Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

Part II

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Based on work in collaboration with

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Outline

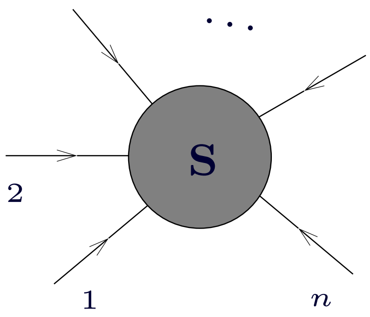
- On-shell scattering amplitudes in $\mathcal{N}=4$ SYM
- Iterative structure of gluon amplitudes and BDS ansatz
- Dual conformal invariance – hidden symmetry of planar MHV amplitudes
- Wilson loop/MHV amplitude duality in $\mathcal{N}=4$ SYM
- Dual superconformal invariance of MHV and next-to-MHV amplitudes
- Wilson loop/all amplitudes (MHV, NMHV, N^{2}MHV, $\ldots$) duality in $\mathcal{N}=4$ SYM

On-shell scattering amplitudes in $\mathcal{N}=4$ SYM

- ✔ $\mathcal{N}=4$ SYM – (super)conformal gauge theory with the $SU(N_c)$ gauge group

  Asymptotic on-shell states: gluons $G_{\pm 1}(p)$, four gaugino $\Gamma^{A}_{\pm\frac{1}{2}}(p)$, six real scalars $S_0^{AB}(p)$

- ✔ Scattering amplitudes in $\mathcal{N}=4$ SYM

$\mathcal{A}_n =$ 

  - ✗ Quantum numbers of on-shell states $|i\rangle = |p_i, h_i, a_i\rangle$: momentum ($p_i^2=0$), helicity (h_i), color (a_i)
  - ✗ On-shell matrix elements of $S-$matrix
  - ✗ Suffer from IR divergences $\mapsto$ require IR regularization

- ✔ Color-ordered planar partial amplitudes

$$\mathcal{A}_n(\{p_i,h_i,a_i\}) = \mathrm{tr}\left[T^{a_1}T^{a_2}\dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) + \text{[Bose symmetry]}$$

  - ✗ *All-loop planar* amplitudes can be split into (universal) IR divergent and (nontrivial) finite part

$$A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) = \mathrm{Div}(\{s_{i,i+1}\}, 1/\epsilon_{\mathrm{IR}})\,\mathrm{Fin}(\{p_i,h_i\})$$

  - ✗ IR divergences exponentiate [Mueller],[Sen],[Collins],[Sterman],[GK,Radyushkin],...

$$\log\left(\mathrm{Div}(\{s_{i,i+1}\}, 1/\epsilon_{\mathrm{IR}})\right) = -\frac{1}{4}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma^{(l)}_{\mathrm{cusp}}}{(l\epsilon_{\mathrm{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\mathrm{IR}}}\right)\sum_{i=1}^{n}(-s_{i,i+1})^{l\epsilon_{\mathrm{IR}}}$$

  - ✗ *Main goal: identify the finite part of the planar amplitudes*

MHV amplitudes

Classify color-ordered amplitudes $A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n)$ according to their helicity content:

- ✔ Supersymmetry relations:

$$A^{++\dots+} = A^{-+\dots+} = 0\,, \qquad A^{(\mathrm{MHV})} = A^{--+\dots+}\,, \qquad A^{(\mathrm{next-MHV})} = A^{---+\dots+}\,, \quad \dots$$

- ✔ The $n=4$ and $n=5$ planar gluon amplitudes are all MHV

$$\{A_4^{++--}\,,\quad A_4^{+-+-}\,,\quad \dots\}\,,\quad \{A_5^{+++--}\,,\quad A_5^{+-+--}\,,\quad \dots\}$$

- ✔ Next-to-MHV amplitude appear starting from $n=6$ gluon amplitudes

$$A_6^{+++---}\,,\quad A_6^{-+--++}\,,\dots$$

- ✔ Weak-coupling expansion of generic color-ordered amplitudes in ’t Hooft coupling $\lambda = g^2 N_c$

$$A_n = \sum_{\alpha \in \text{Lorentz structures}} \left[A_n^{(0),\alpha} + \lambda\, A_n^{(1),\alpha} + O(\lambda^2) \right]$$

The MHV amplitudes involve *only one* Lorentz structure [Parke,Taylor]

$$A^{(\mathrm{MHV})} = A_n^{(0)} + \lambda\, A_n^{(1)} + O(\lambda^2) = A_n^{(0)} \left[1 + \lambda \frac{A_n^{(1)}}{A_n^{(0)}} + O(\lambda^2) \right]$$

Weak/strong coupling corrections to all MHV amplitudes are described by a single function of ’t Hooft coupling and kinematical invariants!

MHV superamplitude

- On-shell helicity states in $\mathcal{N}=4$ SYM:

$$\pm 1 \text{ (gluons)}, \quad \pm\tfrac{1}{2} \text{ (gluinos)}, \quad 0 \text{ (scalars)}$$

- Can be 'packed' into a single on-shell superstate [Mandelstam],[Brink et el]

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p)$$

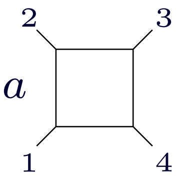
- Combine all MHV amplitudes into a single MHV superamplitude [Nair]

$$\mathcal{A}_n^{\rm MHV} = (\eta_1)^4(\eta_2)^4 \times A\left(G_1^- G_2^- G_3^+ \dots G_n^+\right) + (\eta_1)^4(\eta_2)^3\eta_3 \times A\left(G_1^- \bar{\Gamma}_2\, \Gamma_3 \dots G_n^+\right) + (\eta_1)^4(\eta_2)^2(\eta_3)^2 \times A\left(G_1^- \bar{S}_2\, S_3 \dots G_n^+\right) + \dots$$

Homogenous polynomial in η's of degree 8

$$\mathcal{A}_n^{\rm MHV} = i(2\pi)^4\delta^{(4)}(\sum_{i=1}^n p_i)\underbrace{\frac{\delta^{(8)}(\sum_{i=1}^n \lambda_i^\alpha\,\eta_i^A)}{\langle 1\,2\rangle\langle 2\,3\rangle\dots\langle n\,1\rangle}}_{\text{tree amplitude}} \times \underbrace{M_n^{\rm MHV}(\{s_{i,i+1}\};a)}_{\text{universal function}}$$

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling

$$\log\left[M_4^{\rm MHV}\right] = a$$ 

$$+O(a^2) = \text{Div}(s,t,1/\epsilon_{\rm IR}) + \text{Fin}(s/t)$$ [Green,Schwarz,Brink'82]

- ✔ Bern-Dixon-Smirnov (BDS) conjecture:

$$\text{Fin}(s/t) = a\left[\tfrac{1}{2}\ln^2(s/t) + 4\zeta_2\right] + O(a^2) \quad \overset{\textit{all loops}}{\Longrightarrow} \quad \tfrac{1}{4}\Gamma_{\rm cusp}(a)\ln^2(s/t) + \text{const}$$

  - ✗ Compared to QCD,

    (i) the complicated functions of s/t are replaced by the elementary function $\ln^2(s/t)$;

    (ii) no higher powers of logs appear in Fin(s/t) at higher loops;

    (iii) the coefficient of $\ln^2(s/t)$ is determined by the cusp anomalous dimension $\Gamma_{\rm cusp}(a)$ just like the coefficient of the double IR pole.

  - ✗ The conjecture has been verified up to three loops [Anastasiou,Bern,Dixon,Kosower'03],[Bern,Dixon,Smirnov'05]
  - ✗ A similar conjecture exists for n-gluon MHV amplitudes [Bern,Dixon,Smirnov'05]
  - ✗ It has been confirmed for $n=5$ at two loops [Cachazo,Spradlin,Volovich'04], [Bern,Czakon,Kosower,Roiban,Smirnov'06]
  - ✗ Agrees with the strong coupling prediction from the AdS/CFT correspondence [Alday,Maldacena'06]

- ✔ Surprising features of the finite part of the MHV amplitudes in planar $\mathcal{N}=4$ SYM:

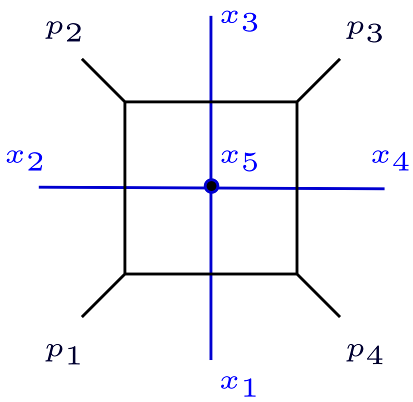
  Why should finite corrections exponentiate? and be related to the cusp anomaly of Wilson loop?

Dual conformal symmetry

Examine one-loop 'scalar box' diagram

- ✔ Change variables to go to a *dual 'coordinate space'* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$



$$= \int \frac{d^4k\,(p_1+p_2)^2(p_2+p_3)^2}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} = \int \frac{d^4x_5\; x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Check conformal invariance by inversion $x_i^\mu \to x_i^\mu / x_i^2$

[Broadhurst],[Drummond,Henn,Smirnov,Sokatchev]

- ✔ The integral is invariant under conformal $SO(2,4)$ transformations in the dual space!
- ✔ The symmetry *is not related* to conformal $SO(2,4)$ symmetry of $\mathcal{N}=4$ SYM
- ✔ All scalar integrals contributing to A_4 up to four loops possess the dual conformal invariance!
- ✔ If the dual conformal symmetry survives to all loops, it allows us to determine four- and five-gluon planar scattering amplitudes to all loops! [Drummond,Henn,GK,Sokatchev],[Alday,Maldacena]
- ✔ Dual conformality is slightly broken by the infrared regulator
- ✔ For *planar* integrals only!

From gluon amplitudes to Wilson loops

Properties of gluon scattering amplitudes in $\mathcal{N}=4$ SYM:

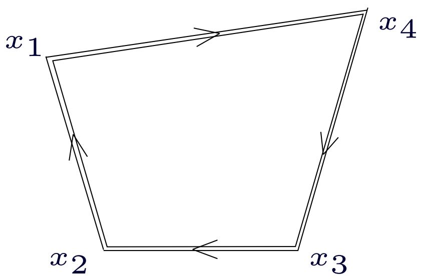
(1) IR divergences of M_4 are in one-to-one correspondence with UV div. of *cusped Wilson loops*

(2) Perturbative corrections to M_4 possess a hidden *dual conformal symmetry*

☞ *Is it possible to identify the object in $\mathcal{N}=4$ SYM for which both properties are manifest ?*

Yes! The expectation value of light-like Wilson loop in $\mathcal{N}=4$ SYM [Drummond-Henn-GK-Sokatchev]

$$W(C_4) = \frac{1}{N_c}\langle 0|\mathrm{Tr}\,\mathrm{P}\exp\left(ig\oint_{C_4} dx^\mu A_\mu(x)\right)|0\rangle\,, \qquad C_4 =$$



✔ Gauge invariant functional of the integration contour C_4 in Minkowski space-time

✔ The contour is made out of 4 light-like segments $C_4 = \ell_1\cup\ell_2\cup\ell_3\cup\ell_4$ joining the cusp points x_i^μ

$$x_i^\mu - x_{i+1}^\mu = p_i^\mu = \text{on-shell gluon momenta}$$

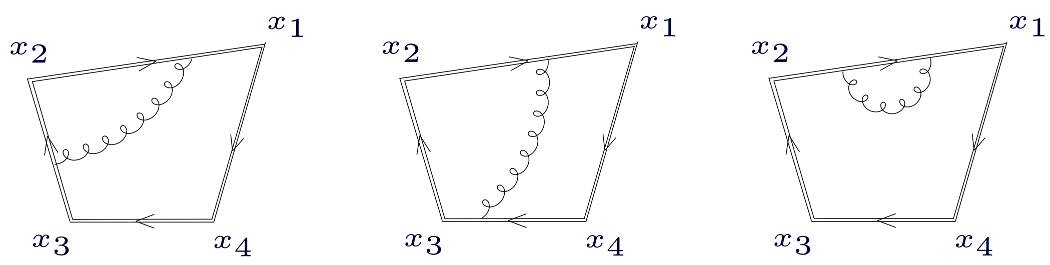
✔ The contour C_4 has four light-like cusps $\mapsto$ $W(C_4)$ has UV divergencies

✔ Conformal symmetry of $\mathcal{N}=4$ SYM $\mapsto$ conformal invariance of $W(C_4)$ in dual coordinates x^μ

MHV scattering amplitudes/Wilson loop duality I

The one-loop expression for the light-like Wilson loop (with $x_{jk}^2 = (x_j - x_k)^2$)

$\ln W(C_4) =$



$$= \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm UV}^2} \left[(-x_{13}^2\mu^2)^{\epsilon_{\rm UV}} + (-x_{24}^2\mu^2)^{\epsilon_{\rm UV}} \right] + \frac{1}{2}\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{const} \right\} + O(g^4)$$

The one-loop expression for the gluon scattering amplitude

$$\ln M_4(s,t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm IR}^2} \left[(-s/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} + (-t/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} \right] + \frac{1}{2}\ln^2\left(\frac{s}{t}\right) + \text{const} \right\} + O(g^4)$$

✔ Identity the light-like segments with the on-shell gluon momenta $x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$:

$$x_{13}^2\,\mu^2 := s/\mu_{\rm IR}^2\,, \qquad x_{24}^2\,\mu^2 := t/\mu_{\rm IR}^2\,, \qquad x_{13}^2/x_{24}^2 := s/t$$

- ☞ UV divergencies of the light-like Wilson loop match IR divergences of the gluon amplitude
- ☞ the finite $\sim \ln^2(s/t)$ corrections coincide to one loop!

MHV scattering amplitudes/Wilson loop duality II

MHV amplitudes are dual to light-like Wilson loops [Drummond,Henn,GK,Sokatchev], [Brandhuber,Heslop,Travaglini]

$$\ln M_n^{(\mathrm{MHV})} = \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = \text{light-like } n-\text{(poly)gon}$$

- ✔ At strong coupling, the relation holds to leading order in $1/\sqrt{\lambda}$ [Alday,Maldacena]
- ✔ At weak coupling, the duality relation was verified for:
  - ✗ $n = 4$ (rectangle) to two loops [Drummond,Henn,GK,Sokatchev]
  - ✗ $n \geq 5$ to one loop [Brandhuber,Heslop,Travaglini]
  - ✗ $n = 5$ (pentagon) to two loops [Drummond,Henn,GK,Sokatchev]
- ✔ For arbitrary coupling, conformal symmetry of light-like Wilson loops in $\mathcal{N} = 4$ SYM + duality relation impose constraints on the finite part of the MHV amplitudes
- ✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the MHV amplitudes
  $\mathbb{D} =$ dilatations, $\mathbb{K}^\mu =$ special conformal transformations [Drummond,Henn,GK,Sokatchev]

$$\mathbb{D}\, F_n \equiv \sum_{i=1}^{n} (x_i \cdot \partial_{x_i}) F_n = 0$$

$$\mathbb{K}^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\mathrm{cusp}}(a) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Finite part of MHV amplitudes

The consequences of the conformal Ward identity for the finite part of the Wilson loop/ MHV scattering amplitudes: [Drummond, Henn, GK, Sokatchev]

- ✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)
  $\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\rm cusp}(a)\ln^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\rm cusp}(a)\sum_{i=1}^{5}\ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right)\ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

  Exactly the functional forms of the BDS ansatz for the 4- and 5-point MHV amplitudes!

- ✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x^2_{13}x^2_{46}}{x^2_{14}x^2_{36}}, \qquad u_2 = \frac{x^2_{24}x^2_{15}}{x^2_{25}x^2_{14}}, \qquad u_3 = \frac{x^2_{35}x^2_{26}}{x^2_{36}x^2_{25}}$$

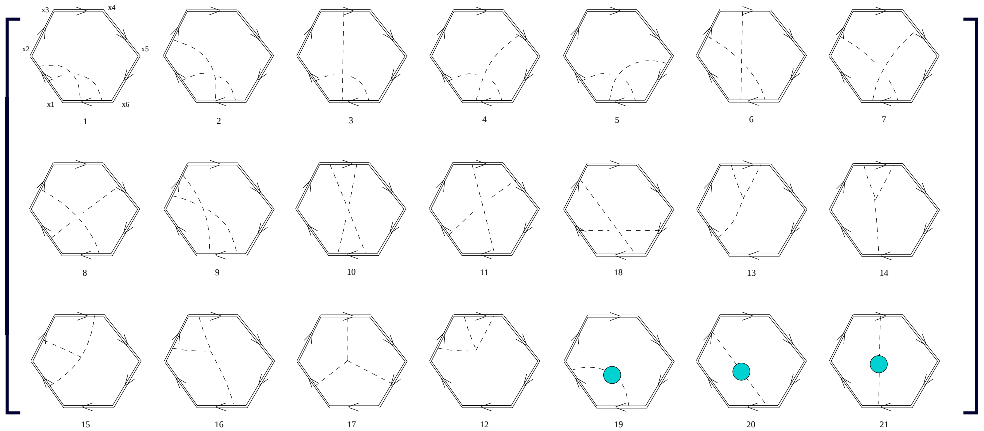
  Hence the general solution of the Ward identity for $W(C_n)$ with $n \geq 6$ contains *an arbitrary function* of the conformal cross-ratios.

- ✔ The BDS ansatz is a solution of the conformal Ward identity for *arbitrary* n but does it actually work for $n \geq 6$ [Alday, Maldacena] [Bartels, Lipatov, Sabio Vera]? if not *what is a missing function of* $u_{1,2,3}$*?*

Discrepancy function

✔ We computed the two-loop hexagon Wilson loop $W(C_6)$ ... [Drummond, Henn, GK, Sokatchev'07]

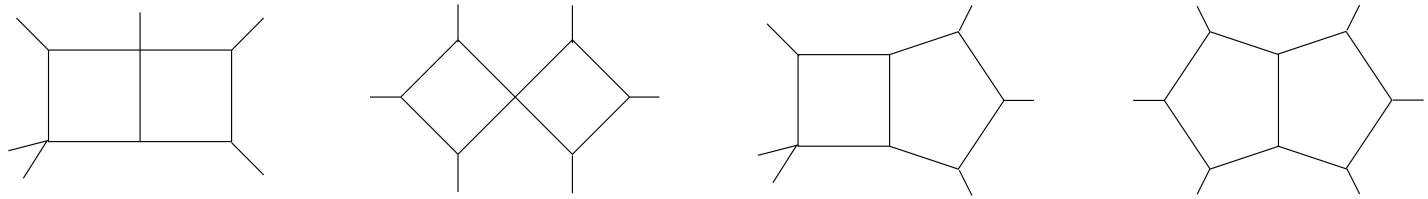
$$\ln W(C_6) =$$



... and found a **discrepancy** $\ln W(C_6) \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

✔ Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich computed 6-gluon amplitude to 2 loops

$$\mathcal{M}_6^{(\mathrm{MHV})} =$$



$$+ \dots$$

... and found a **discrepancy** $\ln \mathcal{M}_6^{(\mathrm{MHV})} \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

☞ The BDS ansatz **fails** for $n = 6$ starting from two loops.

☞ *What about Wilson loop duality?* $\ln \mathcal{M}_6^{(\mathrm{MHV})} \stackrel{?}{=} \ln W(C_6)$

6-gluon amplitude/hexagon Wilson loop duality

✔ Comparison between the DHKS discrepancy function $\Delta_{\rm WL}$ and the BDKRSVV results for the six-gluon amplitude $\Delta_{\rm MHV}$:

Kinematical point	(u_1, u_2, u_3)	$\Delta_{\rm WL} - \Delta_{\rm WL}^{(0)}$	$\Delta_{\rm MHV} - \Delta_{\rm MHV}^{(0)}$
$K^{(1)}$	$(1/4, 1/4, 1/4)$	$< 10^{-5}$	-0.018 ± 0.023
$K^{(2)}$	$(0.547253, 0.203822, 0.88127)$	-2.75533	-2.753 ± 0.015
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74460	-4.7445 ± 0.0075
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.09138	4.12 ± 0.10
$K^{(5)}$	$(4/81, 4/81, 4/81)$	9.72553	10.00 ± 0.50

evaluated for different kinematical configurations, e.g.

$$K^{(1)}: \quad x_{13}^2 = -0.7236200\,, \quad x_{24}^2 = -0.9213500\,, \quad x_{35}^2 = -0.2723200\,, \quad x_{46}^2 = -0.3582300\,, \quad x_{36}^2 = -0.4825841\,,$$
$$x_{15}^2 = -0.4235500\,, \quad x_{26}^2 = -0.3218573\,, \quad x_{14}^2 = -2.1486192\,, \quad x_{25}^2 = -0.7264904\,.$$

✔ Two nontrivial functions coincide with an accuracy $< 10^{-4}$!

✌ *The Wilson loop/MHV amplitude duality holds at $n = 6$ to two loops!!*

✌ *We expect that the duality relation should also hold for arbitrary n to all loops!!!*

What about next-to-MHV amplitudes?

MHV superamplitude

- All tree MHV amplitudes can be combined into a single (Nair) superamplitude by introducing Grassmann variables η_i^A (with $A = 1, \ldots, 4$), one for each external particle.

- Perturbative corrections to all MHV amplitudes are factorized into a universal factor $M_n^{(\mathrm{MHV})}$

- The all-loop generalization of the MHV superamplitude as

$$\mathcal{A}_n^{\mathrm{MHV}}(p_1, \eta_1; \ldots; p_n, \eta_n) = i(2\pi)^4 \frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\, \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} M_n^{(\mathrm{MHV})},$$

- The all-loop MHV amplitudes appear as coefficients in the expansion of $\mathcal{A}_{n;0}^{\mathrm{MHV}}$ in powers of η_i. In particular, the gluon MHV amplitude arises as

$$\mathcal{A}_n^{\mathrm{MHV}} = (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^n p_i\Big) \sum_{1\le j<k\le n} (\eta_j)^4 (\eta_k)^4 A_n^{(\mathrm{MHV})}(1^+ \ldots j^- \ldots k^- \ldots n^+) + \ldots\,, \qquad (1)$$

- The function $M_n^{(\mathrm{MHV})}$ is dual to light-like Wison loop

$$\ln M_n^{(\mathrm{MHV})} = \ln W_n + O(\epsilon, 1/N^2)\,,$$

- The MHV superamplitude possesses a much bigger, dual superconformal symmetry which acts on the dual coordinates x_i^μ and their superpartners $\theta_{i\,\alpha}^A$ [Drummond, Henn, GK, Sokatchev]

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu\,, \qquad \lambda_i^\alpha\, \eta_i = \theta_i^\alpha - \theta_{i+1}^\alpha$$

Next-to-MHV amplitudes

- ✔ Are known to have a much more complicated structure compared with MHV amplitudes
- ✔ Simplest example: the six-gluon nMHV amplitudes A^{+++---}, A^{++-+--} and A^{+-+-+-}

$$A^{+++---} = A_{6;0} + g^2 A_{6;1} + O(g^4)\,,$$

- ✗ Involves few Lorentz structures, each coming with its own perturbative corrections

[Bern,Dixon,Dunbar,Kosower'94]

$$A_{6;0} = \frac{1}{2}\left[B_1 + B_2 + B_3\right]$$

$$A_{6;1} = c_\Gamma N \left[B_1 F_6^{(1)} + B_2 F_6^{(2)} + B_3 F_6^{(3)}\right].$$

- ✗ Expressions for B_i in the dual coordinates $p_i = x_i - x_{i+1}$

$$B_1 = i\,\frac{(x_{14}^2)^3}{\langle 12\rangle\langle 23\rangle[45][56]\langle 1|x_{14}|4]\langle 3|x_{36}|6]}$$

$$B_2 = \left(\frac{[23]\langle 56\rangle}{x_{25}^2}\right)^4 B_1\big|_{i\to i-2} + \left(\frac{\langle 4|x_{41}|1]}{x_{25}^2}\right)^4 B_1\big|_{i\to i+1}\,,$$

$$B_3 = \left(\frac{[12]\langle 45\rangle}{x_{36}^2}\right)^4 B_1\big|_{i\to i+2} + \left(\frac{\langle 6|x_{63}|3]}{x_{36}^2}\right)^4 B_1\big|_{i\to i-1}$$

- ✗ $F_6^{(i)}$ = combination of box (IR-divergent) integrals evaluated within the dim. regularization

Do NMHV amplitudes have some (hidden) symmetry? Yes! Dual superconformal symmetry!

Six-point next-to-MHV superamplitude

$$\mathcal{A}_6^{\rm NMHV} = \mathcal{A}_6^{\rm MHV} \left[\tilde{c}_{146}\, \delta^{(4)}(\Xi_{146})\,(1 + aV_{146} + O(\epsilon)) + \text{(cyclic)}\right] \,,$$

- Supercovariant Ξ_{146} is a linear combination of three Grassmann $\eta-$variables

$$\Xi_{146} = \langle 61\rangle\langle 45\rangle\big(\eta_4[56] + \eta_5[64] + \eta_6[45]\big)\,,$$

- 'Even' Lorentz factor $\tilde{c}_{146}$ in the dual coordinates

$$\tilde{c}_{146} = \frac{1}{2}\langle 34\rangle\langle 56\rangle\Big(x_{14}^2\langle 1|\,x_{14}|4]\langle 3|\,x_{36}|6](\langle 45\rangle\langle 61\rangle)^3[45][56]\Big)^{-1}\,,$$

- The scalar function $V_{146} =$ linear combination of scalar box integrals

$$V_{146} = -\ln u_1 \ln u_2 + \frac{1}{2}\sum_{k=1}^{3}\Big[\ln u_k \ln u_{k+1} + \mathrm{Li}_2(1-u_k)\Big] = \text{conformal invariant!}$$

conformal ratios in the dual coordinates $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}\,,\quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}\,,\quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$

- From $n=6$ NMHV superamplitude to six-gluon NMHV amplitudes

$$\mathcal{A}_6^{\rm NMHV} = (2\pi)^4\delta^{(4)}\big(\sum_{i=1}^{6} p_i\big)\left[(\eta_1)^4(\eta_2)^4(\eta_3)^4 A(1^-2^-3^-4^+5^+6^+) + \dots\right]$$

Reproduces all known results [Bern,Dixon,Dunbar,Kosower'94] *for one-loop six-point NMHV amplitudes!*

$n-$point Next-to-MHV superamplitude

- The dual superconformal symmetry also allows us to understand the complicated structure of $n-$point NMHV amplitudes.

- In a close analogy with the MHV amplitude $\mathcal{A}_n^{\rm MHV}$, all NMHV amplitudes can be combined into a single superamplitude $\mathcal{A}_n^{\rm NMHV}$.

- The ratio of the two superamplitudes is given by a linear combination of *superinvariants*

$$\mathcal{A}_n^{\rm NMHV} = \mathcal{A}_n^{\rm MHV}\left(\sum_{p,q,r=1}^{n} c_{pqr}\,\delta^{(4)}\left(\Xi_{pqr}\right)\left[1 + aV_{pqr} + O(\epsilon)\right] + O(a^2)\right)$$

  Ingredients: 'odd' supercovariants Ξ_{pqr}, 'even' spinor made c_{pqr}, conformal invariant V_{pqr} made of scalar boxes

- The gluon NMHV amplitudes arise as coefficients in front of $(\eta_i)^4(\eta_j)^4(\eta_k)^4$, i.e.

$$\mathcal{A}_n^{\rm NMHV} = (2\pi)^4\delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big)\sum_{i,j,k}(\eta_i)^4(\eta_j)^4(\eta_k)^4 A_n^{\rm (NMHV)}(1^+\dots i^-\dots j^-\dots k^-\dots n^+) + \dots$$

- *Reproduces all known results* [Bern,Dixon,Dunbar,Kosower'04],[Risanger'08] *for $n-$point NMHV amplitudes!*

- The dual conformal invariance of the superamplitudes $\mathcal{A}_n^{\rm MHV}$ and $\mathcal{A}_n^{\rm NMHV}$ is broken by infrared divergences in such a way that *their ratio remains conformal* as $\epsilon \to 0$.

All $\mathcal{N}=4$ superamplitudes to all loops

Our proposal for $n-$particle superamplitude

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} + \mathcal{A}_n^{\rm NMHV} + \mathcal{A}_n^{\rm N^2MHV} + \ldots + \mathcal{A}_n^{\overline{\rm MHV}}$$

- ✔ The tree superamplitude $\mathcal{A}_n^{(\rm tree)}$ is covariant under superconformal transformations in the dual superspace (x,λ,θ)
- ✔ At loop level, this symmetry becomes anomalous due to IR divergences
- ✔ The dual superconformal symmetry is restored in the ratio of superamplitudes $\mathcal{A}_n$ and $\mathcal{A}_n^{\rm MHV}$

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} \times \Big[R_n(x_i,\lambda_i,\theta_i^A) + O(\epsilon)\Big]$$

The ratio function

$$R_n = 1 + R_n^{\rm NMHV} + R_n^{\rm N^2MHV} + \ldots$$

is *IR finite* and, most importantly, it is *superconformal invariant*!

- ✔ Wilson loop/superamplitude duality involves a new ingredient

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A)/W_n(x_i) = \mathcal{A}_n^{\rm MHV\,(tree)} \times \big[R_n(x_i,\lambda_i,\theta_i^A) + O(\epsilon)\big]$$

Wilson loop $W_n(x_i)$ takes care of anomalous contribution, $R_n =$ dual superconformal invariant

$$\mathbb{K}^\mu\, R_n(x_i,\lambda_i,\theta_i^A) = \mathbb{D}\, R_n(x_i,\lambda_i,\theta_i^A) = 0$$

Conclusions and open questions

- ✔ Various MHV amplitudes possess the dual conformal symmetry at both weak and strong coupling (is not a symmetry of the full $\mathcal{N} = 4$ SYM!)
- ✔ This symmetry is a part of much bigger dual superconformal symmetry of all planar superamplitudes in $\mathcal{N} = 4$ SYM
- ✔ The symmetry becomes manifest within the Wilson loops/superamplitudes duality
- ✔ We do not understand the origin of this symmetry but we do know how to make use of it (anomalous conformal Ward identities)
- ✔ The fact that the DHKS discrepancy function for the $n = 6$ Wilson loop coincides with the BDKRSVV discrepancy function for the six-gluon amplitude indicates that there exists yet another hidden symmetry
- ✔ We have now good reasons to believe that the Wilson loop/superamplitude duality holds for all superamplitudes to all loops... but
  - ✗ What is the origin of the dual superconformal symmetry?
  - ✗ Who controls a nontrivial discrepancy function of conformal ratios?
  - ✗ What is a dual description of the superconformal ratio function $R_n(x_i, \lambda_i, \theta_i)$?

  Should be related to integrability of planar $\mathcal{N} = 4$ SYM. More work is needed!