The S-Matrix Reloaded: Twistors, Unitarity, Gauge Theories and Gravity

Potsdam 2006: Integrability in Gauge and String Theory

Zvi Bern, UCLA

Based on papers with: I. Bena, C. Berger, N.E.J Bjerrum-Bohr, M. Czakon, L. Dixon, D. Dunbar, D. Forde, H. Ita, D. Kosower, R. Roiban and V. Smirnov.

Outline

"A method is more important that a discovery, since the right method can lead to new and even more important discoveries" -- L.D. Landau

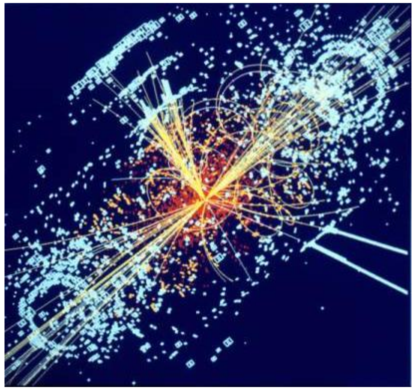
The past two years have seen a significant advance in our ability to compute scattering amplitudes.

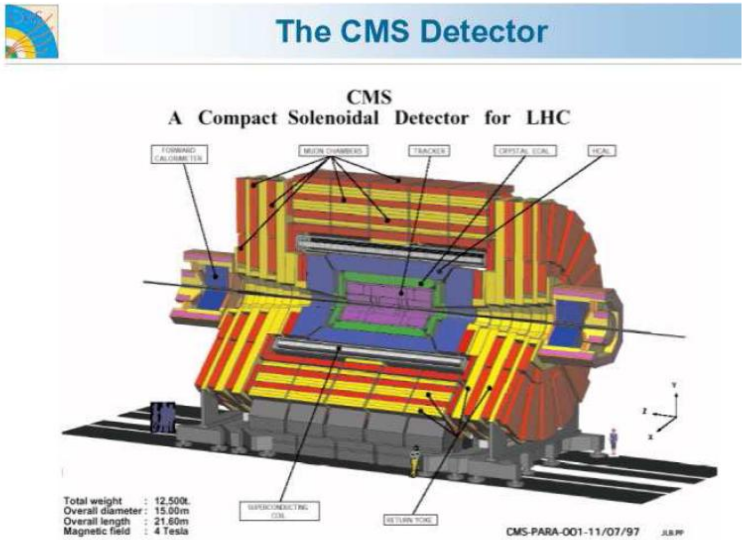
- **The call of the LHC: multi-parton scattering at loop level.**
- **Can we resum (planar) $\mathcal{N} = 4$ super-Yang-Mills theory?**
- **The structure of perturbative quantum gravity. Reexamine standard wisdom on quantum gravity.**

LHC Physics

The LHC will start operations in 2007.

We will have lots of multi-particle processes. Want reliable predictions.

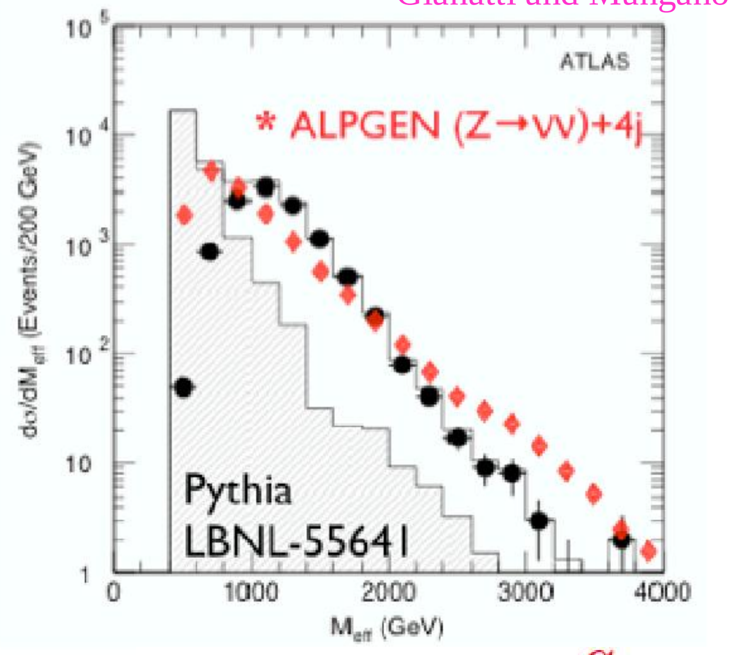




Example: Susy Search

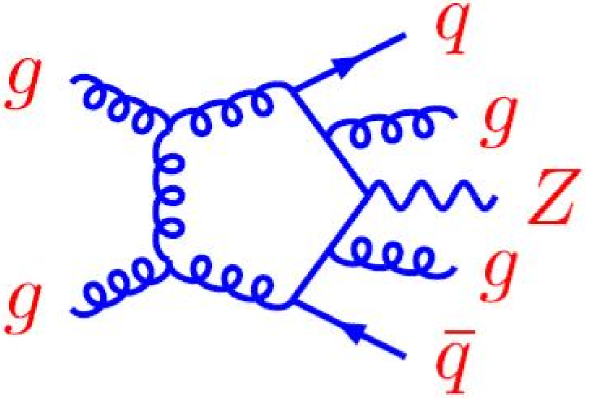
Gianatti and Mangano

Early ATLAS TDR studies using PYTHIA overly optimistic.



- ALPGEN is based on LO matrix elements and much better at modeling hard jets.

- What will disagreement between ALPGEN and data mean? Hard to tell. Need NLO.



Such a calculation is well beyond anything that has been done using Feynman diagrams

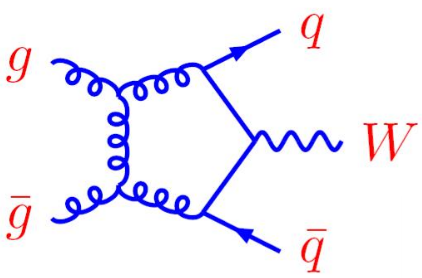
We need $pp \to Z + 4$ jets at NLO

State-of-the-Art NLO QCD

Five point is *still* state-of-the art for QCD cross-sections:

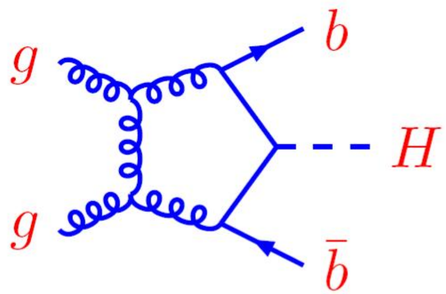
Typical examples:

$pp \to W, Z + 2$ jets



Bern, Dixon, Kosower
Dixon, Kunszt, and Signer
Campbell and Ellis: MCFM

$pp \to \bar{b}bH$ or $pp \to \bar{t}tH$



Reina, Dawson, Jackson and Wackeroth
Beenakker, Dittmaier, Kramer, Plumper, Spira

Brute force calculations give GB expressions – numerical stability?

Amusing numbers: 6g: 10,860 diagrams, 7g: 168,925 diagrams

Much worse difficulty: integral reduction generates nasty dets.

$$\frac{1}{\det(k_i \cdot k_j)^n}$$ “Grim” determinant

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What needs to be done at NLO?

Experimenters to theorists: “Please calculate the following at NLO”

Run II Monte Carlo Workshop, April 2001

Single boson	Diboson	Triboson	Heavy flavour
$W + \leq 5j$	$WW + \leq 5j$	$WWW + \leq 3j$	$t\bar{t} + \leq 3j$
$W + b\bar{b} + \leq 3j$	$WW + b\bar{b} + \leq 3j$	$WWW + b\bar{b} + \leq 3j$	$t\bar{t} + \gamma + \leq 2j$
$W + c\bar{c} + \leq 3j$	$WW + c\bar{c} + \leq 3j$	$WWW + \gamma\gamma + \leq 3j$	$t\bar{t} + W + \leq 2j$
$Z + \leq 5j$	$ZZ + \leq 5j$	$Z\gamma\gamma + \leq 3j$	$t\bar{t} + Z + \leq 2j$
$Z + b\bar{b} + \leq 3j$	$ZZ + b\bar{b} + \leq 3j$	$WZZ + \leq 3j$	$t\bar{t} + H + \leq 2j$
$Z + c\bar{c} + \leq 3j$	$ZZ + c\bar{c} + \leq 3j$	$ZZZ + \leq 3j$	$t\bar{b} + \leq 2j$
$\gamma + \leq 5j$	$\gamma\gamma + \leq 5j$		$b\bar{b} + \leq 3j$
$\gamma + b\bar{b} + \leq 3j$	$\gamma\gamma + b\bar{b} + \leq 3j$		
$\gamma + c\bar{c} + \leq 3j$	$\gamma\gamma + c\bar{c} + \leq 3j$		
	$WZ + \leq 5j$		
	$WZ + b\bar{b} + \leq 3j$		
	$WZ + c\bar{c} + \leq 3j$		
	$W\gamma + \leq 3j$		
	$Z\gamma + \leq 3j$		

Theorists to experimenters:

“In your dreams”

A key theoretical problem for LHC is NLO

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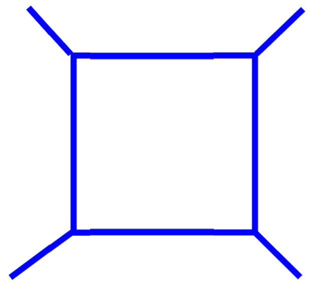
More Realistic NLO Wishlist

Les Houches 2005

process ($V \in \{Z, W, \gamma\}$)	background to
1. $pp \to V\,V\,\mathrm{jet}$	$t\bar{t}H$, new physics
2. $pp \to H + 2\,\mathrm{jets}$	H production by vector boson fusion (VBF)
3. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
4. $pp \to t\bar{t} + 2\,\mathrm{jets}$	$t\bar{t}H$
5. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to V\,V + 2\,\mathrm{jets}$	VBF$\to H \to VV$
7. $pp \to V + 3\,\mathrm{jets}$	various new physics signatures
8. $pp \to V\,V\,V$	SUSY trilepton

Bold action required!

Consider an integral



$$\int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-\epsilon}} \frac{\ell^\mu\, \ell^\nu\, \ell^\rho\, \ell^\lambda}{\ell^2\,(\ell-k_1)^2\,(\ell-k_1-k_2)^2\,(\ell+k_4)^2}$$

Evaluate this integral via Passarino-Veltman reduction. Result is …

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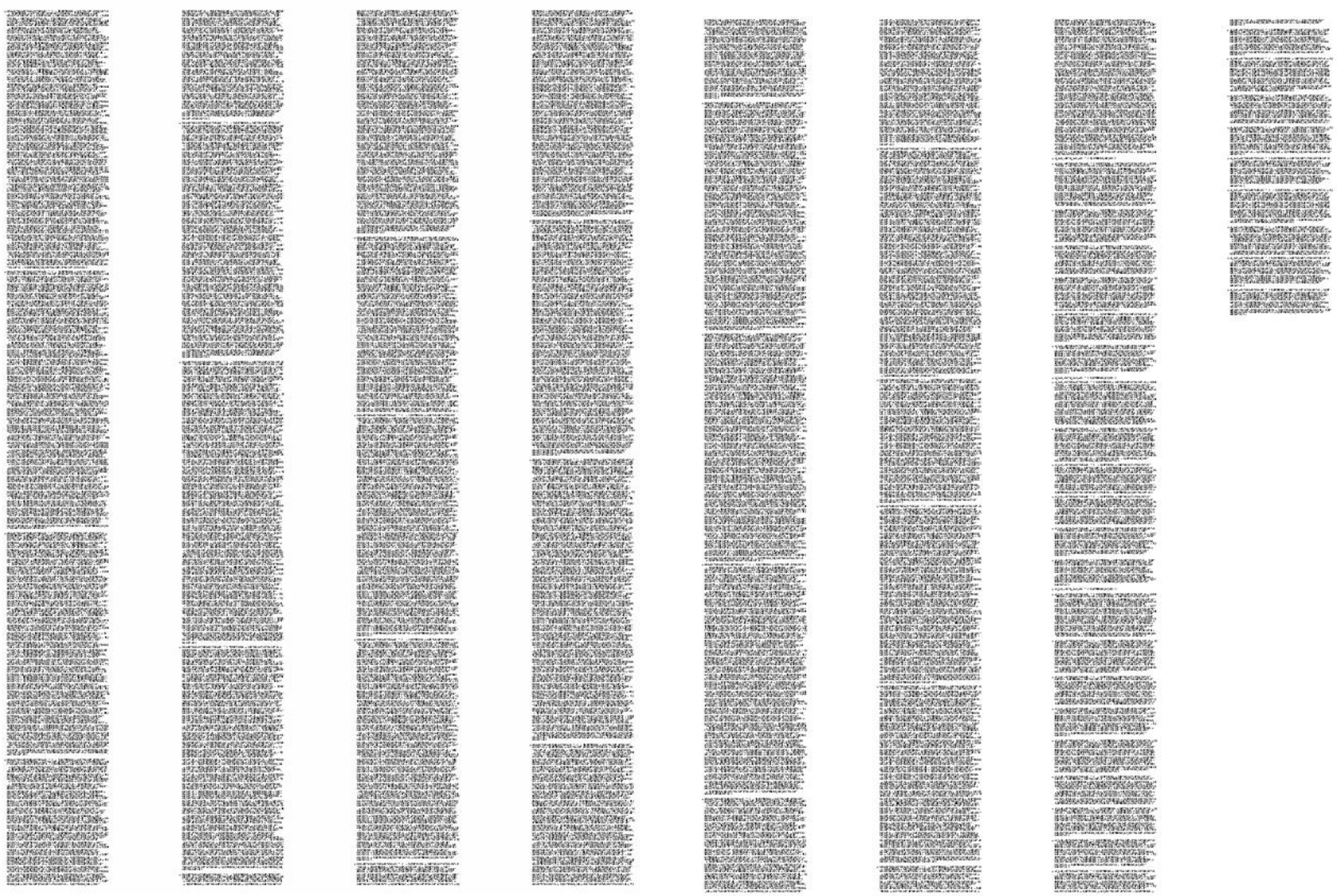
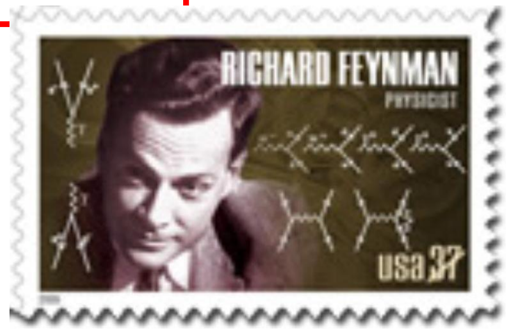
Result of performing the integration

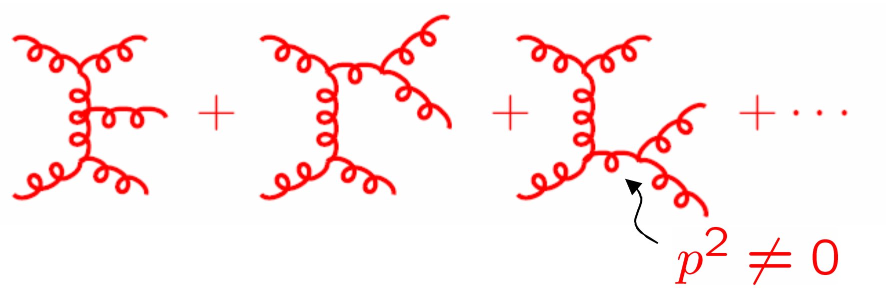


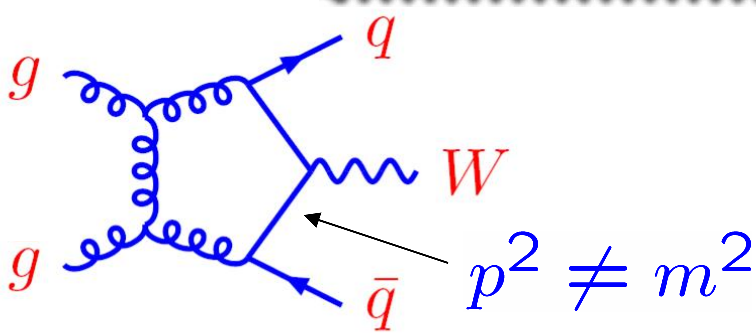
Numerical stability is a key issue.
Clearly, there should be a better way

Why are Feynman diagrams clumsy for high loop or multiplicity processes?



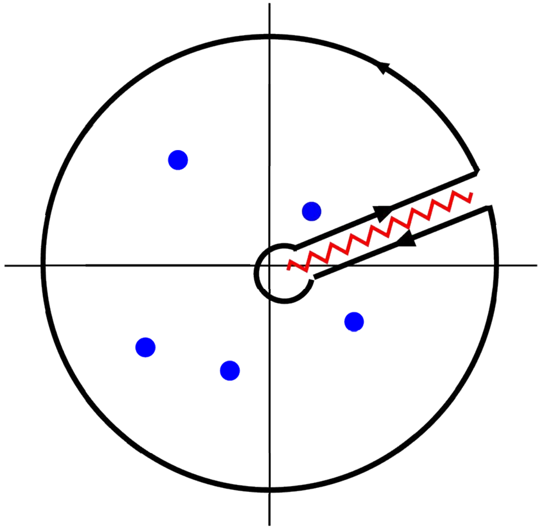
- Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.

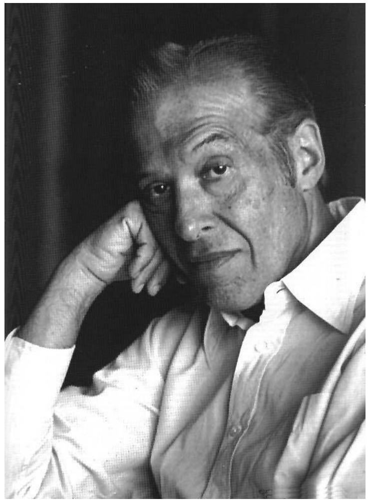




- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

- **All steps should be in terms of gauge invariant on-shell states. $p^2 = m^2$ On shell formalism.**
- **Radical rewrite of gauge theory needed.**





"One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane."

J. Schwinger in "Particles, Sources and Fields" Vol 1

On-shell Formalisms

With on-shell formalisms we can exploit analytic properties

- **Curiously, a practical on-shell formalism was constructed at loop level prior to tree level: unitarity method.** Bern, Dixon, Dunbar, Kosower **(1994)**

- **Solution at tree-level had to await Witten's twistor inspiration. (2004)**

  -- MHV vertices Cachazo, Svrcek Witten; Brandhuber, Spence, Travaglini

  -- On-shell recursion Britto, Cachazo, Feng, Witten

- **Combining unitarity method with on-shell recursion gives loop-level on-shell bootstrap. (2006)** Berger, Bern, Dixon, Forde, Kosower

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Spinors and Twistors

Spinor helicity for gluon polarizations in QCD:

$$\varepsilon_\mu^+(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\,k\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\,q]}$$

$$\epsilon^{ab}\lambda_{ja}\lambda_{lb} \longleftrightarrow \langle j\,l\rangle = \langle k_{j\,-}|k_{l+}\rangle = \sqrt{2k_j\cdot k_l}\;e^{i\phi} = \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l)$$

$$\epsilon_{\dot a\dot b}\tilde\lambda_j^{\dot a}\tilde\lambda_l^{\dot b} \longleftrightarrow [j\,l] = \langle k_{j\,+}|k_{l-}\rangle = -\sqrt{2k_j\cdot k_l}\;e^{-i\phi} = \tfrac{1}{2}\bar{u}(k_j)(1-\gamma_5)u(k_l)$$

Penrose twistor transform:

$$\widetilde{A}(\lambda_i,\mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2}\,\exp\left(\sum_j \mu_j^{\dot a}\widetilde{\lambda}_{j\dot a}\right) A(\lambda_i,\widetilde{\lambda}_i)$$

Early work from Nair

Witten's remarkable twistor-space link:

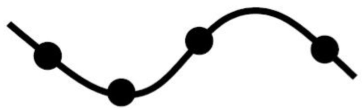
Witten; Roiban, Spradlin and Volovich

QCD scattering amplitudes ⟷ Topological String Theory

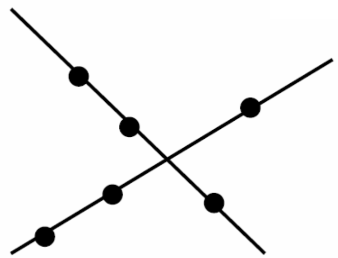
Amazing Simplicity

Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

$$d = q - 1 + L, \quad q = \#\ \text{negative helicities}, \quad L = \#\ \text{loops},$$



Connected picture



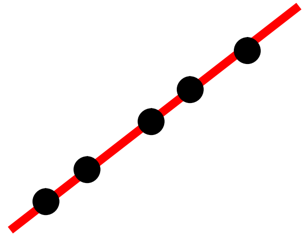
Disconnected picture

Witten
Roiban, Spradlin and Volovich
Cachazo, Svrcek and Witten
Gukov, Motl and Neitzke
Bena Bern and Kosower

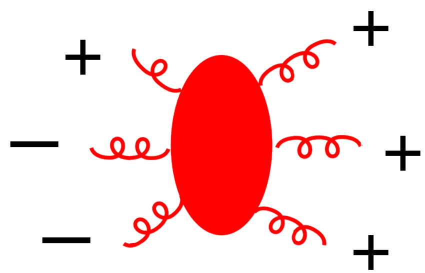
Structures imply an amazing simplicity in the scattering amplitudes. Amplitudes are much much simpler than anyone imagined.

MHV Amplitudes

Parke and Taylor (1984)



At tree level Parke and Taylor conjectured a very simple form for *n*-gluon scattering.



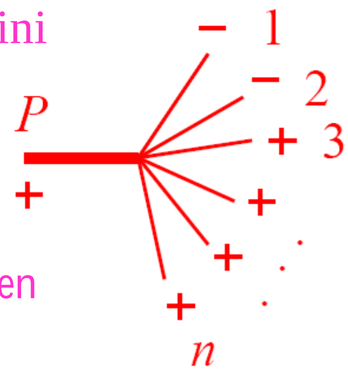
$$A(1^-,2^-,3^+,\ldots,n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

$$\mathcal{A}(1^-,2^-,3^+,\ldots,n^+) = \sum_{\text{perms}} \text{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]A(1^-,2^-,3^+,\ldots,n^+)$$

Proven by Berends and Giele

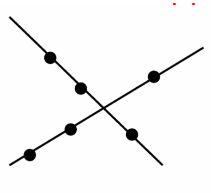
Amazingly, this simplicity continues to loops and to general helicities.

Bern, Dixon, Dunbar, Kosower
Cachazo, Svrcek, Witten; Bern, Dixon, Kosower
Brandhuber, Spence and Travaglini



These MHV amplitudes can be thought of as vertices for building new amplitudes.

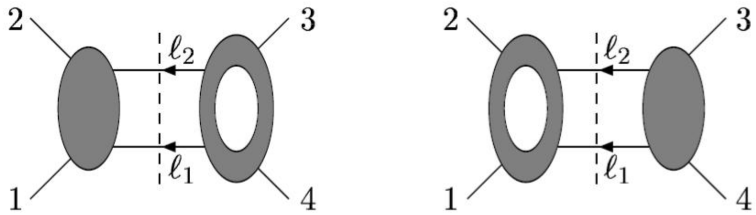
Cachazo, Svrcek and Witten



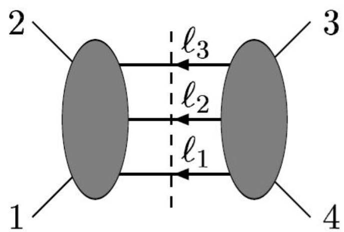
Bern, Dixon, Dunbar and Kosower

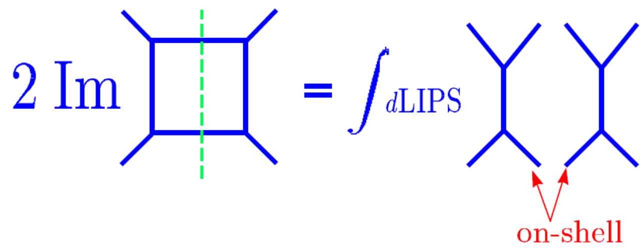
Unitarity Method

Two-particle cut:



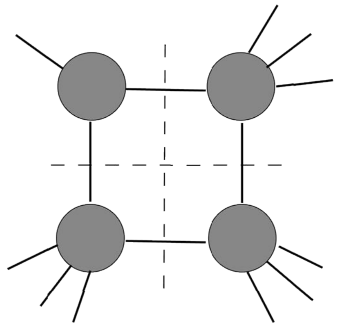
Three- particle cut:





Generalized unitarity:

Bern, Dixon and Kosower



As observed by Britto, Cachazo and Feng quadruple cut freezes integral:
Coefficients of box integrals always easy.

Generalized cut interpreted as cut propagators not canceling.

Recent improvements for bubble and triangle contributions

Britto, Buchbinder, Cachazo and Feng; Britto, Feng and Mastrolia

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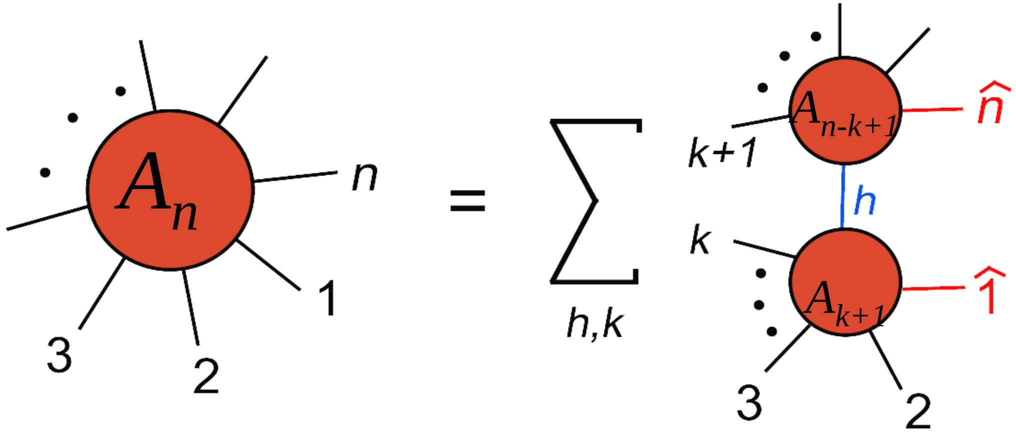
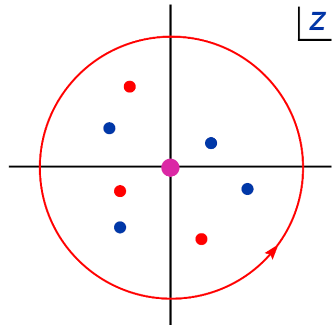
Tree-Level On-Shell Recursion

New representations of tree amplitudes from IR consistency of one-loop amplitudes in $N = 4$ super-Yang-Mills theory. Bern, Del Duca, Dixon, Kosower; Roiban, Spradlin, Volovich

Using intuition from twistors and generalized unitarity: Britto, Cachazo, Feng
BCF + Witten





$$p_1^\mu(z) = p_1^\mu - \frac{z}{2}\langle 1^- | \gamma^\mu | n^- \rangle \qquad p_n^\mu(z) = p_n^\mu + \frac{z}{2}\langle 1^- | \gamma^\mu | n^- \rangle$$

On-shell conditions maintained by shift.

Proof relies on so little. Power comes from generality

- **Cauchy's theorem** Britto, Cachazo, Feng and Witten
- **Basic field theory factorization properties**
- **Applies as well to massive theories** Badger, Glover, Khoze and Svrcek

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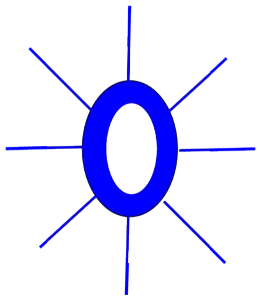
Loop-Level On-Shell Bootstrap

Berger, Bern, Dixon, Forde and Kosower

Shifted amplitude function of a complex parameter



$$A(z)$$

$$p_1^\mu(z) = p_1^\mu - \frac{z}{2}\langle 1^-|\gamma^\mu|2^-\rangle$$

$$p_2^\mu(z) = p_2^\mu + \frac{z}{2}\langle 1^-|\gamma^\mu|2^-\rangle$$

Shift maintains on-shellness and momentum conservation

$$A(z) = \sum \text{polylog terms}$$ ← **Use unitarity method** **(in special cases on-shell recursion)**

$$+\sum_i \frac{\text{Res}_i}{(z - z_i)}$$ ← **Use on-shell recursion**

$$+\sum_i a_i\, z^i$$ ← **Use auxiliary on-shell recursion in another variable**

See David Kosower's talk

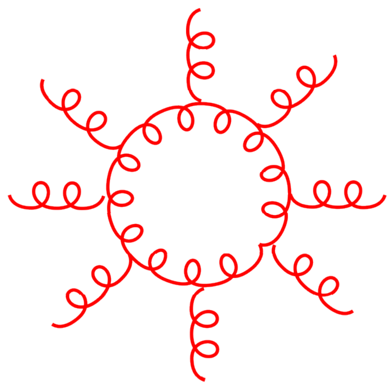
Numerical Results for *n* Gluons

Choose specific points in phase-space – see hep-ph/0604195

Scalar loop contributions

	Helicity	$1/\epsilon$	ϵ^0
6 points	$++++++$	0	$0.1024706290 + i\,0.5198025397$
	$-+++++$	0	$2.749806130 + i\,1.750985849$
	$--++++$	$-9.370119558 + i\,1.547789294$	$-45.80779561 + i\,13.03695870$
	$---+++$	$-0.2614534328 - i\,0.6288641470$	$0.3883482043 - i\,5.830791857$
7 points	$+++++++$	0	$0.1815778027 + i\,1.941357266$
	$-++++++$	0	$22.52927821 + i\,5.464377788$
	$--+++++$	$-34.85372799 + i\,15.11569825$	$-176.2169235 + i\,87.93931019$
	$---++++$	$0.3564513374 - i\,0.4914226070$	$0.7087164424 - i\,11.32916632$
8 points	$++++++++$	0	$-0.0009856214410 + i\,0.002143695508$
	$-+++++++$	0	$0.001078316199 + i\,0.03129931739$
	$--++++++$	$-0.05330088846 - i\,0.04051789981$	$0.05513350697 + i\,0.1659518861$
	$---+++++$	$-0.003622640270 - i\,0.0007910999246$	$0.02719752089 - i\,0.02586206549$
	$----++++$	$-0.002273559586 - i\,0.001209645382$	$0.01154855076 - i\,0.0008935357840$

Naive diagram count



+ 3,017,489
other diagrams

Modest growth in complexity as number of legs increases

At 6 points these agree with numerical results of Ellis, Giele and Zanderighi

$\mathcal{N}$ = 4 Super-Yang-Mills to All Loops

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In 1974 ‘t Hooft proposed that we can solve QCD in planar (‘t Hooft) limit.
This is too hard. $\mathcal{N}$ = 4 sYM is much more promising.

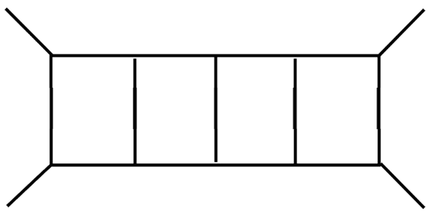
- Heuristically, we expect magical simplicity especially in planar limit with large ‘t Hooft coupling – dual to weakly coupled gravity in AdS space.

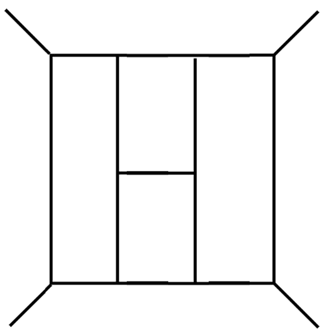
Can we solve (planar) $\mathcal{N}$ = 4 super-Yang-Mills theory?

Initial Goal: Resum amplitude to all loop orders.

What we need

1. Intuition and bold guesses.
2. Sufficiently powerful methods for confirming and guiding guesses.
   (a) Unitarity method. Bern, Dixon, Dunbar and Kosower
   (b) A loop integration package: MB. Czakon
3. Faith and optimism.



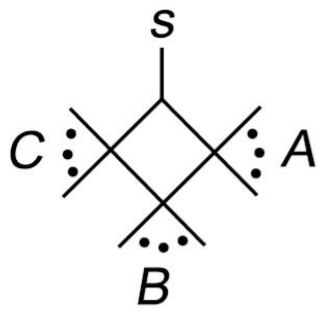


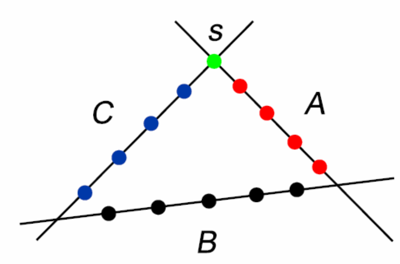
Example: Twistor Space Hint

At one-loop the coefficients of all integral functions have beautiful twistor space interpretations

Box integral | **Twistor space support**

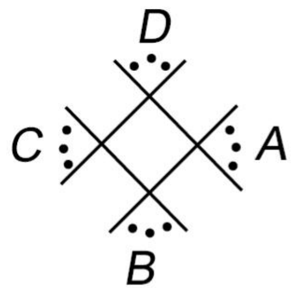
Three negative helicities

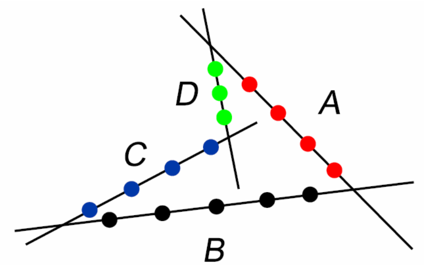




Bern, Dixon and Kosower
Britto, Cachazo and Feng

Four negative helicities





The existence of such twistor structures implies loop-level simplicity. Supports notion that we should be able to evaluate amplitudes to *all* loop orders.

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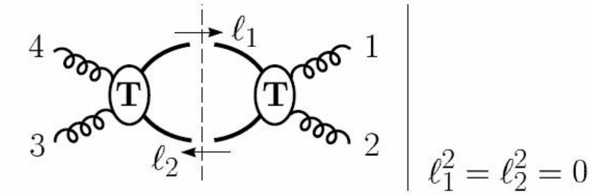
$\mathcal{N} = 4$ Multi-loop Amplitude

Bern, Rozowsky and Yan

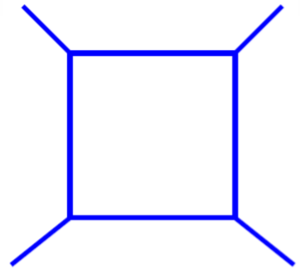
Consider one-loop in $\mathcal{N} = 4$:



The basic *D*-dimensional two-particle sewing equation

$$\sum_{\mathcal{N}=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{st A_4^{\text{tree}}(1,2,3,4)}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2}$$

Applying this at one-loop gives



$$\mathcal{A}_4^{\text{1-loop}}(1,2,3,4) = -st A_4^{\text{tree}} \mathcal{I}_4^{\text{1-loop}}(s,t)$$

Agrees with known result of Green, Schwarz and Brink.

The two-particle cuts algebra recycles to all loop orders!

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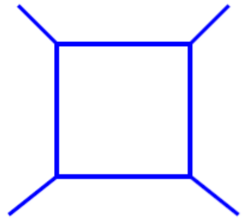
Loop Iteration of the Amplitude

Four-point one-loop $D = 4 - 2\epsilon$, $N = 4$ amplitude:

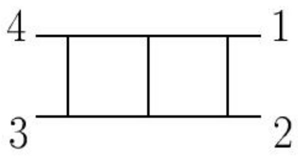
$$A_4^{\text{1-loop}}(s,t) = -st A_4^{\text{tree}} \mathcal{I}_{\text{1-loop}}(s,t)$$

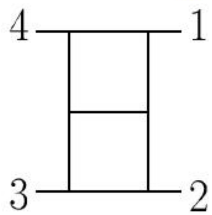
$$I^{\text{1-loop}}(s,t) \sim \frac{1}{st}\left[\frac{2}{\epsilon^2}\Big((-s)^{-\epsilon} + (-t)^{-\epsilon}\Big) - \ln^2\left(\frac{t}{s}\right) - \pi^2\right] + \mathcal{O}(\epsilon)$$



To check for iteration use evaluation of two-loop integrals.

$$A_4^{\text{2-loop}}(1^-,2^-,3^+,4^+) = -st\, A_4^{\text{tree}}(1^-,2^-,3^+,4^+)\,\Big(s\,\mathcal{I}_4^{\text{2-loop}}(s,t) + t\,\mathcal{I}_4^{\text{2-loop}}(t,s)\Big)$$

$-\,st\,A_4^{\text{tree}}\Big\{\, s$ 

$+\ t$ 

$\Big\}$

Planar contributions

Obtained via unitarity method

Bern, Rozowsky, Yan

Integrals known and involve 4th order polylogarithms.

V. Smirnov

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Loop Iteration of the Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification.

Anastasiou, Bern, Dixon, Kosower

$$M_4^{2\text{-loop}}(s,t) = \frac{1}{2}\Big(M_4^{1\text{-loop}}(s,t)\Big)^2 + f(\epsilon) M_4^{1\text{-loop}}(s,t)\Big|_{\epsilon\to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

where

$$M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}}\,, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

$f(\epsilon)$ is universal function related to IR singularities $\quad D = 4 - 2\epsilon$

Thus we have succeeded to express two-loop four–point planar amplitude as iteration of one-loop amplitude.

Recent confirmation directly on integrands. Cachazo, Spradlin and Volovich

Generalization to *n* Points

Anastasiou, Bern, Dixon, Kosower

Can we guess the *n*-point result? Expect simple structure.

Trick: use collinear behavior for guess

Bern, Dixon, Kosower

Have calculated two-loop splitting amplitudes.

Following ansatz satisfies all collinear constraints

$$M_n^{2\text{-loop}}(\epsilon) = \frac{1}{2}\left(M_n^{1\text{-loop}}(\epsilon)\right)^2 + f(\epsilon) M_n^{1\text{-loop}}(2\epsilon) - \frac{1}{2}\zeta_2^2$$

Valid for planar MHV amplitudes

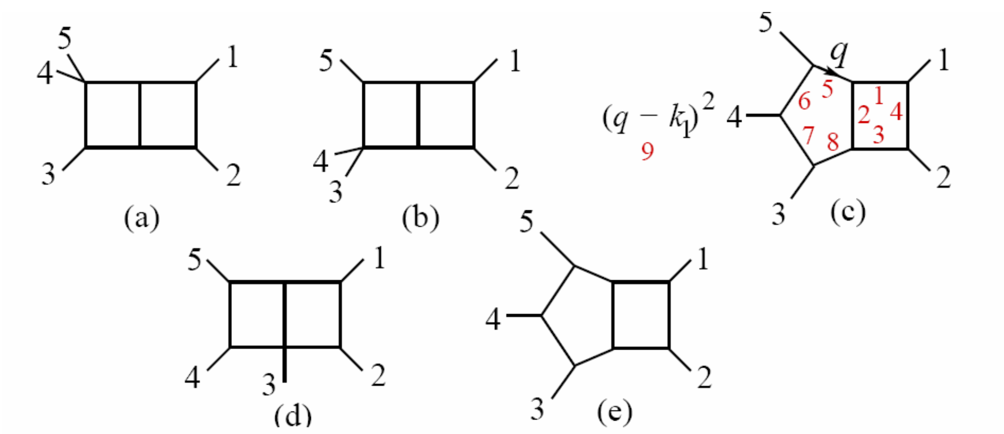
$$M_n^{\rm loop} = A_n^{\rm loop}/A_n^{\rm tree}\,, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

***D* = 4 – 2ϵ**

Has correct analytic properties

Five–point Consistency Check

As a non-trivial consistency check, worked out 5-point two-loop amplitudes. Cachazo, Spradlin and Volovich
Bern, Czakon, Kosower, Roiban, Smirnov

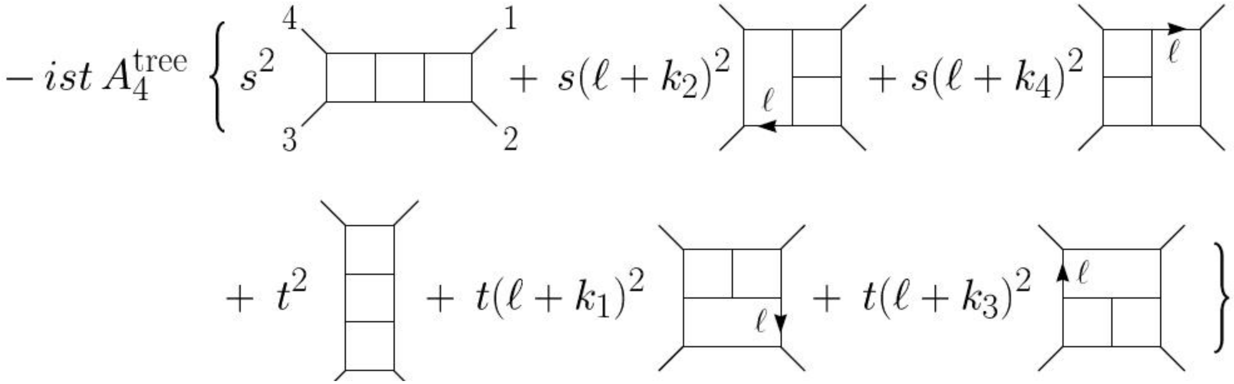


To deal with the loop integrals we used Czakon's wonderful MB integration package.

High precision numerical confirmation of iteration. Analytic proof would be better.

Three-loop Generalization

From unitarity method we get three-loop planar integrand:



Bern, Rozowsky, Yan

Use Mellin-Barnes integration technology and apply hundreds of harmonic polylog identities:

V. Smirnov
Vermaseren and Remiddi

$$M_4^{\text{3-loop}}(\epsilon) = -\frac{1}{3}\Big[M_4^{\text{1-loop}}(\epsilon)\Big]^3 + M_4^{\text{1-loop}}(\epsilon)\, M_4^{\text{2-loop}}(\epsilon) + f^{\text{3-loop}}(\epsilon)\, M_4^{\text{1-loop}}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

Bern, Dixon, Smirnov

where

$$f^{\text{3-loop}}(\epsilon) = \frac{11}{2}\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)\,,$$

and

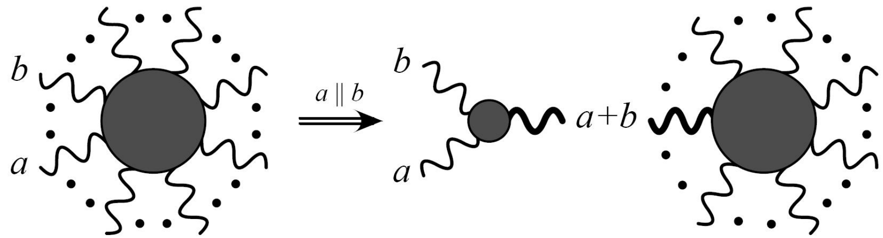
$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,.$$

Answer actually does not actually depend on c_1 and c_2. Five-point calculation would determine these.

Three-loop Generalization to *n* Points

Anastasiou, Bern, Dixon, Kosower

Repeat two-loop discussion, but at three loops.



Although we haven't calculated the three-loop splitting function, by now it is clear it too should iterate.

Same logic as at two loops immediately gives three-loop generalization:

$$M_n^{\text{3-loop}}(\epsilon) = -\frac{1}{3}\left[M_n^{\text{1-loop}}(\epsilon)\right]^3 + M_n^{\text{1-loop}}(\epsilon)\, M_n^{\text{2-loop}}(\epsilon) + f^{\text{3-loop}}(\epsilon)\, M_n^{\text{1-loop}}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

Valid for planar MHV amplitudes

All-Leg All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop all legs (at least for MHV amplitudes)?

- Remarkable formula from Magnea and Sterman tells us IR singularities to all loop orders. Guides construction.
- Collinear limits gives us the key analytic information, at least for MHV amplitudes.

$O(\epsilon)$

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\Big)\Big]$$

constant

$$a = \frac{N_c \alpha_s}{2\pi} \qquad f^{(l)} = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} + \cdots$$

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)}$$

- Soft anomalous dimension
- Or leading twist high spin anomalous dimension
- Or cusp anomalous dimension
- Or high moment limit of Altarelli-Parisi splitting kernel

Expression for Finite Remainder

After subtracting IR singularities finite remainder of the all loop order planar amplitude is:

All loop resummation of finite remainder

$$\mathcal{F}_n = \exp\left[\frac{1}{4}\gamma_K F_n^{(1)} + C\right]$$

An unnamed constant

Soft or cusp anomalous dimension

Same anomalous dimension guessed by Eden and Staudacher using integrability

One- loop finite remainder. Complicated function of kinematic variables

$$\gamma_K = \sum_{l=1}^{\infty} a^{(l)} \gamma_K^{(l)}$$

It seems likely that the simplicity uncovered here is connected to integrability.

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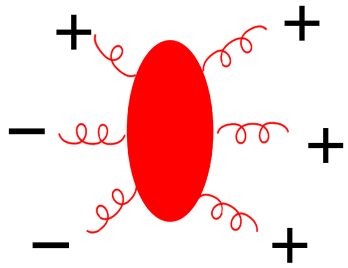
Finite Remainder

$$\mathcal{F}_n = \exp\left[\frac{1}{4}\gamma_K F_n^{(1)} + C\right] \qquad F_n^{(1)}(0) = \frac{1}{2}\sum_{i=1}^{n} g_{n,i}$$



$$g_{n,i} = -\sum_{r=2}^{\lfloor n/2\rfloor -1} \ln\left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}}\right)\ln\left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}}\right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2$$

$$D_{2m+1,i} = -\sum_{r=2}^{m-1} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right),$$

$$L_{2m+1,i} = -\frac{1}{2}\ln\left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}}\right)\ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}}\right)$$

$$t_i^{[r]} = (k_i + \cdots + k_{i+r-1})^2$$

- All loop resummation of a one-loop amplitude in planar limit.
- In QCD this type of function contributes to physical quantites such as jet rates.
- IR divergences cancel against similar divergences from real emission diagrams.

Link to Integrability

It is suspected that $N = 4$ super-Yang-Mills is integrable in the planar limit.

Minhan and Zarembo;
Beisert, Krisjansen, Staudacher
and many others

Recent proposal for soft/cusp anomalous dimension in $N = 4$ SYM to *all* perturbative orders, based on integrability.

Eden, Staudacher, hep-ph/0603157

$$
\begin{aligned}
f(g) = \; & 4\, g^2 \\
& - \frac{2}{3}\, \pi^2\, g^4 \\
& + \frac{11}{45}\, \pi^4\, g^6 \\
& - \left(\frac{73}{630}\, \pi^6 - 4\, \zeta(3)^2 \right) g^8 \\
& + \left(\frac{887}{14175}\, \pi^8 - \frac{4}{3}\, \pi^2\, \zeta(3)^2 - 40\, \zeta(3)\zeta(5) \right) g^{10} \\
& - \left(\frac{136883}{3742200}\, \pi^{10} - \frac{8}{15}\, \pi^4\, \zeta(3)^2 - \frac{40}{3}\, \pi^2\, \zeta(3)\zeta(5) \right. \\
& \qquad \left. - 210\, \zeta(3)\zeta(7) - 102\, \zeta(5)^2 \right) g^{12} \\
& + \ldots .
\end{aligned}
$$

$$\mathcal{F}_n = \exp\left[\frac{1}{4} \gamma_K F_n^{(1)} + C \right]$$

← Generating function for γ_K

Satisfies maximal transcendentality conjecture to be discussed by Lipatov

If we know soft anomalous dimension then we know finite remainder of *all*-loop MHV planar amplitudes, up to overall constant.

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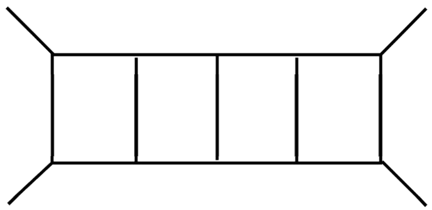
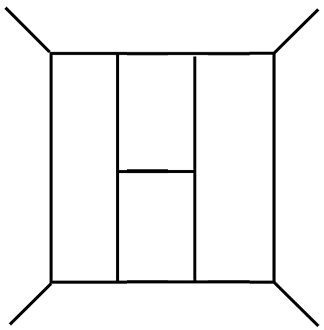
Is ES Conjecture Correct?

On path of checking our iteration formula at four loops we will extract the 4-loop anomalous dimension. γ_K appears in coefficient of $1/\epsilon^2$ IR singularity.





etc.

We are in the midst of computing this: stay tuned…

Connection of Gravity and Gauge Theory

At tree level Kawai, Lewellen and Tye have presented a relationship between closed and open string amplitudes.
In field theory limit relationship is between gravity and gauge theory

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Gravity amplitude

where we have stripped all coupling constants

Color stripped gauge theory amplitude

$$\mathcal{A}_4^{\rm tree} = g^2 \sum_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

Full gauge theory amplitude

Holds for any external states.
See review: gr-qc/0206071

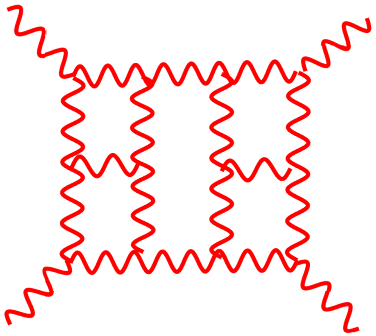
Progress in gauge theory can be imported into gravity theories

Divergences in Supergravity

Conventional wisdom states that it impossible to construct a finite quantum field theory of gravity

- Flaw with *all* previous studies of divergences. Rely on powercounting, taking into account only supersymmetry.
- We now have a much deeper understanding: hidden structures, dualities, twistors, connection to sYM via KLT.
- Perturbative $N = 8$ supergravity inherits its property from $N = 4$ sYM.
  Is it finite, contrary to prevailing wisdom?

Suppose we wanted to check this with Feynman diagrams:



First potential divergence is at 5 loops
This single diagram has $\sim 10^{30}$ terms prior to evaluating any integrals.
Impossible to evaluate via diagrams!

Gravity

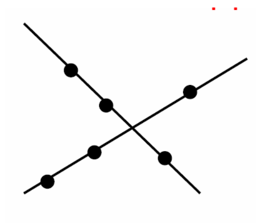
• We may use KLT relations in conjunction with the unitarity method to check the divergence structure of gravity theories.

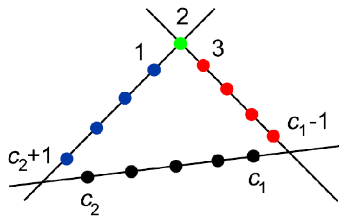
•Strategy already used to demonstrate that $N = 8$ sugra is less divergent than previously thought. First potential divergence will be at least 5 loops!

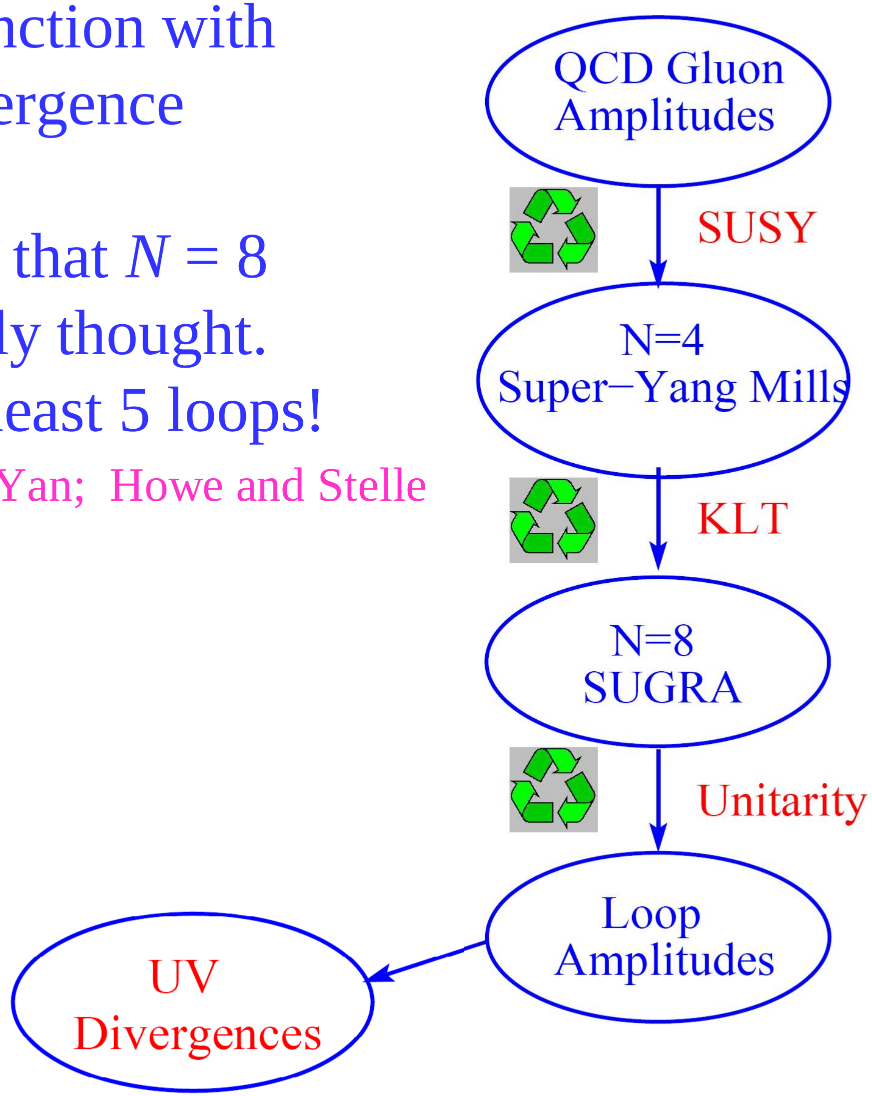
Bern, Dixon, Dunbar, Rozowsky, and Yan; Howe and Stelle

• Similar twistor structures exist in gravity as in gauge theory.

Witten; Bern, Bjerrum-Bohr, Dunbar







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Summary

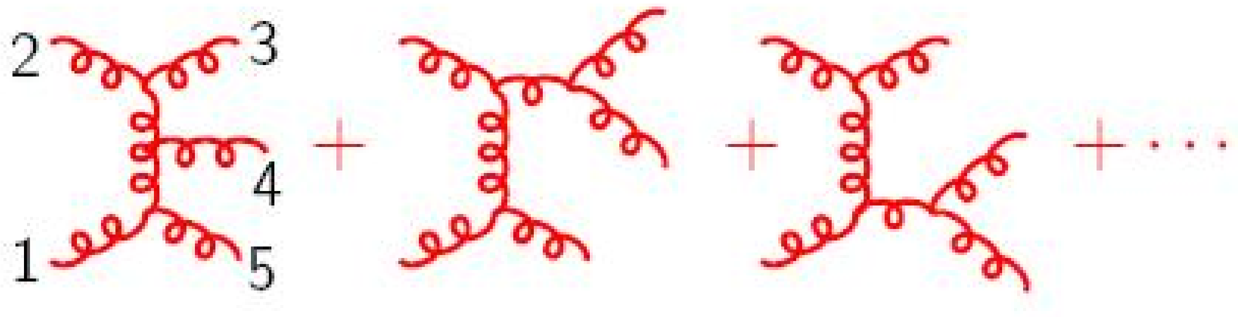
- Motivation for studying amplitudes:
  - (a) LHC demands QCD loop calculations.
  - (b) Can we solve (planar) $N = 4$ super-Yang-Mills?
  - (c) Is $N = 8$ supergravity finite, contrary to accepted wisdom? Demands explicit computations.
- Inspiration from twistor space: amazing simplicity.
- On-shell methods – unitarity and factorization.
- Explicit conjecture for resumming the MHVamplitudes of $N = 4$ super-Yang-Mills theory to *all* loop orders.
- Expect to have $N = 4$ four-loop soft anomalous dimension soon.

There are a variety of exciting avenues for further exploration of amplitudes in QCD, super-Yang-Mills theory and supergravity.

Extra Transparancies

Tree-level example: Five gluons

Consider the five-gluon amplitude



If you evaluate follow the textbooks you find…

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Result of evaluation (actually only a small part of it):

$$k_1 \cdot k_4\,\varepsilon_2 \cdot k_1\,\varepsilon_1 \cdot \varepsilon_3\,\varepsilon_4 \cdot \varepsilon_5$$

A Remarkable Twistor String Formula

The following formula encapsulates the entire tree-level S-matrix of *N* = 4 super-Yang-Mills:

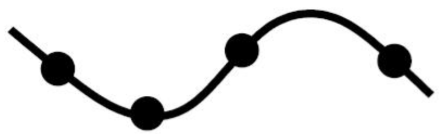
Witten
Roiban, Spradlin and Volovich

Integral over the Moduli and curves

$$A_n = i(2\pi)^4 g_{\mathrm{YM}}^{n-2} \sum_{d=1}^{n-3} \int d\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2\left(\lambda_i^\alpha - \xi_i P_i^\alpha\right) \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot\alpha}\right) \delta^4\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \eta_{iA}\right)$$

$$P_i^\alpha = \sum_{k=0}^{d} a_k^\alpha \sigma_i^k$$

Degree *d* polynomial in the moduli a_k

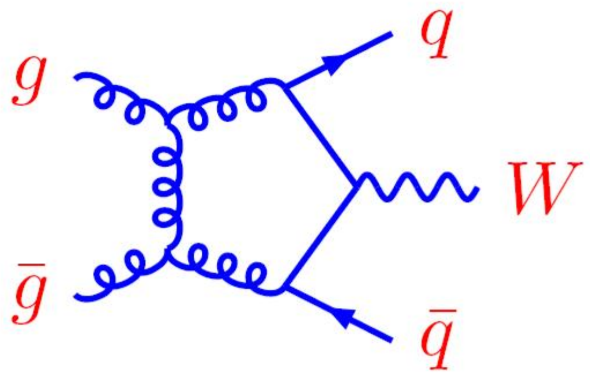


Strange formula from Feynman diagram viewpoint.

But it's true: impressive checks by Roiban, Spradlin and Volovich

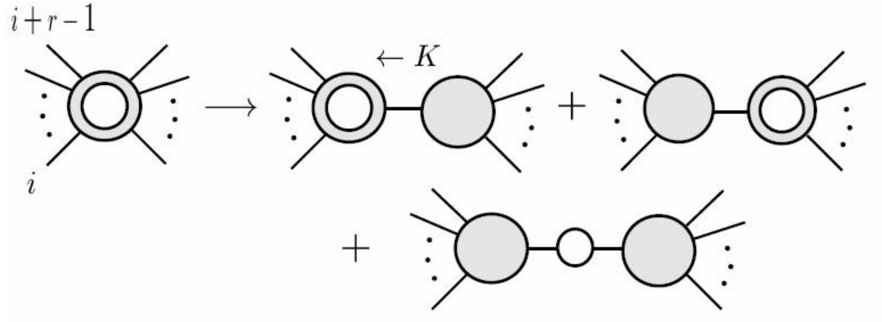
Early On-Shell Bootstrap

Bern, Dixon, Kosower (1997)



$$e^+e^- \to Z \to 4 \text{ partons}$$

$$pp \to W, Z + 2 \text{ partons}$$



Early Approach:

- Use Unitarity Method with $D = 4$ helicity states. Efficient means for obtaining logs and polylogs.
- Use factorization properties to find rational function contributions.

Key problems preventing widespread applications:

- Difficult to find rational functions with desired factorization properties.
- Systematization unclear – key problem.

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Other theories

Khoze, hep-th/0512194

Two classes of (large N_c) conformal gauge theories "inherit" the same large N_c perturbative amplitude properties from N=4 SYM:

1. Theories obtained by orbifold projection
– product groups, matter in particular bi-fundamental rep's

Bershadsky, Johansen, hep-th/9803249

2. The N=1 supersymmetric "beta-deformed" conformal theory
– same field content as N=4 SYM, but superpotential is modified:

$$ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \to ig\,\mathrm{Tr}(e^{i\pi\beta_R}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta_R}\Phi_1\Phi_3\Phi_2)$$

Leigh, Strassler, hep-th/9503121

Supergravity dual known for this case, deformation of $AdS_5 \times S^5$

Lunin, Maldacena, hep-th/0502086

Breakdown of inheritance at five loops (!?) for more general marginal perturbations of N=4 SYM? Khoze, hep-th/0512194

Beyond three loops

Recent proposal for soft/cusp anomalous dimension in N=4 SYM
to all perturbative orders (!), based on integrability. Eden, Staudacher, hep-ph/0603157

$$f(g) = 4\,g^2 - 16\,g^4 \int_0^\infty dt\,\hat{\sigma}(t)\,\frac{J_1(\sqrt{2}\,g\,t)}{\sqrt{2}\,g\,t}$$

where

$$\hat{\sigma}(t) = \frac{t}{e^{\,t}-1}\left[\frac{J_1(\sqrt{2}\,g\,t)}{\sqrt{2}\,g\,t} - 2\,g^2 \int_0^\infty dt'\ \hat{K}(\sqrt{2}\,g\,t, \sqrt{2}\,g\,t')\,\hat{\sigma}(t')\right]$$

is the solution to an integral equation with Bessel-function kernel

$$\hat{K}(t,\,t') = \frac{J_1(t)\,J_0(t') - J_0(t)\,J_1(t')}{t - t'}$$

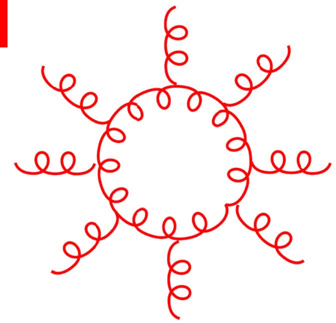
Perturbative expansion:

$$f(g) = 4\,g^2 - \frac{2}{3}\,\pi^2\,g^4 + \frac{11}{45}\,\pi^4\,g^6 - \left(\frac{73}{630}\,\pi^6 - 4\,\zeta(3)^2\right)\,g^8 + \ldots$$

?

Progress Towards the Dream

Results with on-shell methods:



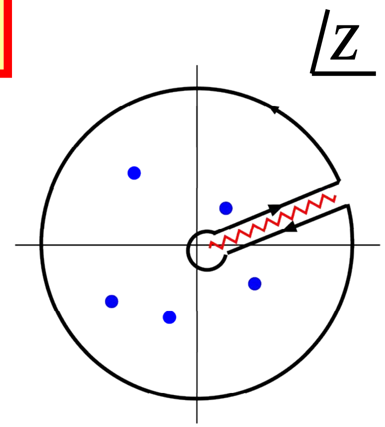
- Complete QCD amplitudes with $n > 5$ legs.

see David Darren Forde's talk

- Logarithmic contributions via on-shell recursion.
- Improved ways to obtain logarithmic contributions via unitarity method. All six-gluon helicities.

Key Feature: Modest growth in complexity as *n* increases.

Loop-Level Recursion



New Features:

- **Presence of branch cuts.**
- **Unreal poles – poles appear with complex momenta.**

$$\frac{[a\,b]}{\langle a\,b\rangle}$$ Pure phase for real momenta

- **Double poles.**

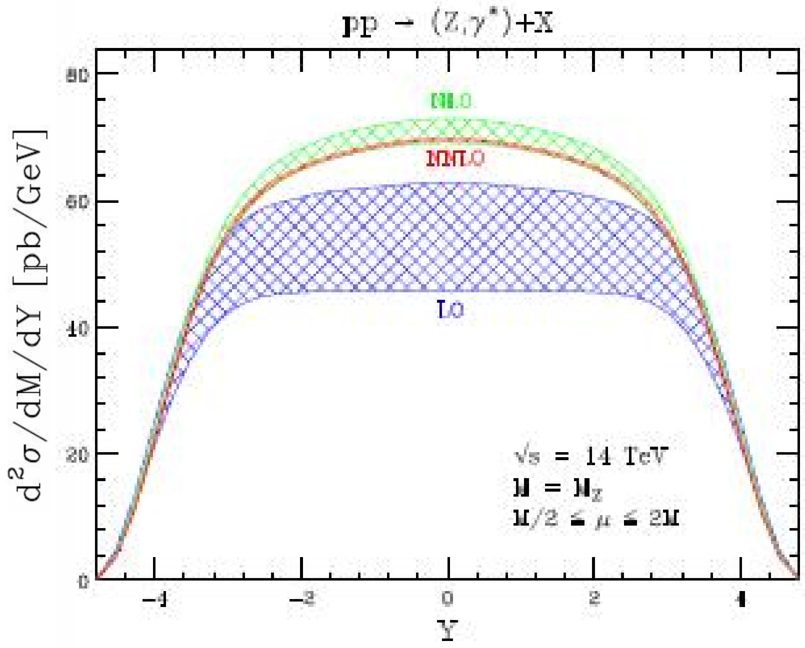
$$\frac{[a\,b]}{\langle a\,b\rangle^2}$$

- **Spurious singularities that cancel only against polylogs.**

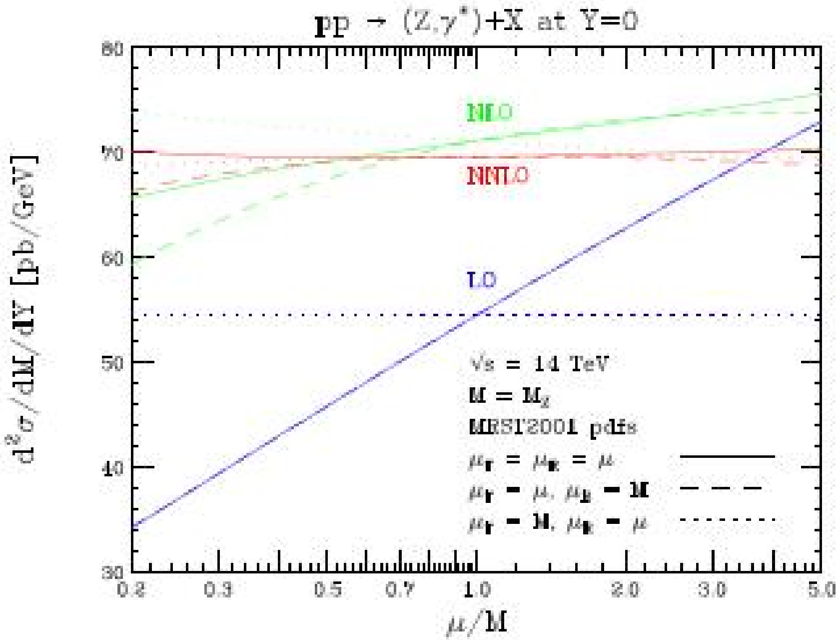
- **Double count between cuts and recursion.**

See Carola Berger's and Darren Forde's talks

The Gold Standard: NNLO Drell-Yan Rapidity Distributions



Anastasiou, Dixon, Melnikov and Petriello



- Amazingly good stabilty
- Theoretical uncertainties less than 1%

- One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane

J. Schwinger in “Particles, Sources and Fields” Vol 1