The S-Matrix of AdS/CFT

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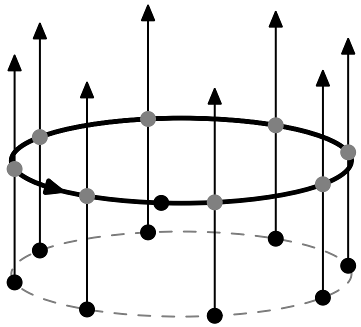

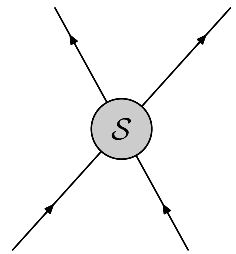

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Introduction

AdS/CFT Conjecture

- $\mathcal{N} = 4$ gauge theory (exactly) dual to IIB superstrings on $AdS_5 \times S^5$.
- Spectra should agree. Would like to test.
- Weak coupling: Planar gauge theory states described by a spin chain.
- Strong coupling: Planar quantum strings described by sigma model.
- Both models appear integrable. Make full use.

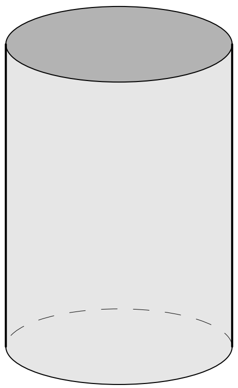
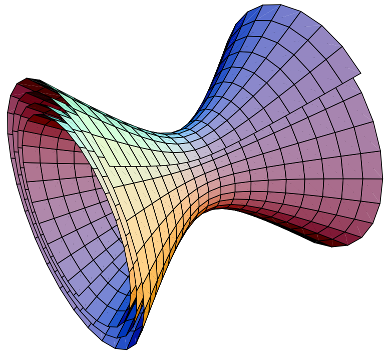
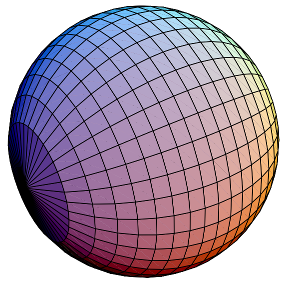
Outline

- 1D Particle model from planar string and gauge theory.
- Nature of the Hamiltonian.
- Symmetry: Centrally extended $\mathfrak{su}(2|2)$.

- Construction of S-matrix, Yang-Baxter equation.
- Relation to Shastry's R-matrix for the Hubbard chain.
- S-matrix of AdS/CFT. Abelian phase.

Planar IIB Superstrings on $AdS_5 \times S^5$

IIB superstrings propagate on the curved superspace $AdS_5 \times S^5$

 $\longrightarrow$  $\times$  $\times$ fermi

Subspaces

$$S^5 = \frac{\mathrm{SO}(6)}{\mathrm{SO}(5)} = \frac{\mathrm{SU}(4)}{\mathrm{Sp}(2)}\,, \qquad AdS_5 = \frac{\widetilde{\mathrm{SO}}(2,4)}{\mathrm{SO}(1,4)} = \frac{\widetilde{\mathrm{SU}}(2,2)}{\mathrm{Sp}(1,1)}\,, \qquad \text{fermi} = \mathbb{R}^{32}\,.$$

Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{\widetilde{\mathrm{PSU}}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)}\,.$$

The $\mathcal{N} = 4$ Spin Chain and $\mathfrak{psu}(2,2|4)$

Basis of spins: All fields of the theory & derivatives

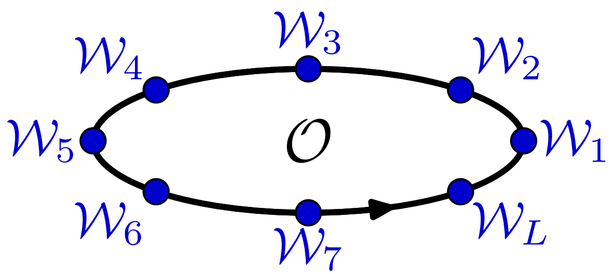
$$\mathcal{W} \in \{\mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F}\}.$$

One irreducible non-compact lowest-weight module $\mathbb{V}_{\mathrm{F}}$ of $\mathfrak{psu}(2,2|4)$.

Basis of states: Single-trace operators (cyclicity of gauge invariant states)

$$\mathcal{O} = \operatorname{Tr} \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \ldots \mathcal{W}_L = \pm \operatorname{Tr} \mathcal{W}_2 \mathcal{W}_3 \ldots \mathcal{W}_L \mathcal{W}_1 = \ldots .$$

Spin chain with non-compact spin representation $\mathbb{V}_{\mathrm{F}}$



Hamiltonian as a Symmetry Generator

The Hamiltonian ...

- ... generates time translations on AdS_5,
- ... is an isometry of the string background,
- ... is part of superconformal symmetry,
- ... is part of the symmetry algebra $\widetilde{\mathrm{PSU}}(2,2|4)$.

Unusual spin chain model where ...

- ... the Hamiltonian does not commute with the symmetry generators,
- ... the Hamiltonian is part of the non-abelian symmetry algebra,
- ... energies determine how states transform under symmetry,
- ... energies are labels for multiplets of states,
- ... the multiplet structure is dynamical,
- ... the multiplets have a continuous (energy) label.

Split $\mathfrak{H}(g) = \mathfrak{H}_0 + \delta\mathfrak{H}(g)$: Classical energy $\mathfrak{H}_0$ and energy shift $\delta\mathfrak{H}(g)$.

Light Cone Gauge and a Particle Model

Perform light cone gauge using time from AdS_5 and great circle from S^5.

- Vacuum: Point-particle moving along time and great circle. [Berenstein Maldacena Nastase]
- Excitations: 4 coordinates on AdS_5 and 4 coordinates on S^5.
- Fermions: $1/2$ are momenta, $1/2$ are gauged away, 8 remain.

QM particle model of 8 bosonic and 8 fermionic flavours on the circle.

Residual symmetry

Unbroken symmetries of the vacuum state:

- $\widetilde{\mathrm{SO}}(4,2) \simeq \widetilde{\mathrm{SU}}(2,2)$ reduces to $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$.
- $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$ reduces to $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$.
- $\widetilde{\mathrm{PSU}}(2,2|4)$ reduces to $(\mathrm{PSU}(2|2) \times \mathrm{PSU}(2|2)) \ltimes \mathbb{R}$.

Excitations transform in representations of $(\mathrm{PSU}(2|2) \times \mathrm{PSU}(2|2)) \ltimes \mathbb{R}$.

Coordinate Space Bethe Ansatz

Consider spin chain states with few "excitations".

Ferromagnetic vacuum of an *infinite* chain: protected state with scalar $\mathcal{Z}$

$$|0\rangle = |\ldots \mathcal{Z}\mathcal{Z}\mathcal{Z}\ldots\rangle, \qquad \delta\mathfrak{H}\,|0\rangle = 0.$$

One-excitation states with excitation $\mathcal{A}$ at position a, momentum p

$$|\mathcal{A}(p)\rangle = \sum_a e^{ipa}|\ldots \mathcal{Z}\ldots \overset{\overset{a}{\downarrow}}{\mathcal{A}}\ldots \mathcal{Z}\ldots\rangle, \qquad \delta\mathfrak{H}\,|\mathcal{A}(p)\rangle = \delta E_{\mathcal{A}}(p)|\mathcal{A}(p)\rangle.$$

$(4{+}4|\,4{+}4)$ flavours of single excitations $\mathcal{A} \in \{\phi^i, \mathcal{D}^\mu \mathcal{Z}|\,\psi^a, \dot{\psi}^{\dot{a}}\}$. [Berenstein Maldacena Nastase]

The remaining (infinitely many) spin orientations in module $\mathbb{V}_{\mathrm{F}}$

- are multiple excitations,
- arise when two or more elementary excitations coincide on a single site.

Coordinate space Bethe ansatz leads to a particle model with $8|8$ flavours.

Residual Symmetry

1D particle model with particles of $8|8$ flavours.

States transform under $(30|32)$ generators of $\mathfrak{psu}(2,2|4)$.

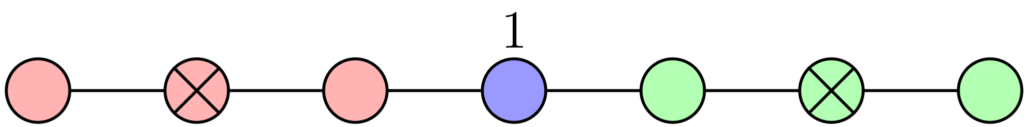
- $(8|8)$ generators create excitations with $p=0$,
- $(8|8)$ generators annihilate excitations with $p=0$,
- $(14|16)$ remaining generators transform the excitations.

Regroup excitations

$$\mathcal{A} \in \left\{ \begin{array}{cc|cc} \phi^{11} & \phi^{12} & \psi^{11} & \psi^{12} \\ \phi^{21} & \phi^{22} & \psi^{21} & \psi^{22} \\ \hline \dot\psi^{11} & \dot\psi^{12} & \mathcal{D}^{11} & \mathcal{D}^{12} \\ \dot\psi^{21} & \dot\psi^{22} & \mathcal{D}^{21} & \mathcal{D}^{22} \end{array} \right\}.$$

Residual symmetry

$\mathbb{R} \ltimes \big(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)\big) \ltimes \mathbb{R}$.



Hamiltonian $\mathfrak{H}$ is central charge of both $\mathfrak{psu}(2|2) \ltimes \mathbb{R}$.

$\mathfrak{su}(2|2)$ Residual Algebra

1D particle model with $2|2$ flavours (ϕ^a, ψ^α). Inhomogeneous spin chain.

Residual algebra $\mathfrak{su}(2|2)$ preserves particle number. Generators:

- $\mathfrak{R}^a{}_b$ internal $\mathfrak{su}(2)$ rotation generator,
- $\mathfrak{L}^\alpha{}_\beta$ spacetime $\mathfrak{su}(2)$ rotation generator,
- $\mathfrak{Q}^\alpha{}_b$ supersymmetry generator,
- $\mathfrak{S}^a{}_\beta$ superboost generator,
- $\mathfrak{C}$ central charge. Particle number and energy: $C = \frac{1}{2}K + \frac{1}{2}\delta E$.

Algebra: $\mathfrak{R}^a{}_b$, $\mathfrak{L}^\alpha{}_\beta$ transform indices. Anticommutator of supercharges

$$\{\mathfrak{Q}^\alpha{}_a, \mathfrak{S}^b{}_\beta\} = \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta \mathfrak{C},$$

$$\{\mathfrak{Q}^\alpha{}_a, \mathfrak{Q}^\beta{}_b\} = 0,$$

$$\{\mathfrak{S}^\alpha{}_a, \mathfrak{S}^b{}_\beta\} = 0.$$

Excitations should transform in $(\mathbf{2}|\mathbf{2})$ representation of $\mathfrak{su}(2|2)$.

Extension to $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$

Problems:

- Central charge acts as energy: $C = \frac{1}{2}K + \frac{1}{2}\delta E$ with δE continuous.
- The only $(2|2)$-dimensional representations are the fundamentals.
- The fundamental representations have $C = \pm\frac{1}{2}$.
- The asymptotic states have discrete C: No anomalous dimensions?

Resolution:

- Enlarge algebra by two central charges $\mathfrak{P}, \mathfrak{K}$: $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ [NB hep-th/0511082]

$$\{\mathfrak{Q}^\alpha{}_a, \mathfrak{Q}^\beta{}_b\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathfrak{P}, \qquad \{\mathfrak{S}^a{}_\alpha, \mathfrak{S}^b{}_\beta\} = \varepsilon^{ab}\varepsilon_{\alpha\beta}\mathfrak{K}.$$

- Closure of algebra for the $(\mathbf{2}|\mathbf{2})$ representation requires

$$\vec{C}^2 = C^2 - PK = \tfrac{1}{4}\,.$$

- Family of $(\mathbf{2}|\mathbf{2})$ representations with continuous C (mass shell).

Fundamental Representation of $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$

Ansatz for $(\mathbf{2}|\mathbf{2})$ representation with canonical action of $\mathfrak{R}^a{}_b, \mathfrak{L}^\alpha{}_\beta$:

$$\begin{aligned}
\mathfrak{Q}^\alpha{}_a|\phi^b\rangle &= a\,\delta^b_a|\psi^\alpha\rangle,\\
\mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle &= b\,\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^b\mathcal{Z}^+\rangle,\\
\mathfrak{S}^a{}_\alpha|\phi^b\rangle &= c\,\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^\beta\mathcal{Z}^-\rangle,\\
\mathfrak{S}^a{}_\alpha|\psi^\beta\rangle &= d\,\delta^\beta_\alpha|\phi^a\rangle,\\
\mathfrak{C}|\mathcal{X}\rangle &= C|\mathcal{X}\rangle,\\
\mathfrak{P}|\mathcal{X}\rangle &= P|\mathcal{X}\mathcal{Z}^+\rangle,\\
\mathfrak{K}|\mathcal{X}\rangle &= K|\mathcal{X}\mathcal{Z}^-\rangle.
\end{aligned}$$

Closure requires $ad-bc=1$, $C=\frac{1}{2}(ad+bc)$, $P=ab$, $K=cd$.

Denote this multiplet by $\langle\vec{C}\rangle=\langle C,P,K\rangle$.

Reduction to $\mathfrak{su}(2|2)$ for Composite States

To recover $\mathfrak{su}(2|2)$ we need $\mathfrak{P}, \mathfrak{K}$ to annihilate physical states.

Idea: Identify them with gauge transformations $\delta\mathcal{W} = [\epsilon, \mathcal{W}]$.

For $\mathfrak{P}$ use gauge parameter $\epsilon = g\alpha\mathcal{Z}$.

- Vacuum spin $\mathcal{Z}$ trivially invariant: $\mathfrak{P}|0\rangle = 0$.
- Particle transforms as $\mathfrak{P}|\mathcal{X}\rangle = g\alpha\,|[\mathcal{Z}^+, \mathcal{X}]\rangle = g\alpha\,|\mathcal{Z}^+\mathcal{X}\rangle - g\alpha\,|\mathcal{X}\mathcal{Z}^+\rangle$.

Symbols $\mathcal{Z}^\pm$ mean insertion/removal of vacuum site $\mathcal{Z}$. Implies

$$|\mathcal{Z}^\pm\mathcal{X}\rangle = e^{\mp ip}\,|\mathcal{X}\mathcal{Z}^\pm\rangle.$$

Likewise, $\mathfrak{K}|\mathcal{X}\rangle = g\alpha^{-1}\,|[\mathcal{Z}^-, \mathcal{X}]\rangle$ to remove vacuum site. Action of $\mathfrak{P}, \mathfrak{K}$:

$$\begin{aligned} \mathfrak{P}|\mathcal{X}\rangle &= P\,|\mathcal{Z}^+\mathcal{X}\rangle, \\ \mathfrak{K}|\mathcal{K}\rangle &= K\,|\mathcal{Z}^-\mathcal{X}\rangle \end{aligned} \qquad \text{with} \qquad \begin{aligned} P &= g\alpha\quad\;\,(1 - e^{+ip}), \\ K &= g\alpha^{-1}(1 - e^{-ip}). \end{aligned}$$

Cyclicity condition $P = K = 0$ for physical states with zero momentum.

Solution for the Representation

Momentum constraint and energy eigenvalue C (central charge)

$$\prod_{k=1}^{K} e^{ip_k} = 1, \qquad C = \sum_{k=1}^{K} C_k, \quad C_k = \pm\tfrac{1}{2}\sqrt{1 + 16g^2 \sin^2(\tfrac{1}{2}p_k)}\,.$$

Introduce spectral parameters $x_k^\pm$ with

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}\,.$$

Then momentum and energy of an excitation read

$$e^{ip_k} = \frac{x_k^+}{x_k^-}\,, \qquad C_k = \frac{1}{2} + \frac{ig}{x_k^+} - \frac{ig}{x_k^-} = -igx_k^+ + igx_k^- - \frac{1}{2}\,.$$

S-Matrix as an Invariant Operator

S-matrix is a two-particle permutation operator

$$\mathcal{S}_{kl}|\ldots\mathcal{X}_k\mathcal{X}'_l\ldots\rangle \mapsto *|\ldots\mathcal{X}''_l\mathcal{X}'''_k\ldots\rangle\,.$$

invariant under $\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$: $[\mathfrak{J}_k+\mathfrak{J}_l,\mathcal{S}_{kl}]=0$.

From manifest $\mathfrak{su}(2)\times\mathfrak{su}(2)$ symmetries

$$\begin{aligned}
\mathcal{S}_{12}|\phi_1^a\phi_2^b\rangle &= A_{12}|\phi_2^{\{a}\phi_1^{b\}}\rangle + B_{12}|\phi_2^{[a}\phi_1^{b]}\rangle + \tfrac{1}{2}C_{12}\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi_2^\alpha\psi_1^\beta\mathcal{Z}^-\rangle,\\
\mathcal{S}_{12}|\psi_1^\alpha\psi_2^\beta\rangle &= D_{12}|\psi_2^{\{\alpha}\psi_1^{\beta\}}\rangle + E_{12}|\psi_2^{[\alpha}\psi_1^{\beta]}\rangle + \tfrac{1}{2}F_{12}\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi_2^a\phi_1^b\mathcal{Z}^+\rangle,\\
\mathcal{S}_{12}|\phi_1^a\psi_2^\beta\rangle &= G_{12}|\psi_2^\beta\phi_1^a\rangle + H_{12}|\phi_2^a\psi_1^\beta\rangle,\\
\mathcal{S}_{12}|\psi_1^\alpha\phi_2^b\rangle &= K_{12}|\psi_2^\alpha\phi_1^b\rangle + L_{12}|\phi_2^b\psi_1^\alpha\rangle.
\end{aligned}$$

with ten coefficient functions $A_{12},\ldots,L_{12}$ and $A_{12}=A(p_1,p_2)$,. . . .

Coefficients

Commutators with fermionic generators relate all coefficients: [NB hep-th/0511082]

$$A_{12} = S^0_{12}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\,,$$

$$G_{12} = S^0_{12}\,\frac{1}{\xi_1}\,\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\,, \qquad B_{12} = S^0_{12}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\quad 1 - 2\,\frac{1 - 1/x_2^- x_1^+}{1 - 1/x_2^- x_1^-}\,\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}\Bigg)\,,$$

$$H_{12} = S^0_{12}\,\frac{\xi_2\gamma_1}{\xi_1\gamma_2}\,\frac{x_2^+ - x_2^-}{x_2^- - x_1^+}\,, \qquad C_{12} = S^0_{12}\,\frac{2\xi_2\gamma_2\gamma_1}{\alpha x_2^- x_1^-}\,\frac{1}{1 - 1/x_2^- x_1^-}\,\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\,,$$

$$K_{12} = S^0_{12}\,\frac{\gamma_2}{\gamma_1}\,\frac{x_1^+ - x_1^-}{x_2^- - x_1^+}\,, \qquad E_{12} = -S^0_{12}\,\frac{\xi_2}{\xi_1}\quad 1 - 2\,\frac{1 - 1/x_2^+ x_1^-}{1 - 1/x_2^+ x_1^+}\,\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\Bigg)\,,$$

$$L_{12} = S^0_{12}\,\xi_2\,\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\,, \qquad F_{12} = -S^0_{12}\,\frac{2\alpha(x_2^+ - x_2^-)(x_1^+ - x_1^-)}{\xi_1\gamma_2\gamma_1 x_2^+ x_1^+}\,\frac{1}{1 - 1/x_2^+ x_1^+}\,\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\,,$$

$$D_{12} = -S^0_{12}\,\frac{\xi_2}{\xi_1}\,.$$

Spectral parameters $x_k^\pm = x^\pm(p_k)$ functions of particle momenta p_k.

- Overall factor $S^0_{12} = S^0(p_1, p_2)$ cannot be determined by symmetry.
- S-matrix not of difference form $\mathcal{S}(p_1, p_2) \neq \mathcal{S}(f(p_1) - f(p_2))$.

Uniqueness and Representations

Why is the S-matrix uniquely determined by symmetry alone? [NB to appear]

Consider the tensor product of two fundamentals

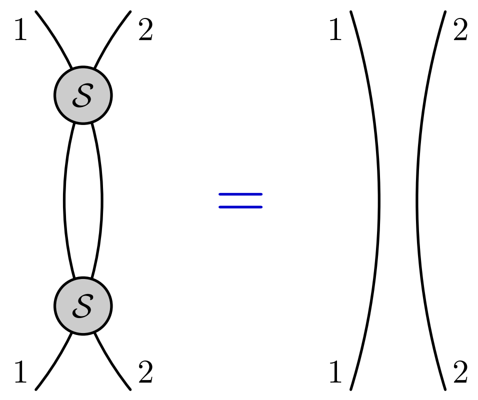
$$\langle\vec{C}\rangle \otimes \langle\vec{C}'\rangle = \{0, 0; \vec{C} + \vec{C}'\}.$$

The one resulting representations is irreducible!

- Fundamental representation $\langle\vec{C}\rangle$ is short/atypical: 4 components. $\{0, 0; \vec{C}\}$ is long/typical (unless $\vec{C}^2 = 1$): 16 components. $4 \times 4 = 16$.
- Shortening condition for $\{m, n; \vec{C}\}$ is quadratic $\vec{C}^2 = \frac{1}{4}(n + m + 2)^2$. Generally not met because $(\vec{C} + \vec{C}')^2 = \frac{1}{2} + 2\vec{C} \cdot \vec{C}' \neq 1$.
- Very unusual to obtain only one irrep in tensor product.

Unitarity

Does the S-matrix satisfy the unitarity condition $\mathcal{S}_{21}\mathcal{S}_{12} = 1$?



It maps between identical spaces

$$\mathcal{S}_{21}\mathcal{S}_{12} : \langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle \to \langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle.$$

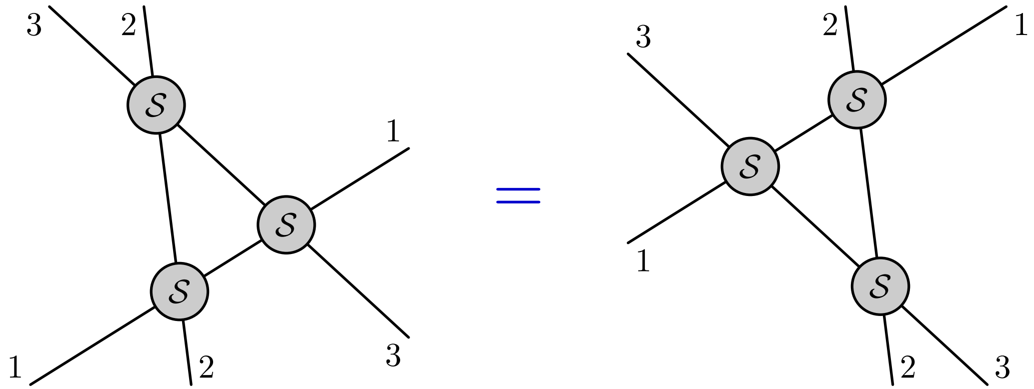
But $\langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle = \{0, 0; \vec{C}_1 + \vec{C}_2\}$ is irreducible, so $\mathcal{S}_{21}\mathcal{S}_{12} \sim \mathcal{I}$!

Show $\mathcal{S}_{21}\mathcal{S}_{12} = \mathcal{I}$ for one component $|\psi^1\psi^1\rangle$: $\mathcal{S}_{12}$ acts as factor D_{12}.

Overall factor: $D_{12}D_{21} = S^0_{12}S^0_{21}$. Unitarity requires $S^0_{12}S^0_{21} \overset{!}{=} 1$.

Yang-Baxter Relation

Does it satisfy the Yang-Baxter equation $\mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23} = \mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12}$?



Consider the tensor product

$$\langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle \otimes \langle \vec{C}_3 \rangle = \{0, 1; \vec{C}_1 + \vec{C}_2 + \vec{C}_3\} \oplus \{1, 0; \vec{C}_1 + \vec{C}_2 + \vec{C}_3\}.$$

Then show $\mathcal{S}_{21}\mathcal{S}_{31}\mathcal{S}_{32}\mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23} = \mathcal{I}$ for one representative state in each multiplet to prove YBE: $|\phi^1\phi^1\phi^1\rangle$ and $|\psi^1\psi^1\psi^1\rangle$. No mixing here!
S-matrix $\mathcal{S}_{12}$ acts as factor A_{12} and D_{12}, respectively. Trivially satisfied.

Nested Bethe Ansatz

To describe a state we started with

- a homogeneous spin chain infinitely many spin orientations

$$|\ldots \mathcal{Z}\mathcal{Z}\phi^a\mathcal{Z}\mathcal{Z}\mathcal{Z}\phi^b\mathcal{Z}\mathcal{Z}\mathcal{Z}\ldots\mathcal{Z}\mathcal{Z}\psi^c\mathcal{Z}\mathcal{Z}\ldots\rangle^0.$$

We can now describe a state by

- a set of K momenta p_k,
- an inhomogeneous spin chain with $(2|2)$ spin orientations

$$|\phi_1^a\phi_2^b\psi_3^c\rangle^{\mathrm{I}}.$$

We should repeat this procedure to trade more orientations for momenta.

In the end, we would like to describe a state by continuous numbers only

- a set of K parameters p_k (expressed through $x_k^\pm$),
- a set of N parameters y_k,
- a set of M parameters w_k.

Achieved by nested Bethe ansatz.

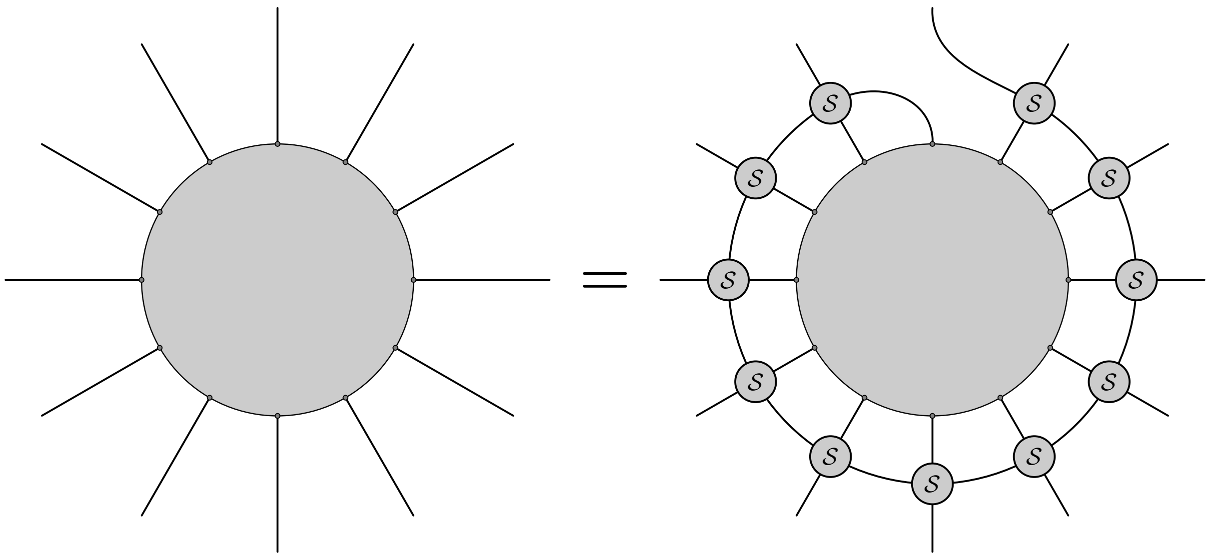
Diagonalised Scattering

Components of the diagonalised S-matrix [NB hep-th/0511082]

$$S_{12}^{\mathrm{I},0} = \frac{x_1^+}{x_1^-}, \qquad S_{12}^{\mathrm{II},\mathrm{I}} = \xi_2 \frac{y_1 - x_2^-}{y_1 - x_2^+}, \qquad S_{12}^{\mathrm{III},\mathrm{II}} = \frac{w_1 - y_2 - 1/y_2 - \frac{i}{2}g^{-1}}{w_1 - y_2 - 1/y_2 + \frac{i}{2}g^{-1}},$$

$$S_{12}^{\mathrm{I},\mathrm{I}} = S_{12}^0 \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}, \quad S_{12}^{\mathrm{II},\mathrm{II}} = 1, \qquad S_{12}^{\mathrm{III},\mathrm{III}} = \frac{w_1 - w_2 + ig^{-1}}{w_1 - w_2 - ig^{-1}}.$$

Periodicity condition: Bethe equation



$$1 = \prod_{\substack{j=1 \\ j\neq k}}^{K+M+N} S_{jk}.$$

Hubbard Chain

- Spin chain model of 2 bosonic and 2 fermionic spin orientations.
- It has $\mathfrak{su}(2)\times\mathfrak{su}(2)$ symmetry (for odd length only $\mathfrak{su}(2)$).
- It has a coupling constant U.
- It does not fit into scheme of conventional integrable spin chains.
- Shastry's R-matrix for this model is not of difference form.

Is there a relationship? Compare to Lieb-Wu equations: [Lieb, Wu Phys. Rev. Lett. 20, 1445 (1968)][NB to appear]

$$1=\prod_{j=1}^{K}\frac{1}{\xi_j}\frac{y_k-x_j^+}{y_k-x_j^-}\prod_{j=1}^{M}\frac{y_k+1/y_k-w_j+\frac{i}{2}g^{-1}}{y_k+1/y_k-w_j-\frac{i}{2}g^{-1}},$$

$$1=\prod_{j=1}^{N}\frac{w_k-y_j-1/y_j+\frac{i}{2}g^{-1}}{w_k-y_j-1/y_j-\frac{i}{2}g^{-1}}\prod_{\substack{j=1\\j\neq k}}^{M}\frac{w_k-w_j-ig^{-1}}{w_k-w_j+ig^{-1}},$$

$$1=\exp(-ik_kK)\prod_{j=1}^{M}\frac{2\sin k_k-2\Lambda_j+\frac{i}{2}U}{2\sin k_k-2\Lambda_j-\frac{i}{2}U},$$

$$1=\prod_{j=1}^{N}\frac{2\Lambda_k-\sin k_j+\frac{i}{2}U}{2\Lambda_k-\sin k_j-\frac{i}{2}U}\prod_{\substack{j=1\\j\neq k}}^{M}\frac{2\Lambda_k-2\Lambda_j-iU}{2\Lambda_k-2\Lambda_j+iU}.$$

Need to identify

$$g^{-1}=U,\qquad w_k=2\Lambda_k,\qquad y_k=-i\exp(ik_k).$$

Furthermore, $x_k^+=i\xi_k$, $x_k^-=-i/\xi_k$ and take the limit $\xi_k\to\infty$.

Shastry's R-matrix

Form of present S-matrix similar to Shastry's R-matrix. [Shastry PRL 56,2453]

Compare coeffs $A,\ldots,L$ to $\alpha_{1,\ldots,10}$ from Ramos-Martins. [Ramos, Martins hep-th/9605141]

Adjusting the parameters $x_k^\pm, \xi_k, \gamma_k$ to a_k, b_k, h_k appropriately, we find

$$\frac{\alpha_2}{\alpha_1}=\frac{A_{12}}{D_{12}}, \qquad \frac{\alpha_3}{\alpha_1}=\frac{D_{12}+E_{12}}{2D_{12}}, \qquad \frac{\alpha_4}{\alpha_1}=\frac{A_{12}+B_{12}}{2D_{12}},$$

$$\frac{\alpha_5}{\alpha_1}=\frac{H_{12}}{D_{12}}=\frac{K_{12}}{D_{12}}, \qquad \frac{\alpha_6}{\alpha_1}=-\frac{D_{12}-E_{12}}{2D_{12}}, \qquad \frac{\alpha_7}{\alpha_1}=\frac{A_{12}-B_{12}}{2D_{12}},$$

$$\frac{\alpha_8}{\alpha_1}=\frac{G_{12}}{D_{12}}, \qquad \frac{\alpha_9}{\alpha_1}=-\frac{L_{12}}{D_{12}}, \qquad \frac{\alpha_{10}}{\alpha_1}=-\frac{C_{12}}{2D_{12}}=-\frac{F_{12}}{2A_{12}}.$$

R-matrix is equivalent to present S-matrix (up to twist)!

- R-Matrix has hidden $\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$ supersymmetry.
- Supersymmetry broken in Hubbard chain by choice of $x_k^\pm, \xi_k$.
- Hubbard chain related to exceptional Lie superalgebra $\mathfrak{d}(2|1;\alpha)$.
- AdS/CFT scattering related to R-matrix of Hubbard chain.
- Different from earlier relation by Rej-Serban-Staudacher. [Rej, Serban Staudacher]

The S-Matrix for AdS/CFT

Generalise to full symmetry $\mathfrak{psu}(2,2|4)$ of AdS/CFT.

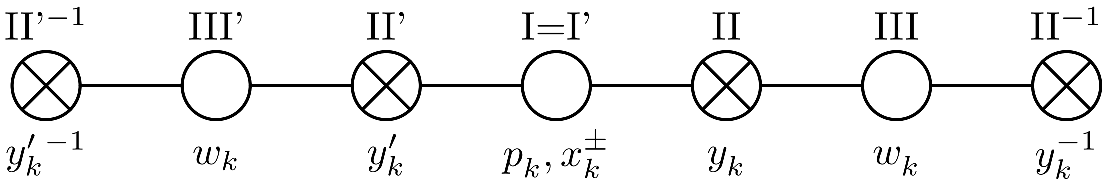
Particles transform as $(\mathbf{2}|\mathbf{2}) \otimes (\mathbf{2}|\mathbf{2})'$ under $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)') \ltimes \mathbb{R}^3$

$$\langle \vec{C} \rangle \otimes \langle \vec{C} \rangle'.$$

Scattering of multiplets factorises. Complete S-matrix is a product

$$\mathcal{S}_{12}^{\mathrm{AdS/CFT}} = \mathcal{S}_{12}\,\mathcal{S}'_{12}.$$

Diagonalised excitations: Five types $w'_k, y'_k, p_k, y_k, w_k$.



Momenta p_k shared between $\mathfrak{psu}(2|2)$'s.

Abelian Phase

The phase $S^0(p_k, p_j)$ is unconstrained. Perturbatively (?) [NB, Klose hep-th/0510124]

$$(S^0_{kj})^2 = \frac{1 - 1/x_k^- x_j^+}{1 - 1/x_k^+ x_j^-} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \exp\bigl(-2i\theta_{kj}\bigr),$$

$$\theta_{kj} = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{rs}(g^2)\bigl(q_{r,k}\, q_{s,j} - q_{s,k}\, q_{r,j}\bigr),$$

$$q_{r,k} = \frac{1}{r-1}\left(\frac{i}{(x_k^+)^{r-1}} - \frac{i}{(x_k^-)^{r-1}}\right).$$

- Proposal for gauge theory: $\beta_{rs} = 0$ from Feynman diagrams. [NB, Dippel Staudacher]
  Confirmed at three loops. Exact?! [NB hep-th/0310252] [NB, Staudacher hep-th/0504190] [Rej, Serban Staudacher] [Eden Staudacher]
  Works in non-compact sectors. [Moch Vermaseren Vogt] [Kotikov, Lipatov Onishchenko Velizhanin] [Staudacher hep-th/0412188] [Bern Dixon Smirnov]
- Proposal for string theory: $\beta_{rs} = g\delta_{r+1,s} + \mathcal{O}(g^0)$. [Arutyunov Frolov Staudacher]
  Corrections for sigma model loops needed. [NB, Tseytlin hep-th/0509084] [Hernández López] [Freyhult Kristjansen]

Conclusions

⋆ **Planar AdS/CFT Correspondence**

- Exciting spin chain model from $\mathcal{N} = 4$ gauge theory.
- Strings theory & coordinate space Bethe ansatz for gauge theory: Particle model with $(2|2) \times (2|2)$ particle flavours.
- Residual symmetry is two copies of $\mathfrak{su}(2|2)$ with central extensions.
- Unique S-matrix constructed. YBE automatically satisfied.

⋆ **Hubbard Chain**

- BE contain Lieb-Wu equations. S-Matrix contains Shastry's R-matrix.
- R-matrix supersymmetric. Scattering in AdS/CFT like Hubbard chain.

⋆ **Open Questions**

- Prove integrability for gauge and string theory.
- Find abelian phase S^0_{12} consistent with crossing symmetry.
- Understand better & generalise $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ chain.