On $\mathrm{AdS}_5 \times \mathrm{S}^5$ String S-matrix

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"Integrability in Gauge and String Theory"
July 24-28 2006

hep-th/0508140 with Fernando Alday and Sergey Frolov
hep-th/0510208 and hep-th/0604043 with Sergey Frolov

Plan

1. **Superstring as integrable coset model**
   - Superalgebra $\mathfrak{psu}(2,2|4)$
   - Lagrangian
   - Lax representation
   - Light-cone type gauges
2. **The "$\mathfrak{su}(1|1)$" sector of string theory**
   - Uniform static gauge
   - Uniform light-cone gauge
   - Spectrum and the Bethe ansatz
3. **Crossing symmetry for $\mathrm{AdS}_5 \times \mathrm{S}^5$ string S-matrix**
   - String and gauge theory Bethe ansatze
   - Functional relation for the dressing factor
   - Perturbative check of the functional equation
4. **Open problems**

Motivation

✓ Green-Schwarz string on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is invariant

- ∗ reparametrizations
- ∗ local fermionic (kappa) symmetry
- ∗ global supergroup PSU(2,2|4)

Metsaev and Tseytlin '98

✓ Unclear how to construct an exact quantization but possible to guess Bethe type ansätze capturing quantum physics at leading order

Frolov, Staudacher and G.A. '04

✓ Impose a light-cone type gauge

- ∗ $x^+ = \tau$, $p_+ = P_+$ is uniform along the string
- ∗ Gauge-fixed Hamiltonian depends on $\sqrt{\lambda}$ and P_+

✓ Different limits

- ∗ BMN limit: $\lambda \to \infty$, $\frac{\sqrt{\lambda}}{P_+}$=fixed
- ∗ Flat-space limit: $\lambda \to \infty$, $\frac{\sqrt[4]{\lambda}}{P_+}$=fixed
- ∗ Decompactification limit: λ=fixed, $P_+ \to \infty$

✓ Use the gauge-fixed Hamiltonian to study $\frac{1}{P_+}$ corrections to these limits; extract string S-matrix

Superstring as integrable coset model

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}\,.$$

- What is $\mathfrak{psu}(2,2|4)$?

$$M_{8\times 8}=\begin{pmatrix} A & X \\ Y & D \end{pmatrix}\leftarrow \mathfrak{su}(2,2|4)\,.$$

✓ A, D are even (bosonic)

✓ X, Y are odd (fermionic)

✓ $\mathrm{str}M=\mathrm{tr}A-\mathrm{tr}D=0$

✓ $HM+M^{\dagger}H=0,$
H–hermitian with $\mathrm{diag}(1,1,-1,-1;1,1,1,1)$

Bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is

$$\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1)$$

$$\mathfrak{psu}(2,2|4) = \mathfrak{su}(2,2|4)/\mathfrak{u}(1)$$

No realization in terms of 8×8 matrices!

$$[M_1, M_2] = M_3 + i\mathbb{I}r, \quad r \in \mathbb{R}$$

The superalgebra $\mathfrak{su}(2,2|4)$ has a $\mathbb{Z}_4$ grading

$$M = M^{(0)} \oplus M^{(1)} \oplus M^{(2)} \oplus M^{(3)}$$

defined by the automorphism $M \to \Omega(M)$

$$\Omega(M) = \begin{pmatrix} KA^tK & -KY^tK \\ KX^tK & KD^tK \end{pmatrix},$$

where we choose the 4×4 matrix K to be

$$K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$M^{(0)} \sim \mathrm{so}(4,1) \times \mathrm{so}(5)$

- Lagrangian

$$\mathscr{L} = \mathscr{L}_{\text{sigma-model}} + \mathscr{L}_{wz}$$

$$\mathbf{A} = -g^{-1}\mathrm{d}g = \underbrace{\mathbf{A}^{(0)} + \mathbf{A}^{(2)}}_{\text{even}} + \underbrace{\mathbf{A}^{(1)} + \mathbf{A}^{(3)}}_{\text{odd}} .$$

$$\mathscr{L} = -\tfrac{1}{2}\sqrt{\lambda}\gamma^{\alpha\beta}\mathrm{str}\Big(\mathbf{A}_\alpha^{(2)}\mathbf{A}_\beta^{(2)}\Big) - \underbrace{\kappa\epsilon^{\alpha\beta}\mathrm{str}\Big(\mathbf{A}_\alpha^{(1)}\mathbf{A}_\beta^{(3)}\Big)}_{\text{Wess-Zumino term}},$$

Bena, Polchinski and Roiban '03

- Virasoro constraints

$$\mathrm{str}\Big(\mathbf{A}_\alpha^{(2)}\mathbf{A}_\beta^{(2)}\Big) - \tfrac{1}{2}\gamma_{\alpha\beta}\ \mathrm{str}\Big(\mathbf{A}_\delta^{(2)}\mathbf{A}_\rho^{(2)}\Big)\gamma^{\delta\rho} = 0$$

- Integrability

Lax operator

$$\mathscr{L}_\alpha = \ell_0 \mathbf{A}^{(0)}_\alpha + \ell_1 \mathbf{A}^{(2)}_\alpha + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho}\mathbf{A}^{(2)}_\rho + \ell_3(\mathbf{A}^{(1)}_\alpha + \mathbf{A}^{(3)}_\alpha) + \ell_4(\mathbf{A}^{(1)}_\alpha - \mathbf{A}^{(3)}_\alpha)\,,$$

where ℓ_i are constants.

$$\mathscr{D}_\alpha = \partial_\alpha - \mathscr{L}_\alpha$$

Equations of motion $\Longrightarrow \ell = \ell(\lambda)$, λ is a spectral parameter.

$$[\mathscr{D}_\alpha, \mathscr{D}_\beta] = 0 \qquad in \qquad \mathfrak{su}(2,2|4)$$

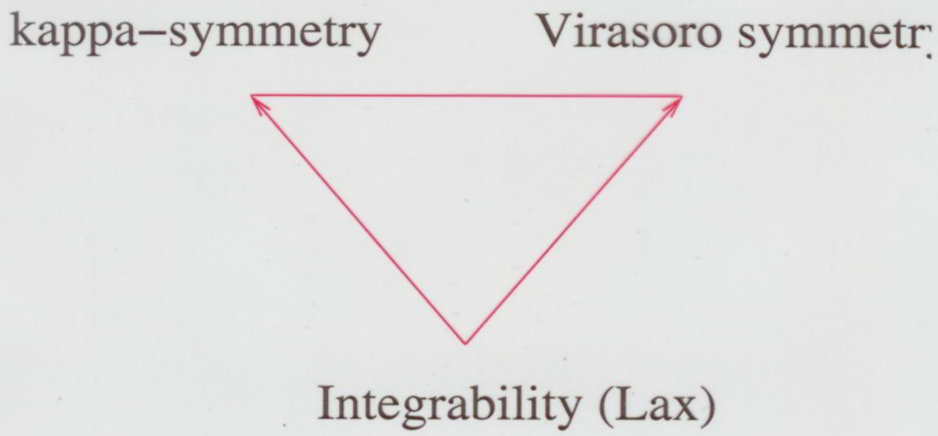
Bena, Polchinski and Roiban '03

- **Global (Noether) Symmetry** $\mathfrak{psu}(2,2|4)$

$$\mathscr{L} \to \mathscr{L}' = h\mathscr{L}h^{-1} + \mathrm{d}hh^{-1}\,, \qquad h = g$$

Expanding this connection around zero

$$\mathscr{L}'_\alpha = \lambda \underbrace{\mathcal{L}_\alpha}_{\text{c.c.}} + \ldots \qquad \underbrace{\partial_\alpha\Big(\epsilon^{\alpha\beta}\mathcal{L}_\beta\Big) = 0}_{at \quad order \quad \lambda}$$



The magic triangle of symmetries

- Integrability (existence of the Lax representation) requires $\kappa = \pm\frac{\sqrt{\lambda}}{2}$.

- It is this value of κ which is required by κ-symmetry

- Kappa-symmetry variation of the Lax connection $\delta_\kappa \mathscr{L}$ is a gauge transformation

$$\delta_\kappa \mathscr{L}_\alpha = [\Lambda(\kappa), \mathscr{L}_\alpha] - \partial_\alpha \Lambda(\kappa)$$

if and only if Virasoro constraints are satisfied.

Gauges

The $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric is

$$ds^2 = f_{\rm a}(z)\mathrm{d}t^2 + f_{\rm s}(y)\mathrm{d}\phi^2 + g^{\rm a}_{ij}\mathrm{d}z^i \mathrm{zd}z^j + g^{\rm s}_{ij}\mathrm{d}y^i\mathrm{d}y^j$$

Isometries

$$t \longrightarrow t + \mathrm{const}\,, \qquad \phi \longrightarrow \phi + \mathrm{const}$$

$$p_t \equiv E^0, \qquad p_\phi \equiv J^0$$

Two conserved charges

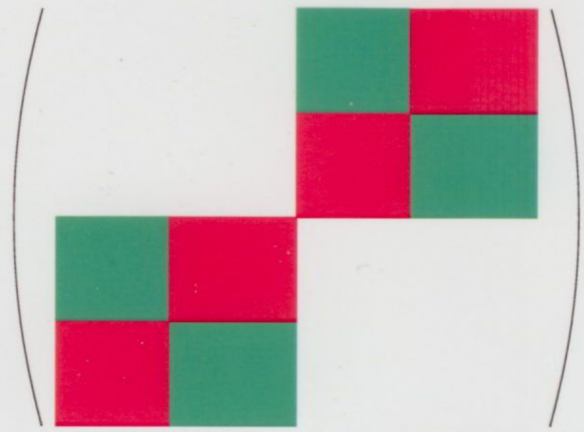
$$E = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, E^0 \qquad J = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, J^0$$

Introduce the light-cone coordinates and the light-cone momenta

$$\begin{aligned} t &= x_+ - x_-\,, \quad \phi = x_+ + x_- \\ p_t &= \frac{1}{2}(p_+ + p_-)\,, \quad p_\phi = \frac{1}{2}(p_+ - p_-) \end{aligned}$$

Fixing κ-symmetry

Fermions =



Fixing κ-symmetry – the "green" fermions are switched off

Subalgebra which leaves the Hamiltonian

$$\mathrm{H} = -p_-$$

invariant comprises two copies of $\mathfrak{su}(2|2)$. It is natural to call these gauges as

$\mathrm{SU}(2|2) \times \mathrm{SU}(2|2)$ Gauges

$\mathrm{SU}(2|2)\times\ \mathrm{SU}(2|2)$ Gauges

This should provide a "stringy" realization of the $\mathfrak{su}(2|2)$-invariant S-matrix

General construction of $\mathfrak{su}(2|2)$-invariant S-matrix

- Symmetry

$$[\mathcal{J}_1 + \mathcal{J}_2, S_{12}] = 0\,, \qquad \mathcal{J} = \mathfrak{su}(2,2)$$

- Unitarity

$$S_{12}S_{21} = 1$$

- Yang-Baxter

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$$

✓ BDS ansatz is derived

Beisert '05

✓ Symmetry algebra is extended $\mathfrak{psu}(2|2) \times \mathbb{R}^3$

✓ S-matrix is unique up to a scalar prefactor (the dressing factor)

✓ To derive the dispersion relation $E = E(p)$ off-shell extension of $\mathfrak{su}(2|2)$ is needed
– it involves two new central charges!

SU(2|2)× SU(2|2) Gauges

Static gauge $t = \tau\,,\qquad p_\phi = J$

$$E = H_{\rm 2dim}$$

Computing the spectrum of the world-sheet Hamiltonian we compute the space-time energy = conformal dimensions

Uniform light-cone gauge

$$x_+ = \tau\,,\qquad p_+ = P_+ = E + J = {\rm const}$$

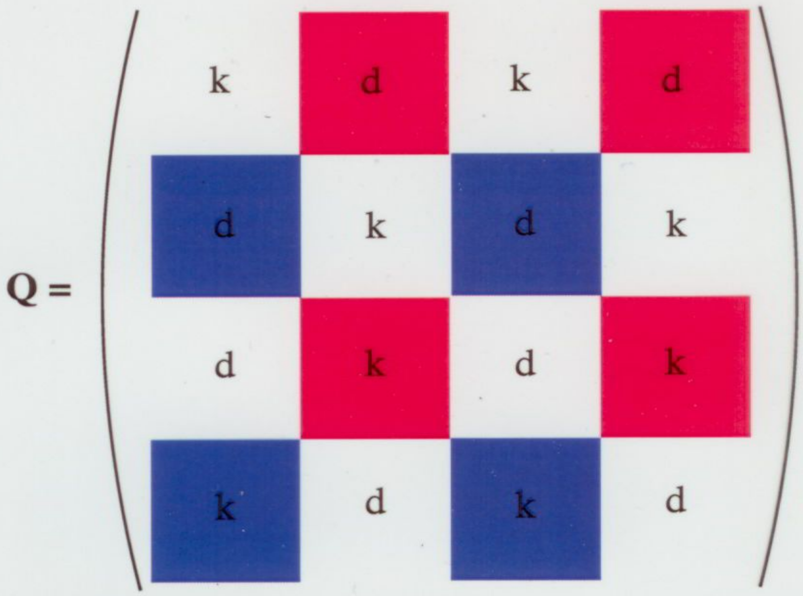
$$H_{\rm 2dim} = E - J$$

$$E = J + H_{\rm 2dim}(\underbrace{E+J}_{P_+={\rm const}})$$

We get non-trivial equation to determine the energy. Level-matching constraint $\mathcal{V} = 0$.

The bosonic unbroken symmetry subalgebra is

$$\mathsf{SO(4)} \times \mathsf{SO(4)} = \underbrace{\mathsf{SU(2)} \times \mathsf{SU(2)}}_{\text{Sphere}} \times \underbrace{\mathsf{SU(2)} \times \mathsf{SU(2)}}_{\text{AdS}}$$



Structure of the light-cone supercharges.
Red and blue blocks are two copies of $\mathfrak{su}(2|2)$ sharing the same central charge which is the Hamiltonian $\mathrm{H} = -p_-$.

We showed that with the level-matching condition omitted the algebra is enlarged by two central charges

$$\begin{aligned} \{Q, Q\} &\sim C_Q \\ \{S, S\} &\sim C_S \\ \{Q, Q^\dagger\} &\sim \mathrm{H} \end{aligned}$$

The charges C_Q and C_S vanish on physical states.

Frolov, Plefka, Zamaklar and G.A., to appear

Quantum String Bethe Ansatz QSBA

Integrability (Lax) ⟶ Finite–gap Solutions (KMMZ) ⟶ Fundamental (discrete) Bethe Equations

$$e^{ip_j L} = \prod_{k \neq j}^{M} \underbrace{S(p_j, p_k)}_{\text{gauge}} e^{i\theta(p_j, p_k)}$$

Frolov, Staudacher and G.A '04

Checks and Properties:

- ✓ Reduces to KMMZ equations in the thermodynamic limit $M, L \longrightarrow \infty$, $\frac{M}{L} = \text{fixed}$
- ✓ Reproduces near-plane wave corrections to the energy
- ✓ Reproduces $\Delta \sim \lambda^{1/4}$ as $\lambda \longrightarrow \infty$
- ✓ For $\lambda \longrightarrow 0$ defines a new long-range integrable spin chain. Beisert '04
- ✓ Derivable from IDSC. Kazakov and Gromov '06
- ✓ Reproduces scattering matrix of giant magnons

Hofman and Maldacena '06

Gauge Choice and Relation to QSBA

✓ It can be seen already at the level of reduced models

✓ Interesting non-trivial reduction – the $\mathfrak{su}(1|1)$ model

t, ϕ and two complex fermions

- In the uniform static gauge one finds the Lagrangian

$$\begin{aligned} \mathcal{L} \;&=\; \sqrt{\lambda}\int_{-\Lambda}^{\Lambda}\mathrm{d}\sigma\Big[-\tfrac{i}{2}\Big(\bar\psi\gamma^\alpha\partial_\alpha\psi-\partial_\alpha\bar\psi\gamma^\alpha\psi\Big)+\bar\psi\psi \\ &\qquad -\tfrac{1}{4}\epsilon^{\alpha\beta}\Big(\bar\psi\partial_\alpha\psi\bar\psi\gamma^3\partial_\beta\psi-\partial_\alpha\bar\psi\psi\partial_\beta\bar\psi\gamma^3\psi\Big) \\ &\qquad +\tfrac{1}{8}\epsilon^{\alpha\beta}(\bar\psi\psi)^2\partial_\alpha\bar\psi\gamma^3\partial_\beta\psi\Big] \end{aligned}$$

where

$$\Lambda=\frac{\pi J}{\sqrt{\lambda}}$$

We see that $J\longrightarrow\infty$ is a decompactification limit

New integrable system of interacting 2dim Dirac fermion

Alday, Frolov and G.A. '05

Perturbative S-matrix over a "bare" vacuum is known

Klose and Zarembo '06

Gauge Choice and Relation to QSBA

- In the uniform light-cone gauge gauge one finds the Lagrangian

$$\mathcal{L} \;=\; \sqrt{\lambda}\int_{-\Lambda}^{\Lambda} \mathrm{d}\sigma\Big[-\tfrac{i}{2}\Big(\bar\psi\gamma^\alpha\partial_\alpha\psi-\partial_\alpha\bar\psi\gamma^\alpha\psi\Big)+\bar\psi\psi\Big]$$

Free massive 2dim Dirac fermion!

$$E = J + H_{\mathrm{2dim}}(\underbrace{E+J}_{P_+=\mathrm{const}})$$

Energy spectrum

$$E - J = \sum_{i=1}^{M}\sqrt{1+\frac{4\lambda n_i^2}{(E+J)^2}}\;.$$

Frolov and G.A. '05

Treatment of the whole model in the uniform light-cone gauge is developed

Frolov, Plefka, Zamaklar '06

Gauge Choice and Relation to QSBA

The dispersion relation we found can be derived form the Bethe ansatz equation

Bethe ansatz

$$e^{ip_k L} = \prod_{j \neq k}^{M} \underbrace{e^{\frac{i}{2}(p_j(e_k - 1) - p_k(e_j - 1))}}_{\text{string S-matrix}}$$

Here $L = J + M/2$ is a "length" of the hypothetical spin chain and

$$e_k = \sqrt{1 + \frac{\lambda p_k^2}{4\pi^2}}$$

is an energy of an elementary excitation. The spectrum is

$$E - J = \sum_{k=1}^{M} e_k$$

Agrees with the AFS proposal in the small-momentum approximation (up to order $1/J$)

$$e_j = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}}$$

String and Gauge Theory Bethe Ansätze

Parametrization of the physical excitations

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i\frac{4\pi}{\sqrt{\lambda}} \equiv 2i\zeta$$

Momentum of physical excitation

$$e^{ip} = \frac{x^+}{x^-} \qquad \Longrightarrow \qquad x^\pm(p)$$

Energy of physical excitation

$$e = 1 + \frac{i}{\zeta}\left(\frac{1}{x^+} - \frac{1}{x^-}\right)$$

Beisert '04

Instead of physical momentum p we introduce another variable x

$$\sin\frac{p}{2} = \frac{\zeta}{x - \frac{1}{x}},$$

This gives

$$x^\pm(x) = x\sqrt{1 - \frac{\zeta^2}{(x-\frac{1}{x})^2}} \pm i\zeta\frac{x}{x - \frac{1}{x}}$$

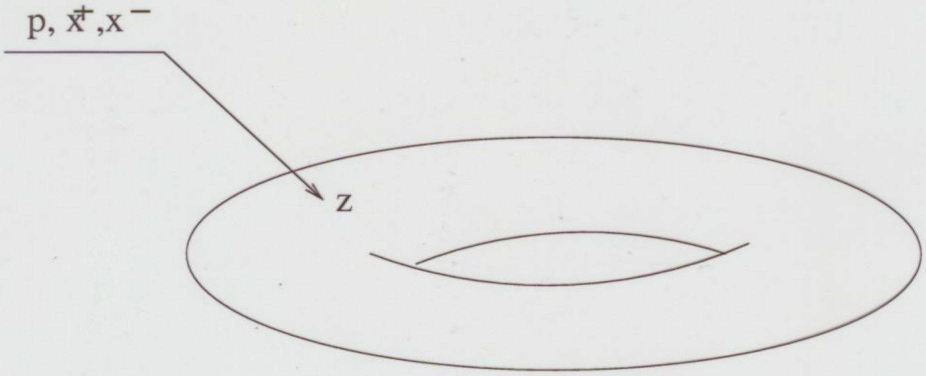
Particle-to-antiparticle transformation becomes inversion $x \longrightarrow 1/x$:

$$x^\pm(1/x) = 1/x^\pm(x)$$

Energy

$$e(x) = \frac{x + \frac{1}{x}}{x - \frac{1}{x}}.$$

String and Gauge Theory Bethe Ansätze



The Bethe ansatz data live on the elliptic curve

Explicit parametrization

$$
\begin{aligned}
x^{\pm} &= \frac{\mathrm{cn}(z,k) \pm i\,\mathrm{sn}(z,k)}{\sqrt{-k}\,\mathrm{sn}(z,k)}(1+\mathrm{dn}(z,k))\,, \\
p &= 2\mathrm{am}(z,k)
\end{aligned}
$$

Periods

$$
2\omega_1 = 4\mathrm{K(k)}\,, \qquad 2\omega_2 = 4\mathrm{K(k)} + 4\mathrm{iK}(1-\mathrm{k})\,.
$$

Modulus

$$
k = -\frac{4}{\zeta^2} = -\frac{\lambda}{\pi^2}
$$

String and Gauge Theory Bethe Ansätze

Rank-one sectors

$$\underbrace{\mathfrak{su}(2)}_{\mathfrak{s}=1}, \qquad \underbrace{\mathfrak{su}(1|1)}_{\mathfrak{s}=0}, \qquad \underbrace{\mathfrak{sl}(2)}_{\mathfrak{s}=-1}$$

Bethe equations

$$e^{ip_j L} = \prod_{k \neq j}^{M} S(x_j, x_k)$$

M number of excitations

$L = J + \frac{\mathfrak{s}+1}{2} M$ "length"

String S-matrix

$$S(x_j, x_k) = \underbrace{\left(\frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \right)^{\mathfrak{s}} \frac{1 - \frac{1}{x_j^+ x_k^-}}{1 - \frac{1}{x_j^- x_k^+}}}_{\text{gauge all loop}} \underbrace{\sigma(x_j, x_k)}_{\text{"stringy"}}$$

Beisert, Dippel and Staudacher '04
Frolov, Staudacher and G.A '04

Describes scattering of world-sheet states in the temporal gauge $t = \tau$ and $p_\phi = J$ in the decompactification limit $J \to \infty$ with λ=fixed.

String and Gauge Theory Bethe Ansätze

Some properties of the dressing factor $\sigma(x_j, x_k)$

- Universal to all sectors and for the whole model
- Cannot be fixed by $\mathfrak{psu}(2,2|4)$ symmetry
- Should tend to one as $\lambda \to 0$

$$\sigma(x_j, x_k) = e^{i\theta(x_j, x_k)}$$

$$\theta(x_j, x_k) \quad = \quad \frac{1}{\zeta} \sum_{r=2}^{\infty} \sum_{n=0}^{\infty} c_{r,r+1+2n}(\zeta)\; q_r(x_{[j})q_{r+1+2n}(x_{k]}) \, .$$

Frolov, Staudacher and G.A '04
Beisert, Klose '05

The BDS local charges

$$q_r(x) = \frac{i}{r-1}\left(\left(\frac{1}{x^+}\right)^{r-1} - \left(\frac{1}{x^-}\right)^{r-1}\right)$$

Functions $c_{r,s}$ are expanded in power series in ζ:

$$c_{r,s}(\zeta) = \delta_{r+1,s} - \zeta \frac{4}{\pi} \frac{(r-1)(s-1)}{(r+s-2)(s-r)} + \cdots$$

Subleading correction due to Hernández and López '06;
Freyhult and Kristjansen '06

Crossing Symmetry in String Theory

Janik's functional equation on dressing factor from crossing symmetry

$$\sigma(x_j, x_k)\sigma(1/x_j, x_k) = h(x_j, x_k)^2$$

Janik '06

$$h(x_j, x_k) = \frac{x_k^-}{x_k^+} \frac{(1 - \frac{1}{x_j^- x_k^-})(x_j^- - x_k^+)}{(1 - \frac{1}{x_j^+ x_k^-})(x_j^+ - x_k^+)}$$

The equation admits different solutions – correspond to S-matrices in different gauges preserving SU(2|2)×SU(2|2) symmetry

Idea:

Confront Janik's equation against the asymptotic expansion of the dressing factor

Expansion: $\zeta \to 0, \qquad x$ fixed

Crossing Symmetry in String Theory

Logarithmic version

$$i\theta(x_j, x_k) + i\theta(1/x_j, x_k) = 2\log h(x_j, x_k)$$

Expansion of the r.h.s.

$$\begin{aligned} 2\log h(x_j, x_k) &= -\zeta\frac{4ix_k(x_k + x_j(-2 + x_j x_k))}{(x_j - x_k)(x_j x_k - 1)(x_k^2 - 1)} \\ &+ \zeta^2\frac{4x_j^2 x_k^2(1 - 4x_j x_k + x_j^2 + x_k^2 + x_j^2 x_k^2)}{(x_j^2 - 1)(x_k^2 - 1)(x_j - x_k)^2(x_j x_k - 1)^2} + \cdots \end{aligned}$$

Dressing phase

$$\begin{aligned} \theta(x_j, x_k) &= \frac{1}{\zeta}\Big[\chi(x_j^-, x_k^-) - \chi(x_j^-, x_k^+) - \chi(x_j^+, x_k^-) + \chi(x_j^+, x_k^+) \\ &\quad - \chi(x_k^-, x_j^-) + \chi(x_k^+, x_j^-) + \chi(x_k^-, x_j^+) - \chi(x_k^+, x_j^+)\Big] \end{aligned}$$

where

$$\chi(x, y) = \sum_{r=2}^{\infty}\sum_{n=0}^{\infty}\frac{c_{r,r+1+2n}(\zeta)}{(r-1)(r+2n)}\frac{1}{x^{r-1}y^{r+2n}} = \chi_0 + \zeta\chi_1 + \cdots$$

Crossing Symmetry in String Theory

Asymptotic expansion of the phase

Leading term

$$\chi_0(x,y) = \frac{1}{y} + \frac{xy-1}{y}\log\left(\frac{xy-1}{xy}\right)$$

Subleading term

$$\begin{aligned}\chi_1(x,y) \quad &= \quad \frac{1}{\pi}\Big[\log\frac{y-1}{y+1}\log\frac{x-\frac{1}{y}}{x-y} \\ &+ \quad \mathrm{Li}_2\frac{\sqrt{y}-\sqrt{\frac{1}{y}}}{\sqrt{y}-\sqrt{x}} - \mathrm{Li}_2\frac{\sqrt{\frac{1}{y}}+\sqrt{y}}{\sqrt{y}-\sqrt{x}} + \mathrm{Li}_2\frac{\sqrt{y}-\sqrt{\frac{1}{y}}}{\sqrt{y}+\sqrt{x}} - \mathrm{Li}_2\frac{\sqrt{y}+\sqrt{\frac{1}{y}}}{\sqrt{y}+\sqrt{x}}\Big]\end{aligned}$$

Plug this into the dressing phase $\theta(x_j, x_k)$ and expand in ζ

$$\begin{aligned}\theta(x_j,x_k) \quad &= \quad \frac{1}{\zeta}\Big[\chi(x_j^-,x_k^-) - \chi(x_j^-,x_k^+) - \chi(x_j^+,x_k^-) + \chi(x_j^+,x_k^+) \\ &\qquad - \chi(x_k^-,x_j^-) + \chi(x_k^+,x_j^-) + \chi(x_k^-,x_j^+) - \chi(x_k^+,x_j^+)\Big]\end{aligned}$$

Janik's equation is satisfied by two leading terms of the asymptotic expansion!

Open Problems

What are analyticity conditions ?

New central charges in string theory?

Origin of the Hopf algebra symmetry?