Different images of integrability in the gauge theories

Plan

1. Examples
   - $N=2$ SUSY YM
   - Anomalous dimensions
   - Regge limit

2. From classical to quantum integrability: Baxter operator

3. Compact versus noncompact spin chains

4. Conclusion/Open questions

Integrability ≡ hidden symmetry of evolution equations!

$$\frac{dx_i}{dt} = \{x_i, H\}$$

Regge limit: $\underline{t = \log s}$ — Lipatov, 93; Faddeev, Korchemsky 94

$N=2$ SYM: $\underline{t = \log \Lambda}$ — Krichever, Marshakov, Mironov, Morozov, A.G. 95; Donagi, Witten, 95

Anomalous dimensions: $t = \log \mu^2$

Holomorphic sector of QCD — Braun, Manashov, Derkachov 98; Belitsky 99

$N=4$ SYM — Minahan, Zarembo, 02; Beisert, Staudacher, 03.

Origin: $\hat{D} = H_{string}$

dilatation operator in gauge theory

String on $AdS_5 \times S_5$ is classically integrable

Polchinski, Bena, Roiban, 04

$\mathcal{N}=2$ SYM

Exact solution for low-energy effective action in holomorphic sector

Seiberg, Witten 94
Nekrasov, 04

- Techically: one log + infinite instanton series

Tools: holomorphy + S-duality + RG. flows

Low energy effective action $\Longleftrightarrow$ classical integrable system

- Softly broken $\mathcal{N}=4 \Rightarrow$ Calogero model

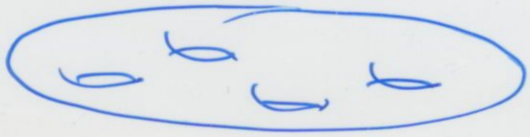
$$H=\sum_{i=1}^{N_c} p_i^2 + M^2 \sum_{i<j}^{N_c} \mathcal{P}(x_i - x_j|\tau)$$

- Pure $\mathcal{N}=2$ SYM $\Rightarrow$ Toda model

$$H=\sum^{N_c} p_i^2 + \Lambda^2 \sum^{N_c} \left(e^{x_{i+1}-x_i} + e^{x_i - x_{i-1}}\right)$$

- $\mathcal{N}=2$ + fundamental matter

$$H=\sum^{N_c} \vec{S}_i \vec{S}_{i+1}$$



Spectral curve $\left(\begin{array}{l} H_k = \text{const} \\ k = 1 \dots N_c \end{array}\right)$

provides solution

$\lambda + \frac{1}{\lambda} = P_{N_c}(x)$; $P_{N_c} = \mathrm{Tr}\, T(x)$

transfer matrix

$H_k = \langle \mathrm{Tr}\, \varphi^k \rangle$

- Action variables of integrable system : $a_i = \oint_{A_i} \log \lambda \, dx$

$a_D = \oint_{B_i} \log \lambda \, dx$

$a_{D_i} = \frac{\partial \mathcal{F}(h_k)}{\partial a_i}$

$\mathcal{F}(a_i)$ - effective action

- Finite-dimensional integrable system (Calogero, Toda, spin chain) should be embedded into two-dimensional integrable model $\in$ KP, 2d Toda

$\mathcal{F} = \log \tau^{KP}_{quasiclas}(t_1 \dots t_k)$; $\tau_{KP} = \langle e^{t_k \mathrm{Tr}\, \varphi^k} \rangle$

Solution to Whitham hyerarchy!

$\wr\wr$

RG flows

Marshakov, Mironov, Morozov, A.G. 98

Edelstein, Marino, Mas 98

$t_1 = \log \Lambda$

- $\frac{\partial \mathcal{F}}{\partial \log \Lambda} = \beta_0 \langle Tr\, \varphi^2 \rangle = H_2 \Longleftrightarrow \frac{\partial S}{\partial t_1} = H$

RG flows can be presented in the Hamiltonian form

- Link to "anomalous dimensions"

$$\frac{\partial^2 \mathcal{F}}{\partial \log \Lambda \, \partial t_k} = \frac{\partial}{\partial \log \Lambda} \langle Tr\, \varphi^k \rangle \approx "\gamma_k" \langle Tr\, \varphi^k \rangle \Rightarrow$$

$\Rightarrow$ combination of θ-functions on the spectral curve

- Riemann surface $\Sigma \Rightarrow$ part of the M5 brane worldvolume $(R^4 \times \Sigma)$ hence it is element of background for the string dual to $N=2$ SYM.

- Meaning of the quantization of the finite-dimensional integrable system

Open problem!

Anomalous dimensions

Quantum(!) integrable systems

$\hat{O} = (O_1 \cdots\cdots O_N)$ - operators which are mixed under RG flows

$$\frac{d\hat{O}}{d\log\mu} = \hat{\gamma}\hat{O} = \hat{H}\hat{O}$$

spin chain Hamiltonian

Classical IS $\longrightarrow$ Quantum IS

Classical spectral curve

$$\det\left(T(\lambda) - \eta\right) = 0 \quad ; \quad T(\lambda) = \operatorname{Tr} L_1(\lambda) \cdots\cdots L_N(\lambda)$$

L_i - local Lax operator at i-th site

$$L_i = \lambda \mathbb{1} + \sum_{\alpha=1-3} S_i^\alpha \sigma^\alpha$$

Expansion of $T(\lambda) \to$ integrals of motion

Most effective quantization scheme - - Baxter equation

Idea: $(x_i, p_i) \Longrightarrow (\mathbb{C}, \mathcal{L}) \Rightarrow \mathrm{Symm}^N[M]$

Equivalent representation of the phase space

- $S_i^0 = iz_ip_i + s$; $S_i^- = -ip_i$; $S_i^+ = iz_i^2p_i + 2sz_i$

$\Downarrow$

(z_i, p_i) – conjugated pair

- $(C, \mathcal{L})$ = spectral curve (moduli) + linear bundle on C (set of N points)

(angles are linearized on the Jacobian of C)

- $Symm^N[M]$ = N points on M (Poisson manifold)

Separated variables

Baxter-Sklyanin scheme:

$(x_i, p_i) \Rightarrow (\lambda_i, \eta_i)$; $P(\lambda, \eta) = 0$.

$\Downarrow$

$\{\lambda_i, \eta_j\} = \delta_{ij}\eta_i$

$$\det(T(\lambda) - \eta) = 0 \Rightarrow \det\left(T(\lambda) - e^{i\partial_\lambda}\right)Q(\lambda) = 0$$

$Q(\lambda)$ – eigenfunction of the Baxter operator $\hat{Q}(\lambda)$

Correct analogy:

First quantized language : $p^2 = m^2$

Second quantized language : $p^2 - m^2 = 0 \Rightarrow (-\partial^2 + m^2)\phi(x) = 0$

This corresponds to the representation of the phase space as $\mathrm{Symm}^N[M]$

$\Psi(x) \sim Q(\lambda_1)\, Q(\lambda_2) \ \ldots \ Q(\lambda_n)$

wave function of the "separated variable"

Example of the Baxter equation (XXX)

$$(\lambda+is)^N Q(\lambda+is) + (\lambda-is)^N Q(\lambda-is) = t(\lambda)\, Q(\lambda)$$

Such equation can be derived since Baxter operator obeys

$$[\hat{Q}_\lambda, \hat{t}(u)] = 0$$

Solution to the Baxter equation $\Rightarrow$ spectrum of the model

$$E \sim \frac{d}{du} \log \left. \frac{Q(u+is)}{Q(u-is)} \right|_{u=0}$$

What is the meaning of $Q(\lambda)$ in the classical limit?

$\hat{Q}(\lambda)$ classically can be identified with the generating function of Bäcklund transformations

$$(X, x) \xrightarrow[F_\lambda(x,y)]{} (Y, y)$$

$$X_i = \frac{\partial F_\lambda}{\partial x_i} \quad ; \quad Y_i = -\frac{\partial F_\lambda}{\partial y_i}$$

$$H(X, x) = H(Y, y)$$

B_λ – generating function of Bäcklund (generates solution from the trivial one)

Baxter operator can be defined via its kernel

$$\hat{Q} : f(x) \to \int d\vec{x}\, Q_\lambda(\vec{y}\,|\,\vec{x})\, f(x)$$

$$Q_\lambda(y\,|\,x) \sim e^{\frac{i}{\hbar} B_\lambda(y|x)}$$

- Link to BA equations

$$(\lambda+is)^N Q(\lambda+is) + (\lambda-is)^N Q(\lambda-is) = t(\lambda) Q(\lambda) \qquad (*)$$

Look for polynomial solutions

$$Q(\lambda) = \prod^{J} (\lambda - \lambda_i)$$

Take $\lambda = \lambda_k$ in $(*)$

$$(\lambda_k+is)^N Q(\lambda_k+is) + (\lambda_k-is)^N Q(\lambda_k-is) = 0 \Big]$$

BA equation

Take $\lambda = \lambda_k + is$

$$(\lambda_k+2is)^N Q(\lambda_k+2is) = t(\lambda_k+is) Q(\lambda_k+is) \Big]$$

This equation encodes the dependence of the solution on the higher integrals of motion !

BA equation is <u>half</u> of the whole data

For instance: BA $\Longrightarrow$ S-matrix

should be supplemented by the second eq

From the second equation:

$$E = 4\log 2 + 2\,\mathrm{Re}\sum^{N} [\psi(1-s-i\xi_k) + \psi(s-i\xi_k) - 2\psi(1) +$$

$$+ 2\,\mathrm{Re}\sum^{N} [\psi(1-s+i\xi_k) + \psi(s+i\lambda_k) - 2\psi(1)]$$

ξ_k - roots of $t(\lambda)$

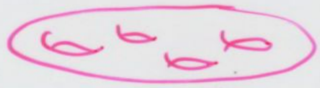
- $Q(\lambda)$ versus stringy picture

Different stringy motions ⇒ (Frolov, Tseytlin, Minahan, Beisert, Staudacher, Zarembo ⋮)
⇒ operators in SYM

Large quantum numbers ⇒ quasiclassics

Spectral curve of the spin chain ≈ Spectral curve for particular σ-model solution (Kazakov, Marshakov, Minahan, Zarembo)

$\Downarrow$ Quantization



Baxter equation

$\Downarrow$

Quantization of finite-gap solutions (generically requires the quantization of the whole σ-model) some developments Dorey, V..

- $Q(\lambda)$ - wave function of the single bit in the string bit model

- $Q(\lambda)$ - generating function for the Backlund transformation in the σ-model (Arutyunov, Staudacher, Zamaklar)

- Towards stringy: $\hat{Q}(\lambda)$

S-matrix (Beisert, Staudacher) — half of the story

σ-model Baxter equation required!

- Subtle points in noncompact sector

$O_{S,J} = \mathrm{Tr}\, \varphi D^{s_1} \varphi \dots D^{s_N} \varphi \qquad S = \sum S_k$

Lorentz spin

$\gamma_{O_{SJ}} \xrightarrow{S \gg \infty} k \log S \qquad 2 \le k \le N$

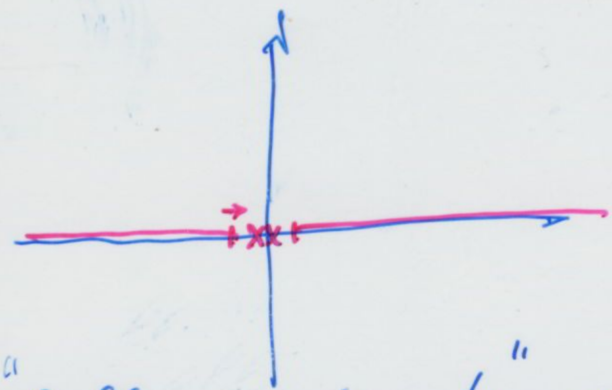
Band structure depends on higher integrals of motion

In some region WKB expansion of Baxter equation fails

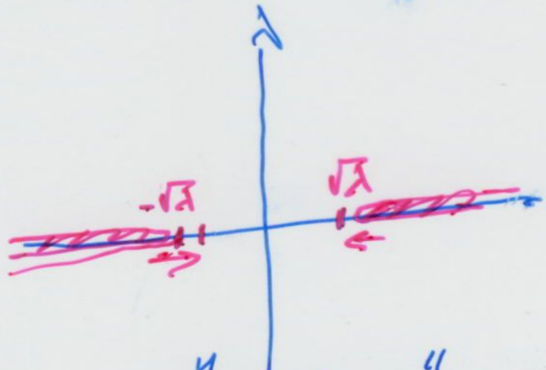
Hidden parameter in noncompact case

$\xi = \dfrac{\log S/L}{L} \qquad \xi \gg 1, \ \gamma_S \to \log S$ (Belitsky, Korchemsky, A.G., 06)

gauge-theory



string theory



"Collision of cuts" at $\lambda = 0$ with double point between them

$\lambda \log S$

"Collision" of cuts with poles

$\sqrt{\lambda} \log S$

(Gubser, Klebanov, Polyakov)

It turns out that the standard **WKB expansion** in this region **is broken**

Reason: Schrodinger equation with the singular potential

More careful analysis yields the desired $\log S$ behaviour.

Higher loops:

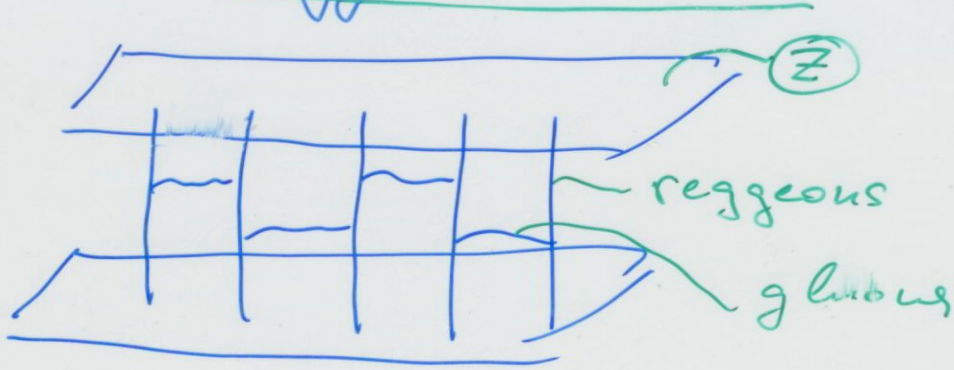
- recent 3-loop answer (Moch, Vermaseren, Kotikov, Lipatov, Onishchenko, Velizhanin)
- attempt to derived all-loop answer for $S \to \infty$ limit (Eden, Staudacher)
- two-loop Baxter equation (Belitsky, Korchemsky, Müller 06)

Nontrivial modification of Baxter equation emerges!

compact case: $\lambda \to f(\lambda, g)$

noncompact: $\lambda \to f(\lambda, g)$ + "dressing factor" $e^{Q_1(0)}$

Regge limit



$$\langle A | e^{-\alpha(t) \log s} | B \rangle \sim s^{-\alpha(t)}$$

evolution operator - $e^{-i\hat{H}t}$ $\qquad t = \log s$

$$H = H(z) + \bar{H}(\bar{z}) \qquad \text{(Lipatov)}$$

$$H(z) = H^{s=0}_{XXX} = \sum_{i}^{N} H_{k,k+1} \qquad N \sim N\text{-reggeon state}$$

$$H_{k,k+1} = H(\vec{S}_k \cdot \vec{S}_{k+1}) \; ; \; H(x) = \psi(j(x)) + \psi(1-j(x)) - 2\psi(1)$$

$$j(j+1) = 2x$$

SL(2,C) group (complex plane) **compared with**

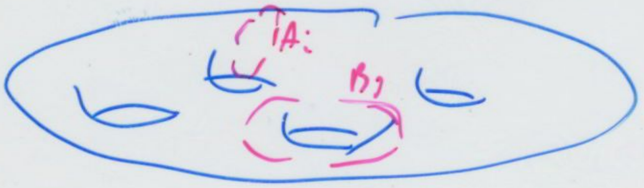
SL(2,R) for anomalous dimensions (light-like line)

- States in principal series of $SL(2,\mathbb{C})$

$$q = -h(h-1), \quad h = \frac{1+n}{2} + i\nu$$

- No Bethe Anzatz solution!
One has to solve Baxter equation

- Consider quasiclassics: large n or ν
Spectral curve: genus $= (N-2)$
the same as for $SL(2,\mathbb{R})$ case



no reality condition

Generic WKB:

$$\mathrm{Re}\int_{A_i} p\,dx = \hbar n_i \;;\; \mathrm{Re}\int_{B_i} p\,dx = \hbar m_i$$

$$E = E(\vec{n}, \vec{m})$$

- S-duality of quasiclassical spectrum (no S-duality in SL(2,R) case)

Derivation of the stringy picture

- Polchinski, Strassler, Brower, Tan or...

Strong coupling limit:
pomeron ($N=2$) = graviton + corrections.

$\alpha(t) = \alpha_0 + \alpha' t \qquad t \to \Box$

Analysis of the spectrum of $\Box$.

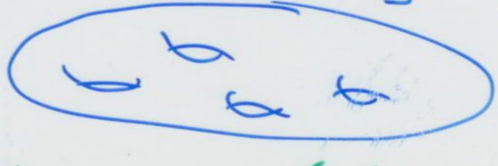
- Link via quasiclassics
(Kogan, Korchemsky, A.G. 02
Belitsky, Korchemsky, A.G. in progress)

Take ν-large
$\Downarrow$

XXX chain

$\xleftrightarrow{\text{similar to } SL(2,R)}$

Finite-gap Solution of σ-model in AdS_3



Stringy solutions $\Rightarrow$
$\Rightarrow \Theta$-functions +
complicated WKB quantization

Conclusions / Questions

- Impressive progress during the last years

?• Matching of integrability in perturbative and nonperturbative sectors

?• Hidden integrals of motion in the gauge theory (Yangians ....)

?• Derivation of Q-operator for σ-model

?• The is perturbative link between $\gamma_S \sim \log S$ and Regge limit.
Stringy interpretation?

?• New apperense of integrability.
MHV dyagrams ∞ number of nullified amplitudes ⇒ hidden quantum numbers

MHV → selfduality

$F = F^* \xrightarrow{\text{reduction}}$ different integrable systems

twistor $\mathbb{C}P_1 \ni$ spectral parameter