplus \mathbf{A}^{(1)} \oplus \mathbf{A}^{(2)} \oplus \mathbf{A}^{(3)}$$

such that $[\mathbf{A}^{(k)}, \mathbf{A}^{(m)}] \subseteq \mathbf{A}^{(k+m)}$ modulo \mathbb{Z}_4 .

Each $\mathbf{A}^{(k)}$ is an eigenspace of Ω

$$\Omega(\mathbf{A}^{(k)}) = i^k \mathbf{A}^{(k)}$$

A projection $A^{(k)}$ of a generic element $A \in \mathfrak{osp}(2, 2|6)$ is

$$A^{(k)} = \frac{1}{4} \left(A + i^{3k} \Omega(A) + i^{2k} \Omega^2(A) + i^k \Omega^3(A) \right) \in \mathfrak{osp}(2, 2|6)$$

Stationary subalgebra of Ω

The stationary subalgebra of Ω is determined by

$$[\Gamma^5, X] = 0, \quad [K_6, Y] = 0$$

and it coincides with $\mathfrak{so}(3, 1) \times \mathfrak{u}(3)$.

- X is generated by $\frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$
- Y can be parametrized as follows

$$Y = \begin{pmatrix} 0 & y_{12} & y_{24} & -y_{23} & y_{26} & -y_{25} \\ -y_{12} & 0 & y_{23} & y_{24} & y_{25} & y_{26} \\ -y_{24} & -y_{23} & 0 & y_{34} & y_{46} & -y_{45} \\ y_{23} & -y_{24} & -y_{34} & 0 & y_{45} & y_{46} \\ -y_{26} & -y_{25} & -y_{46} & -y_{45} & 0 & y_{56} \\ y_{25} & -y_{26} & y_{45} & -y_{46} & -y_{56} & 0 \end{pmatrix}$$

This 9-parametric matrix describes an embedding $\mathfrak{u}(3) \subset \mathfrak{so}(6)$.

The space $A^{(2)}$ – bosonic coset $\text{AdS}_4 \times \mathbb{CP}^3$

The space $A^{(2)}$ is spanned by matrices

$$\Omega(A) = \Upsilon A \Upsilon^{-1} = -A$$

Any such matrix satisfies the remarkable identity

$$A^3 = \frac{1}{8} \text{str}(\Sigma A^2) A + \frac{1}{8} \text{str}(A^2) \Sigma A$$

or

$$A^3 = \frac{1}{8} (\text{tr} A_{\text{AdS}}^2 + \text{tr} A_{\mathbb{CP}}^2) A + \frac{1}{8} (\text{tr} A_{\text{AdS}}^2 - \text{tr} A_{\mathbb{CP}}^2) \Sigma A$$

Here Σ is a diagonal matrix $\Sigma = \Upsilon^2 = (\mathbb{I}_4, -\mathbb{I}_6)$

The Lagrangian

Let g be a coset representative. Construct the one-form

$$A = -g^{-1}dg = A^{(0)} + A^{(2)} + A^{(1)} + A^{(3)}$$

It has zero curvature

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0$$

The sigma model action

$$S = -\frac{R^2}{4\pi\alpha'} \int d\sigma d\tau \mathcal{L}$$

with the Lagrangian density

$$\mathcal{L} = \gamma^{\alpha\beta} \text{str}(A_\alpha^{(2)} A_\beta^{(2)}) + \kappa \epsilon^{\alpha\beta} \text{str}(A_\alpha^{(1)} A_\beta^{(3)})$$

Here $\gamma^{\alpha\beta} = h^{\alpha\beta} \sqrt{-h}$ with $\det \gamma = -1$

Equations of motion

- Bosons

$$\partial_\alpha(\gamma^{\alpha\beta} A_\beta^{(2)}) - \gamma^{\alpha\beta} [A_\alpha^{(0)}, A_\beta^{(2)}] + \frac{1}{2} \kappa \epsilon^{\alpha\beta} \left([A_\alpha^{(1)}, A_\beta^{(1)}] - [A_\alpha^{(3)}, A_\beta^{(3)}] \right) = 0$$

- Fermions

$$P_-^{\alpha\beta} [A_\alpha^{(2)}, A_\beta^{(3)}] = 0,$$

$$P_+^{\alpha\beta} [A_\alpha^{(2)}, A_\beta^{(1)}] = 0.$$

The tensors

$$P_\pm^{\alpha\beta} = \frac{1}{2} (\gamma^{\alpha\beta} \pm \kappa \epsilon^{\alpha\beta})$$

For $\kappa = \pm 1$ the tensors P_\pm are orthogonal projectors:

$$P_+^{\alpha\beta} + P_-^{\alpha\beta} = \gamma^{\alpha\beta}, \quad P_\pm^{\alpha\delta} P_\pm^{\beta\delta} = P_\pm^{\alpha\beta}, \quad P_\pm^{\alpha\delta} P_{\mp\delta}^{\beta} = 0$$

The Lagrangian must be invariant under a local fermionic symmetry (κ -symmetry) which should be capable to remove 8 out of 24 fermions

How to exhibit this symmetry?

Local Fermionic Symmetry

- The action of the global symmetry group $OSP(2, 2|6)$ is realized on a coset element by multiplication from the left
- κ -symmetry transformations can be understood as the *right local action* of a fermionic element $G = \exp \epsilon \in OSP(2, 2|6)$ on a coset representative g

$$gG(\epsilon) = g'g_c,$$

where $\epsilon \equiv \epsilon(\sigma)$ is a local fermionic parameter. Here g_c is a compensating element from $SO(3, 1) \times U(3)$

Local Fermionic Symmetry

Under the local multiplication from the right the connection A transforms

$$\delta_\epsilon A = -d\epsilon + [A, \epsilon]$$

The \mathbb{Z}_4 -decomposition of this equation gives

$$\delta_\epsilon A^{(1)} = -d\epsilon^{(1)} + [A^{(0)}, \epsilon^{(1)}] + [A^{(2)}, \epsilon^{(3)}]$$

$$\delta_\epsilon A^{(3)} = -d\epsilon^{(3)} + [A^{(0)}, \epsilon^{(3)}] + [A^{(2)}, \epsilon^{(1)}]$$

$$\delta_\epsilon A^{(2)} = [A^{(1)}, \epsilon^{(1)}] + [A^{(3)}, \epsilon^{(3)}]$$

where we have assumed that $\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$

Local Fermionic Symmetry

κ -symmetry variation of the Lagrangian

$$\delta_\epsilon \mathcal{L} = \delta\gamma^{\alpha\beta} \text{str}(A_\alpha^{(2)} A_\beta^{(2)}) - 4 \text{str}\left(P_+^{\alpha\beta} [A_\beta^{(1)}, A_\alpha^{(2)}] \epsilon^{(1)} + P_-^{\alpha\beta} [A_\beta^{(3)}, A_\alpha^{(2)}] \epsilon^{(3)}\right)$$

Vanishes on-shell due to the Virasoro constraints

$$\text{str}(A_\alpha^{(2)} A_\beta^{(2)}) - \frac{1}{2} \gamma_{\alpha\beta} \gamma^{\rho\delta} \text{str}(A_\rho^{(2)} A_\delta^{(2)}) = 0$$

and $\gamma_{\alpha\beta} \delta\gamma^{\alpha\beta} = 0$

Take $\kappa = \pm 1$ and for any vector V^α introduce the projections V_\pm^α

$$V_\pm^\alpha = P_\pm^{\alpha\beta} V_\beta$$

so that the variation of the Lagrangian acquires the form

$$\delta_\epsilon \mathcal{L} = \delta\gamma^{\alpha\beta} \text{str}(A_\alpha^{(2)} A_\beta^{(2)}) - 4 \text{str}\left([A_+^{(1),\alpha}, A_{\alpha,-}^{(2)}] \epsilon^{(1)} + [A_-^{(3),\alpha}, A_{\alpha,+}^{(2)}] \epsilon^{(3)}\right)$$

Local Fermionic Symmetry

Some technicalities:

- The condition $P_{\pm}^{\alpha\beta} A_{\beta,\mp} = 0$ the components $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ are proportional

$$A_{\tau,\pm} = -\frac{\gamma^{\tau\sigma} \mp \kappa}{\gamma^{\tau\tau}} A_{\sigma,\pm}$$

As the result, tensorial structures

$$A_{\alpha,-}^{(2)} \cdots A_{\beta,-}^{(2)} \cdots A_{\delta,-}^{(2)}$$

do not depend on the order of indices

- To simplify the treatment, we put $\epsilon^{(3)} = 0$

Local Fermionic Symmetry

Ansatz for the κ -symmetry variation

$$\epsilon^{(1)} = A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} \kappa_{++}^{\alpha\beta} + \kappa_{++}^{\alpha\beta} A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} + A_{\alpha,-}^{(2)} \kappa_{++}^{\alpha\beta} A_{\beta,-}^{(2)} - \frac{1}{8} \text{str}(\Sigma A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)}) \kappa_{++}^{\alpha\beta}$$

Requirements on $\kappa_{++}^{\alpha\beta}$

- $\kappa_{++}^{\alpha\beta} \in \mathfrak{osp}(2, 2|6)$
- $\kappa_{++}^{\alpha\beta} \in \mathbf{A}^{(1)}$

Thus, generically $\kappa_{++}^{\alpha\beta}$ depends on 12 fermionic variables

Local Fermionic Symmetry

Consider now the commutator

$$\begin{aligned}
 [A_{\alpha,-}^{(2)}, \epsilon^{(1)}] &= A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} A_{\delta,-}^{(2)} \kappa_{++}^{\beta\delta} + A_{\alpha,-}^{(2)} \kappa_{++}^{\beta\delta} A_{\beta,-}^{(2)} A_{\delta,-}^{(2)} + A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} \kappa_{++}^{\beta\delta} A_{\delta,-}^{(2)} \\
 &- A_{\beta,-}^{(2)} A_{\delta,-}^{(2)} \kappa_{++}^{\beta\delta} A_{\alpha,-}^{(2)} - \kappa_{++}^{\beta\delta} A_{\beta,-}^{(2)} A_{\delta,-}^{(2)} A_{\alpha,-}^{(2)} - A_{\beta,-}^{(2)} \kappa_{++}^{\beta\delta} A_{\delta,-}^{(2)} A_{\alpha,-}^{(2)} \\
 &- \frac{1}{8} \text{str}(\Sigma A_{\beta,-}^{(2)} A_{\delta,-}^{(2)}) A_{\alpha,-}^{(2)} \kappa_{++}^{\beta\delta} + \frac{1}{8} \text{str}(\Sigma A_{\beta,-}^{(2)} A_{\delta,-}^{(2)}) \kappa_{++}^{\beta\delta} A_{\alpha,-}^{(2)}
 \end{aligned}$$

Most of the terms are cancelled out

$$[A_{\alpha,-}^{(2)}, \epsilon^{(1)}] = [A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)} A_{\delta,-}^{(2)} - \frac{1}{8} \text{str}(\Sigma A_{\beta,-}^{(2)} A_{\delta,-}^{(2)}) A_{\alpha,-}^{(2)}, \kappa_{++}^{\beta\delta}]$$

Due to the remarkable identity

$$[A_{\alpha,-}^{(2)}, \epsilon^{(1)}] = \frac{1}{8} \text{str}(A_{\beta,-}^{(2)} A_{\delta,-}^{(2)}) [\Sigma A_{\alpha,-}^{(2)}, \kappa_{++}^{\beta\delta}].$$

Local Fermionic Symmetry

The κ -symmetry variation of the action

$$\delta_\epsilon \mathcal{L} = \delta\gamma^{\alpha\beta} \text{str}(A_\alpha^{(2)} A_\beta^{(2)}) - 4 \text{str}\left([A_+^{(1),\alpha}, A_{\alpha,-}^{(2)}] \epsilon^{(1)}\right)$$

implies the following transformation law for the metric

$$\delta\gamma^{\alpha\beta} = \frac{1}{2} \text{str}\left(\Sigma A_{\delta,-}^{(2)} [\kappa_{++}^{\alpha\beta}, A_+^{(1),\delta}]\right)$$

The condition $\gamma_{\alpha\beta} \delta\gamma^{\alpha\beta}$ is automatically obeyed as

$$\gamma_{\alpha\beta} \delta\gamma^{\alpha\beta} = \gamma^{\alpha\beta} P_{\alpha\delta}^+ P_{\beta\eta}^+ \kappa^{\delta\eta} = 0$$

Full variation of the metric

$$\delta\gamma^{\alpha\beta} = \frac{1}{2} \text{str}\left(\Sigma A_{\delta,-}^{(2)} [\kappa_{++}^{\alpha\beta}, A_+^{(1),\delta}]\right) + \frac{1}{2} \text{str}\left(\Sigma A_{\delta,+}^{(2)} [\kappa_{--}^{\alpha\beta}, A_-^{(3),\delta}]\right)$$

Local Fermionic Symmetry

Rank of κ -symmetry transformations on-shell?

$$A^{(2)} = \begin{pmatrix} ix\Gamma^0 & 0 \\ 0 & yT_6 \end{pmatrix}$$

The constraint $\text{str}(A_{\alpha,-}^{(2)} A_{\beta,-}^{(2)}) = 0$ then demands that $x = \pm y$

Computing $\epsilon^{(1)}$ one gets

$$\epsilon^{(1)} = x^2 \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon^t C_4 & 0 \end{pmatrix},$$

where ε is the following matrix

$$\varepsilon = \begin{pmatrix} 0 & 0 & i(ik_{13} - k_{16}) & i(ik_{14} - k_{15}) & ik_{14} - k_{15} & ik_{13} - k_{16} \\ 0 & 0 & i(ik_{23} - k_{26}) & i(ik_{24} - k_{25}) & ik_{24} - k_{25} & ik_{23} - k_{26} \\ 0 & 0 & -i(-ik_{33} - k_{36}) & -i(-ik_{34} - k_{35}) & -ik_{34} - k_{35} & -ik_{33} - k_{36} \\ 0 & 0 & -i(-ik_{43} - k_{46}) & -i(-ik_{44} - k_{45}) & -ik_{44} - k_{45} & -ik_{43} - k_{46} \end{pmatrix}$$

Integrability: The Lax Connection

No difference in construction of the Lagrangian $\text{AdS}_5 \times S^5$, the Lax connection found by Bena, Polchinski and Roiban is applicable to our model as well

$$L_\alpha = \ell_0 A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta} \epsilon^{\beta\rho} A_\rho^{(2)} + \ell_3 A_\alpha^{(1)} + \ell_4 A_\alpha^{(3)}$$

- L_α is flat due to e.o.m and this determines all ℓ_i in terms of one parameter z
- L_α is flat provided $\kappa = \pm 1$
- κ -symmetry variation of L_α is a gauge transformation on-shell
- $L_\alpha(z)$ is used to build infinite sets of integrals of motion

Plane-wave Limit

Let z_i be homogenous coordinates on \mathbb{CP}^3 .

Parametrize

$$z_4 = e^{-i\phi/2}, \quad z_3 = (1 - x_4)e^{i\phi/2}, \quad z_1 = \frac{1}{\sqrt{2}}y_1, \quad z_2 = \frac{1}{\sqrt{2}}y_2$$

ϕ is a parameter along the geodesics and the complex y_1, y_2 and the real x_4 denote the five physical fluctuations in \mathbb{CP}^3

The $\text{AdS}_4 \times \mathbb{CP}^3$ background metric admits the following expansion

$$ds^2_{\text{AdS}_4 \times \mathbb{CP}^3} = -dt^2(1 + x_i^2) + dx_i^2 + d\phi^2(1 - x_4^2 - \frac{1}{4}\bar{y}_r y_r) + dx_4^2 + d\bar{y}_r dy_r + \dots$$

Plugging in the point-like string solution with $t = \tau$, $\phi = \tau$ in the string Lagrangian one gets **four fields of mass 1/2** and **four fields of mass 1**. The field x_4 from \mathbb{CP}^3 joins three fields from AdS_4 .

Plane-wave Limit

The bosonic action around particle trajectory $t = \tau$, $\phi = \tau$ is

$$S_B^{(2)} = -\frac{R^2}{4\pi\alpha'} \int d\sigma d\tau \left(\partial^\alpha x_k \partial_\alpha x_k - x_k^2 + \partial^\alpha \bar{y}_r \partial_\alpha y_r - \frac{1}{4} \bar{y}_r y_r \right)$$

Develop now the whole quadratic action (including fermions) starting from the coset representative

$$g = e^X g_B$$

Gauge-fixing κ -symmetry we find that

the sum of the quadratic bosonic and fermionic actions coincides with the light-cone Green-Schwarz action for Type IIA superstrings on the pp-wave background with 24 supersymmetries!

Conclusions

- *Green-Schwarz superstring on $\text{AdS}_4 \times \mathbb{CP}^3$ with κ -symmetry partially fixed is the coset sigma model*

$$\frac{\text{OSP}(2, 2|6)}{\text{SO}(3, 1) \times \text{U}(3)}$$

- *The coset sigma model has κ -symmetry of rank 8*
- *The coset sigma model is classically integrable*
- *Is it a quantum integrable model?*
- *What is the light-cone S-matrix?*