Poisson brackets in AdS/CFT

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Poisson brackets are very important in classical mechanics, in particular because they are the classical analogue of the quantum mechanical commutators.

Integrable systems usually have many Poisson brackets which satisfy some compatibility conditions. Actually, it is enough to have two "compatible" Poisson brackets, and then it is possible to generate an infinite family of them.

In my talk I will discuss this so-called "bihamiltonian structure" for the classical string on a sphere (the nonlinear sigma-model).

The talk is based on my paper:
A.M., hep-th/0511069
and some work in progress.

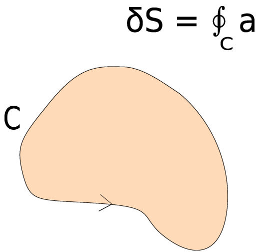
There was a substantial earlier work, for example:
A. Doliwa, P.M. Santini, Phys. Lett. **A185** (1994) 373-384
J.A. Sanders, J.P. Wang, math.AP/0301212
G. Marí Beffa, ...
S.C. Anco, nlin.SI/0512051,0512046
...
I. Bakas, Q-Han Park, H.-J. Shin, hep-th/9512030
...
I will use some ideas from these papers in my talk.
(Not to mention the old classical papers of Pohlmeyer, Eichenherr, Rehren, Neveu, Papanicolaou.)

I will first review the general definition of the symplectic structure, and then describe the canonical Poisson brackets for the nonlinear sigma-model (NLSM).
I will then discuss the "hidden" relativistic symmetry of the NLSM equations and how it acts on the canonical symplectic structure. For this I will need to introduce the "generalized sine-Gordon model". The relativistic symmetry leads to the existence of the non-standard symplectic structures. I will discuss the relativistically invariant non-standard symplectic structure and its geometrical meaning from the point of view of the string worldsheet.

Suppose that we have a classical field theory with the action

$$S = \int d\tau^+ d\tau^- \mathcal{L}[\phi]$$

We usually compute the action over infinite space-time, but let us suppose that we decided to compute the action of a given classical solution ϕ_{cl} in a finite region of τ^+, τ^-.



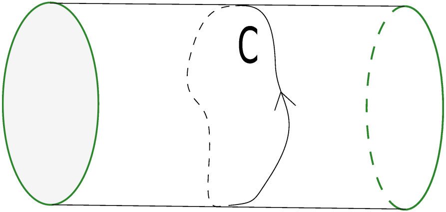
How does the result of this computation depend on the classical solution ϕ_{cl}? Suppose that we change the classical solution by a small amount $\delta\phi_{cl}$. We will get: $\delta S = \int_C a$ where a is some 1-form on the worldsheet. Since a is linear in $\delta\phi_{cl}$, we can also say that a is a 1-form on the phase space (the space of classical solutions). We will say that this is a form of the type $(d\tau)(\delta\phi)$.

Let us consider $\omega = \delta a$. This is a form of the type $d\tau(\delta\phi)^2$. We will assume that ω is defined unambiguously. In principle we could add to a some a' which is closed as a 1-form on the worldsheet. But for a large class of theories if a' is $d\tau\delta\phi$-type and d-closed then it is δ of some d-closed for of the type $(d\tau)$ (a density of the local conserved charge).

Assuming that there are no d-closed forms a' of the type $(d\tau)(\delta\phi)$, other than δ of something, we have $\omega = \delta a$ an unambiguously defined form of the type $(d\tau)(\delta\phi)^2$. Notice that ω is d-closed.

Now consider the theory either on a cylinder (periodic boundary conditions on $\tau^+ - \tau^-$) or some other appropriate boundary condition. The symplectic form is by definition:

$$\Omega = \oint_C \omega$$



This is a closed 2-form on the phase space.

It is also sometimes useful to consider the "symplectic potential" which is defined as:

$$\alpha = \oint_C a$$

such that

$$\delta\alpha = \Omega$$

(But we have to remember that α depends on the choice of the contour C.)

Principal chiral model (PCM) and nonlinear σ-model (NLSM)

These models are both defined by the action of the type:

$$S_{str} = \int d\tau^+ d\tau^- \mathcal{U}(\partial_+ g g^{-1}, \partial_- g g^{-1})$$

where g is a group element belonging to some group G, and $\mathcal{U}$ is some potential. For the PCM we have

$$\mathcal{U}(\partial_+ g g^{-1}, \partial_- g g^{-1}) = -\mathrm{tr}(\partial_+ g g^{-1} \partial_- g g^{-1})$$

For the NLSM we take $\mathbf{g} = Lie(G)$, $\mathbf{g} = \mathbf{g}_{\bar{0}} \oplus \mathbf{g}_{\bar{1}}$,

$$\mathcal{U}(\partial_+ g g^{-1}, \partial_- g g^{-1}) = -\mathrm{tr}\left((\partial_+ g g^{-1})_{\bar{1}} (\partial_- g g^{-1})_{\bar{1}}\right)$$

We will consider $\mathbf{g} = so(N+1)$ and $\mathbf{g}_{\bar{0}} = so(N)$. In this case we have:

$$\mathbf{g}_{\bar{0}} : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

and

$$\mathbf{g}_{\bar{1}} : \begin{bmatrix} 0 & * & * & * & * & * \\ * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This nonlinear sigma-model describes the target space $SO(N+1)/SO(N) = S^N$.

We introduce the notation

$$J_{\pm} = -\partial_{\pm} g g^{-1}$$

So, the action is $\int d\tau^+ d\tau^- \mathcal{U}(J_+, J_-)$.

For the NLSM we will use the notations $J_{\bar{0}}$ and $J_{\bar{1}}$:

$$J = J_{\bar{0}} + J_{\bar{1}}, \quad J_{\bar{0}} \in \mathbf{g}_{\bar{0}}, \quad J_{\bar{1}} \in \mathbf{g}_{\bar{1}}$$

Let us consider the infinitesimal left shift of g:

$$\delta_{\xi} g(\tau^+, \tau^-) = -\xi(\tau^+, \tau^-) g(\tau^+, \tau^-)$$

In terms of J:

$$\delta_{\xi} J = D\xi = d\xi + [J, \xi]$$

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Virasoro constraint

In string theory we use the NLSM with the additional constraint:

$$\mathrm{tr}(J_{\bar{1}+})^2 = \mathrm{tr}(J_{\bar{1}-})^2 = -1 \quad (1)$$

This is called Virasoro constraint

Symplectic structure of PCM

For PCM the symplectic potential is:

$$\alpha_{PCM}(\delta_\xi) = \oint \mathrm{tr}\, \xi \; * J$$

where $*J = *(J_+ d\tau^+ + J_- d\tau^-) = J_+ d\tau^+ - J_- d\tau^-$.
Calculation of $\delta\alpha$ gives:

$$\begin{aligned}\Omega(\delta_\xi, \delta_\eta) \; &= \; \int d\tau^+ \mathrm{tr}\left(2\xi \frac{\partial}{\partial \tau^+}\eta + \xi[J_+, \eta]\right) - \\ &- \; \int d\tau^- \mathrm{tr}\left(2\xi \frac{\partial}{\partial \tau^-}\eta + \xi[J_-, \eta]\right)\end{aligned}$$

Hint: use the formula $\delta\alpha(v_1, v_2) = v_1.\alpha(v_2) - v_2\alpha(v_1) - \alpha([v_1, v_2])$ and the fact that $[\delta_\xi, \delta_\eta] = \delta_{[\xi,\eta]}$.

Symplectic structure of NLSM

For the NLSM the symplectic potential is:

$$\alpha_{NLSM}(\delta_\xi) = \oint \mathrm{tr}\ \xi_{\bar{1}}\ * J_{\bar{1}} \quad (2)$$

The symplectic form Ω evaluated on the vectors δ_ξ and δ_η defined by $\delta_\xi J = d\xi + [J, \xi]$ and $\delta_\eta J = d\eta + [J, \eta]$ is:

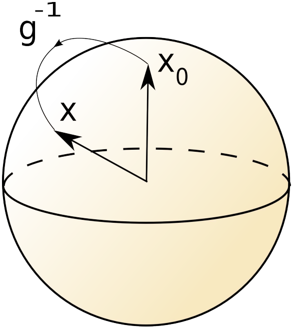
$$\Omega(\delta_\xi, \delta_\eta) = \int d\tau^+ \mathrm{tr}(\xi_{\bar{1}} D_{\bar{0}+} \eta_{\bar{1}}) - \int d\tau^- \mathrm{tr}(\xi_{\bar{1}} D_{\bar{0}-} \eta_{\bar{1}}) \quad (3)$$

where

$$D_{\bar{0}} = d + \mathrm{ad}_{J_{\bar{0}}}$$

When $\xi \in \mathbf{g}_{\bar{0}}$, δ_ξ is a gauge symmetry; it is in the kernel of Ω. (Because Ω does not contain $\xi_{\bar{0}}$.)

In our definition of the NLSM we used a group-valued field $g \in SO(N+1)$. The current J was defined as $J = -dgg^{-1}$.
In fact NLSM describes the minimal embeddings of the worldsheet to S^N, and $g(\tau^+, \tau^-)$ has a simple geometrical meaning:



It is the orthogonal matrix which rotates some fixed $\mathbf{x}_0 \in S^N$ to $\mathbf{x}(\tau^+, \tau^-)$. We have $\mathbf{x} = g^{-1}\mathbf{x}_0$. Notice that g is defined up to $g \simeq g_0 g$ where $g_0 \in SO(N)$. This corresponds to the gauge transformation of J, $\delta_\xi J = d\xi + [J, \xi]$ for $\xi \in \mathbf{g}_{\bar{0}}$.

And the right shift $g \mapsto gC$, $\quad C \in SO(N+1)$, $\quad C = \text{const}$ corresponds to the global rotations of S^N; notice that these global rotations *do not change* J.

NLSM symplectic form in plain English.

The geometrical "translation" of Eq. (3) is:

$$\Omega = \int d\tau^+(\delta\mathbf{x}, D_{\bar{0}+}\delta\mathbf{x}) - \int d\tau^-(\delta\mathbf{x}, D_{\bar{0}-}\delta\mathbf{x}) \tag{4}$$

Here we should understand $D_{\bar{0}}$ as the standard (Levi-Civita) connection in the tangent space to S^N.

This is the canonical symplectic form following from the action $\int d\tau^+ d\tau^-(\partial_+\mathbf{x}, \partial_-\mathbf{x})$.

Generalized sine-Gordon

Consider the space of solutions of the differential equations

$$\partial_+ J_{\bar{1}-} + [J_{\bar{0}+}, J_{\bar{1}-}] = 0 \quad (5)$$

$$\partial_- J_{\bar{1}+} + [J_{\bar{0}-}, J_{\bar{1}+}] = 0 \quad (6)$$

$$\partial_+ J_{\bar{0}-} - \partial_- J_{\bar{0}+} + [J_{\bar{0}+}, J_{\bar{0}-}] + [J_{\bar{1}+}, J_{\bar{1}-}] = 0 \quad (7)$$

with the gauge symmetry

$$\delta J = d\xi_0 + [J, \xi_0], \quad \xi_0 \in \mathbf{g}_{\bar{0}} \quad (8)$$

and the constraint

$$\mathrm{tr}(J_{\bar{1}+})^2 = \mathrm{tr}(J_{\bar{1}-})^2 = -1 \quad (9)$$

Definition. The system of equations (5),(6) and (7) with the gauge symmetry (8) and the constraint (9) is called the generalized sine-Gordon (GSG).

In some sense, the generalized sine-Gordon is equivalent to the NLSM. One only has to add g satisfying $(d+J)g=0$ But this g is almost defined in terms of J, the only ambiguity comes from the integration constants. (Which correspond to $g\mapsto gC$, $C=\text{const}$, it i.e. the global rotations of S^N.)

In terms of J the symplectic structure Eq. (3) is nonlocal:

$$\Omega=\int d\tau^{+}\text{tr}\left((D_{+}^{-1}\delta J_{+})_{\bar{1}}D_{\bar{0}+}(D_{+}^{-1}\delta J_{+})_{\bar{1}}\right)-(+\leftrightarrow -)$$

But if we add g satisfying $(d+J)g=0$ we get the local formula because

$$D_{+}^{-1}\delta J_{+}=\delta g g^{-1}$$

$$\Omega=\int d\tau^{+}\text{tr}\left((\delta g g^{-1})_{\bar{1}}D_{\bar{0}+}(\delta g g^{-1})_{\bar{1}}\right)-(+\leftrightarrow -)$$

In fact adding g is not the only way of getting the system with the local symplectic structure out of the GSG. For example, we can add the $so(N+1)$-valued field Ψ satisfying

$$D\Psi = *J_{\bar{1}}$$

and get the symplectic structure:

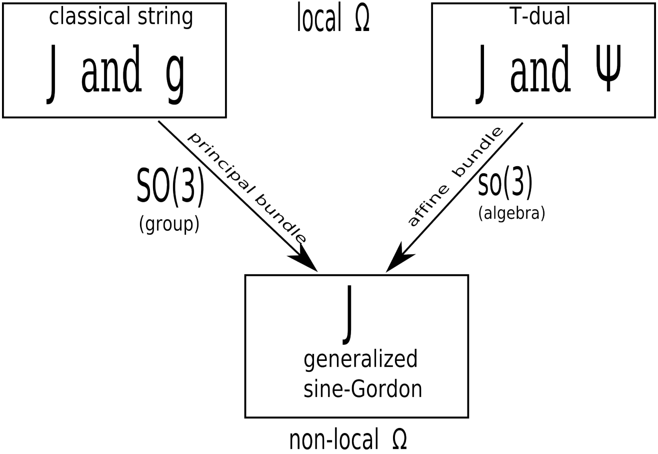
$$\Omega = \oint \delta\Psi\delta J$$

This corresponds to the action

$$S = \int \mathrm{tr}\left(\Psi(dJ + J^2) + J_{\bar{1}} \wedge *J_{\bar{1}}\right)$$

The equation of motion for Ψ implies the existence of g such that $J = -dgg^{-1}$ and the action on-shell is equal to the standard action $\int d\tau^+ d\tau^- \mathrm{tr}\ \left((\partial_+ gg^{-1})_{\bar{1}}(\partial_- gg^{-1})_{\bar{1}}\right)$ and therefore gives the same symplectic structure.
This could be thought of as a "T-dual" of the classical string.

In any case, we have a classical string (or its T-dual) and go to sine-Gordon by forgetting g: $(J, g) \mapsto J$ (or forgetting Ψ in the case of the T-dual). Classical string had a local Poisson bracket, and the corresponding Poisson bracket of GSG becomes nonlocal because we forget some degrees of freedom:



Relativistic symmetry of the GSG

Equations of motion:

$$\partial_+ J_{\bar{1}-} + [J_{\bar{0}+}, J_{\bar{1}-}] = \partial_- J_{\bar{1}+} + [J_{\bar{0}-}, J_{\bar{1}+}] = 0$$
$$\partial_+ J_{\bar{0}-} - \partial_- J_{\bar{0}+} + [J_{\bar{0}+}, J_{\bar{0}-}] + [J_{\bar{1}+}, J_{\bar{1}-}] = 0$$

with the gauge symmetry $\delta J = d\xi_0 + [J, \xi_0], \quad \xi_0 \in \mathbf{g}_{\bar{0}}$
and the Virasoro constraint $\mathrm{tr}(J_{\bar{1}+})^2 = \mathrm{tr}(J_{\bar{1}+})^2 = -1$.

There is an obvious symmetry under the constant shifts of τ^+ and τ^-. But besides shifts, the GSG equations are also symmetric under *boosts*:

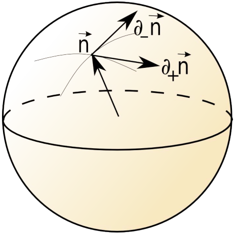
$$J_{\bar{0}\pm}(\tau^+, \tau^-) \mapsto \lambda^{\pm 1} J_{\bar{0}\pm}(\lambda\tau^+, \lambda^{-1}\tau^-)$$
$$J_{\bar{1}\pm}(\tau^+, \tau^-) \mapsto J_{\bar{1}\pm}(\lambda\tau^+, \lambda^{-1}\tau^-)$$

We we will use this relativistic symmetry to show that the classical string has an interesting non-standard symplectic structure.

First of all, we need to understand better the standard symplectic form in terms of the currents J.

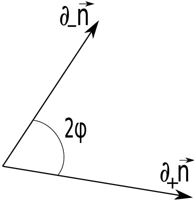
We start by considering the classical string on S^2, which corresponds to the usual sine-Gordon.

Let us first concentrate on the special case of the S^2 sigma-model, *i.e.* $N = 2$, $\mathbf{g} = so(3)$, $\mathbf{g}_{\bar{0}} = so(2)$.



Let $\vec{n}(\tau^+, \tau^-)$ be the S^2-part of the string worldsheet.

The Virasoro constraints: $|\partial_+ \vec{n}| = |\partial_- \vec{n}| = 1$.



The angle 2φ between $\partial_+ \vec{n}$ and $\partial_- \vec{n}$ satisfies the sine-Gordon equation

$$\partial_+ \partial_- \varphi = -\frac{1}{2} \sin 2\varphi$$

The function $\varphi(\tau^+, \tau^-)$ determines the *shape* of the string worldsheet.

The sine-Gordon equation is the equation of motion in the relativistic two-dimensional theory with the action

$$S_{SG} = \int d\tau^+ d\tau^- \left(\partial_+\varphi\partial_-\varphi + \frac{1}{2}\cos 2\varphi \right) \qquad (10)$$

This action gives an exactly solvable relativistic quantum field theory.

But the action (10) does not correspond to the action of the classical string. Therefore the Poisson bracket of the classical string is different from the Poisson bracket of the sine-Gordon theory. In fact, the sine-Gordon theory has an infinite family of symplectic structures on its phase space, and the string symplectic structure corresponds to one of them.

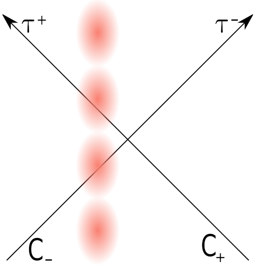
Which one? We will first give a pedestrian derivation of Ω_{str} and then give a more scientific derivation.

Ω_{str} in SG (a pedestrian approach)

The action of the classical string gives the following symplectic structure:

$$\Omega_{str} = \oint \left[d\tau^+ (\delta\vec{n}, \partial_+ \delta\vec{n}) - (\delta\vec{n}, \partial_- \delta\vec{n}) \right]$$

We will concentrate on the left lightcone component:



$$\Omega_{str} = \int_{C_+} d\tau^+ (\delta\vec{n}, \partial_+ \delta\vec{n})$$

Let us choose the following symplectic potential:

$$\alpha = \int d\tau^+ (\delta\vec{n}, \partial_+\vec{n})$$
$$\Omega = \delta\alpha$$

Consider the $O(3)$ invariant vector field on the phase space defined by the functions f_+ and f_-:

$$(\delta_{f_+,f_-}\vec{n}, \partial_+\vec{n}) = f_+, \quad (\delta_{f_+,f_-}\vec{n}, \partial_-\vec{n}) = f_-$$

We have a very simple formula for α:

$$\alpha(\delta_{f_+,f_-}) = \int d\tau^+ f_+$$

Now I want to compute $\Omega = \delta\alpha(\delta_f, \delta_g)$. I will compute $\delta\alpha$ using the general formula:

$$\delta\alpha(v, w) = v.\alpha(w) - w.\alpha(v) - \alpha([v, w])$$

$$\delta_f \mathbf{n} = \frac{4}{\sin^2 2\varphi}\left[(f_+ - f_- \cos 2\varphi)\partial_+ \mathbf{n} + (f_- - f_+ \cos 2\varphi)\partial_- \mathbf{n}\right]$$

Equations of motion for $\vec{n}$ leads to the equations for $\delta\vec{n}$ which in turn leads to the equations for f_+ and f_-:

$$f_- = -\frac{\sin 2\varphi}{2q_+}\partial_+ f_+ + f_+ \cos 2\varphi$$

$$f_+ = -\frac{\sin 2\varphi}{2q_-}\partial_- f_- + f_- \cos 2\varphi$$

We can see that f_- is expressed in terms of f_+ and $\partial_+ f_+$. This allows to compute the action of δ_{f_+,f_-} on $(\partial_+\vec{n}, \partial_-\vec{n})$ in terms of f_+. It is convenient to introduce q_+:

$$q_+ = \partial_+ \varphi$$

Direct computation shows:

$$\delta_f q_+ = \left((1 + \partial_+^2) q_+^{-1} \partial_+ + 4\partial_+ q_+\right) f_+$$

Now the commutator $[\delta_f, \delta_g]$ can be found by an explicit calculation:

$$[\delta_f, \delta_g] = \delta_{[f,g]}$$

where

$$[f,g] = -\langle q_+^{-1}\partial_+ f\rangle \overleftrightarrow{\partial_+} \langle q_+^{-1}\partial_+ g\rangle + 4f \overleftrightarrow{\partial_+} g$$

Therefore the symplectic structure $\Omega(\delta_f, \delta_g)$ is:

$$\Omega(\delta_f, \delta_g) = \int d\tau^+ \left(-\langle q_+^{-1}\partial_+ f\rangle \overleftrightarrow{\partial_+} \langle q_+^{-1}\partial_+ g\rangle + 4f \overleftrightarrow{\partial_+} g \right)$$

In terms of q_+:

$$\Omega = \int d\tau^+ \delta q_+ (\theta_0 + \theta_1)^{-1} \theta_1 (\theta_0 + \theta_1)^{-1} \delta q_+ \qquad (11)$$

where θ_0 and θ_1 are some interesting integro-differential operators:

$$\theta_0 = \partial_+$$
$$\theta_1 = \partial_+^3 + 4\partial_+ q_+ \partial_+^{-1} q_+ \partial_+$$

Given some operator θ we can try to define a Poisson bracket by the formula

$$\{F, G\} = \int d\tau^+ \frac{\delta F}{\delta q_+} \theta \frac{\delta G}{\delta q_+}$$

But if we want to call it a Poisson bracket we should verify that it satisfies the Jacobi identity $\{\{F, G\}, H\} + \text{cycl} = 0$. This leads to some differential equations on the coefficients of θ which are bilinear in θ. We will schematically write these equations as follows:

$$[\![\theta, \theta]\!] = 0 \qquad \text{(the Jacobi equation)}$$

The Jacobi equation is equivalent to the statement that $\Omega = \theta^{-1}$ defines a closed 2-form:

$$\Omega = \int d\tau^+ \delta q_+ \theta^{-1} \delta q_+$$

For example, for θ_0 we have $\Omega_0 = \int d\tau^+ \delta\varphi \partial_+ \delta\varphi$.

It is rather obvious that $\theta_0 = \partial_+$ satisfies $[\![\theta_0, \theta_0]\!] = 0$; in fact, θ_0 defines the standard Poisson bracket of the sine-Gordon model. But we also have:

$$[\![\theta_1 + 2\theta_0 + \theta_0\theta_1^{-1}\theta_0 \ , \ \theta_1 + 2\theta_0 + \theta_0\theta_1^{-1}\theta_0]\!] = 0 \qquad (12)$$

Indeed, we have just shown that $(\theta_0 + \theta_1)^{-1}\theta_1(\theta_0 + \theta_1)^{-1}$ is the canonical symplectic structure of the classical string. And (12) is the corresponding Jacobi identity. (Which follows automatically from the fact that the symplectic form of the classical string is a closed form.)

Consider the boost $\varphi(\tau^+) \mapsto \varphi(\lambda\tau^+)$. We get $\theta_0 \to \lambda\theta_0$ and $\theta_1 \to \lambda^3\theta_1$. The bracket $[\![,]\!]$ is homogeneous under the rescaling, and therefore we should have

$$[\![\theta_1, \theta_1]\!] = 0$$
$$[\![\theta_1, \theta_0]\!] = 0$$

This means that the phase space of the sine-Gordon theory has a family of the Poisson brackets of the form

$$\theta_0 + t\theta_1$$

This satisfies the Jacobi identity

$$[\![\theta_0 + t\theta_1, \theta_0 + t\theta_1]\!] = 0$$

for an arbitrary t. This is known as "bihamiltonian structure".

The Poisson structure of the classical string in terms of θ_0 and θ_1 becomes:

$$\theta_{str} = (\theta_0 + \theta_1)\theta_1^{-1}(\theta_0 + \theta_1)$$

We will now give a slightly more "scientific" derivation of this formula for θ_{str} which will be generalizable from S^2 to S^N.

Scientific approach to θ_{str}

An alternative derivation uses Eq. (2):

$$\alpha(\delta_\xi) = -\int d\tau^+ \mathrm{tr}\,(\xi J_{\bar{1}+})$$

Remember that here δ_ξ is defined by $\delta_\xi J_+ = D_+\xi$.

Let us choose the gauge where $J_{\bar{1}+} = const$. In this gauge, when we compute $\delta\alpha$, we do not have to evaluate $\delta J_{\bar{1}+}$ because $J_{\bar{1}+}$ is a constant matrix. We get:

$$\Omega(\delta_{\xi_1}, \delta_{\xi_2}) = \int d\tau^+ \mathrm{tr}(J_{\bar{1}+}[\xi_1, \xi_2]) \qquad (13)$$

So, let us fix the gauge so that

$$J_{\bar{1}+} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since we have choosen the gauge where $J_{\bar{1}+} = \text{const}$ we should have:

$$J_{+} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2q_{+} \\ 0 & -2q_{+} & 0 \end{pmatrix}$$

$$J_{-} = \begin{pmatrix} 0 & \cos 2\varphi & \sin 2\varphi \\ -\cos 2\varphi & 0 & 0 \\ -\sin 2\varphi & 0 & 0 \end{pmatrix}$$

(Remember that we should have $\partial_{+} J_{\bar{1}-} + [J_{\bar{0}+}, J_{\bar{1}-}] = 0$ and also $\partial_{-} J_{\bar{1}+} + [J_{\bar{0}-}, J_{\bar{1}+}] = 0$.)

Suppose that we have two variations $\delta_{\xi(1)}$ and $\delta_{\xi(2)}$ corresponding to two parameters $\xi^{(1)}$ and $\xi^{(2)}$, $\delta_{\xi(j)} J = D\xi^{(j)}$:

$$\xi^{(1)} = \begin{pmatrix} 0 & \gamma^{(1)} & \alpha^{(1)} \\ -\gamma^{(1)} & 0 & \beta^{(1)} \\ -\alpha^{(1)} & -\beta^{(1)} & 0 \end{pmatrix}$$

$$\xi^{(2)} = \begin{pmatrix} 0 & \gamma^{(2)} & \alpha^{(2)} \\ -\gamma^{(2)} & 0 & \beta^{(2)} \\ -\alpha^{(2)} & -\beta^{(2)} & 0 \end{pmatrix}$$

Then the value of the string symplectic form on these two vectors is, according to (13):

$$\Omega(\delta_{\xi(1)}, \delta_{\xi(2)}) = \int d\tau^+ (\alpha^{(1)}\beta^{(2)} - \alpha^{(2)}\beta^{(1)})$$

Therefore the calculation of the symplectic structure is reduced to solving the equation $D_+\xi = \delta J_+$ for ξ:

$$\partial_+\xi + [J_+, \xi] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\delta q_+ \\ 0 & -2\delta q_+ & 0 \end{pmatrix}$$

We can solve for α, β, γ in terms of δq_+:

$$\alpha = -2\theta_0(\theta_0 + \theta_1)^{-1}\delta q_+$$
$$\beta = 2\theta_0^{-1}\theta_1(\theta_0 + \theta_1)^{-1}\delta q_+$$
$$\gamma = -2\partial_+^{-1} q_+\alpha$$

Now the calculation of the classical string symplectic form gives us:

$$\Omega = \int d\tau^+ \alpha(\delta q_+)\beta(\delta q_+) =$$
$$= \int d\tau^+ \delta q_+(\theta_0 + \theta_1)^{-1}\theta_0(\theta_0 + \theta_1)^{-1}\delta q_+$$

which is of course the same as Eq. (11).

This method works also for the string on $S^N = SO(N+1)/SO(N)$. In this case $\mathbf{g}_{\bar{1}} = so(N+1)$ and $\mathbf{g}_{\bar{0}} = so(N)$, $J_+ = J_{\bar{1}+} + J_{\bar{0}+}$. Let us fix the gauge so that

$$J_{\bar{1}+} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

There is a residual gauge freedom which allows to bring J_+ to the form:

$$J_+ = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & q_+^1 & \dots & q_+^{N-1} \\ 0 & -q_+^1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -q_+^{N-1} & 0 & \dots & 0 \end{pmatrix}$$

Again we have to find $D_+^{-1}(\delta J_+)$ and calculate

$$\Omega = \int d\tau^+ \mathrm{tr}\left(J_{\bar{1}+}[D_+^{-1}(\delta J_+), D_+^{-1}(\delta J_+)]\right)$$

$D_+^{-1}(\delta J_+)$ has the following form:

$$\begin{pmatrix} 0 & \gamma & \vec{\alpha}^T \\ -\gamma & 0 & \vec{\beta}^T \\ -\vec{\alpha} & -\vec{\beta} & \partial_+^{-1}(\vec{q}_+ \otimes \vec{\beta}^T - \vec{\beta} \otimes \vec{q}_+^T) \end{pmatrix}$$

where

$$\vec{\beta} = -(\partial_+ + \vec{q}_+ \partial_+^{-1} \vec{q}_+^T)\vec{\alpha}$$
$$\delta\vec{q}_+ = -\vec{\alpha} - (\partial_+ + \iota(\vec{q}_+)\partial_+^{-1}\vec{q}_+ \wedge)(\partial_+ + \vec{q}_+ \partial_+^{-1} \vec{q}_+^T)\vec{\alpha}$$

Therefore

$$\Omega = \int d\tau^+ (\vec{\alpha}(\delta\vec{q}_+), \vec{\beta}(\delta\vec{q}_+)) = \tag{14}$$
$$= \int d\tau^+ \delta\vec{q}(\theta_0 + \theta_1)^{-1}\theta_1(\theta_0 + \theta_1)^{-1}\delta\vec{q}$$

where

$$\theta_1 = \theta_0 \mathcal{J} \theta_0$$
$$\mathcal{J} = \partial_+ + \vec{q}\partial_+^{-1}\vec{q}^T$$
$$\theta_0 = \text{will explain in a moment}$$

Straightforward expression for θ_0 following from (14) is somewhat clumsy; I will explain θ_0 on the next slide.

So, we have again the Poisson structure of the form:

$$\theta_{str} = (\theta_0 + \theta_1)\theta_1^{-1}(\theta_0 + \theta_1) = \theta_1 + 2\theta_0 + \theta_0\theta_1^{-1}\theta_0$$

The same scaling argument as for S^2 shows that $[\![\theta_{str}, \theta_{str}]\!] = 0$ implies

$$[\![\theta_1, \theta_1]\!] = 0$$
$$[\![\theta_1, \theta_0]\!] = 0$$

But this argument does not tell us that $[\![\theta_0, \theta_0]\!] = 0$.

What is θ_0?

θ_0 is the boost-invariant symplectic structure of the GSG. To describe θ_0, we use the symplectic structure of the WZW model. The (chiral) phase space of the WZW is the space of group-value functions $G(\tau^+)$, and the symplectic structure is:

$$\Omega_{WZW} = \int d\tau^+ \mathrm{tr}\, \delta G G^{-1} \delta(\partial_+ G G^{-1})$$

The symplectic form θ_0^{-1} is the restriction of Ω_{WZW} on the space of functions $G(\tau^+)$ satisfying

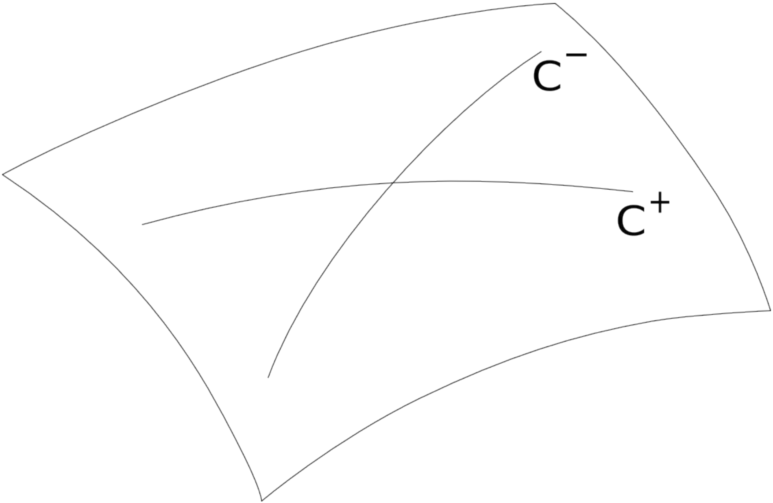
$$\partial_+ G G^{-1} = - \begin{pmatrix} 0 & q_+^1 & \dots & q_+^{N-1} \\ -q_+^1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -q_+^{N-1} & 0 & \dots & 0 \end{pmatrix} \tag{15}$$

It is easy to see that Ω_0 is closed:

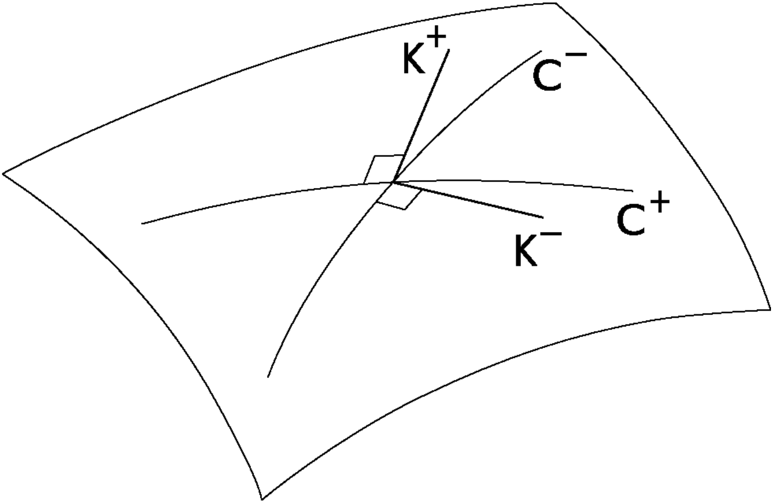
$$\delta\Omega_0 = \int d\tau^+ \frac{1}{3}\partial_+ \mathrm{tr}(\delta G G^{-1})^3 = 0$$

This implies the Jacobi identity $[\![\theta_0, \theta_0]\!] = 0$ for θ_0.

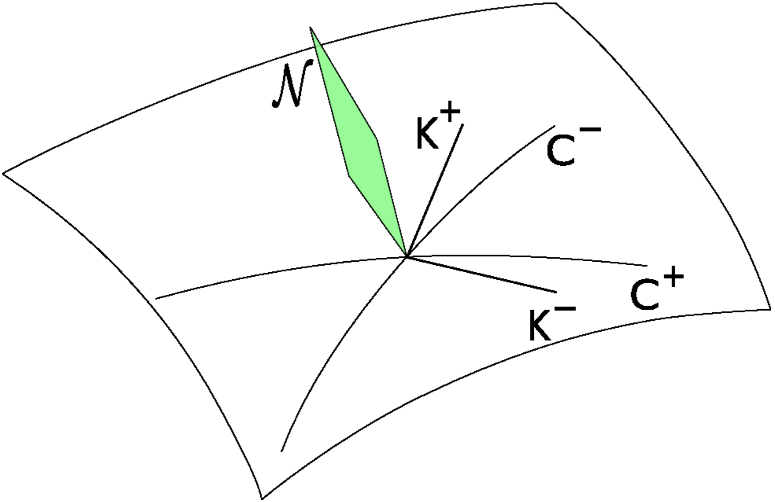
A slight disadvantage of this description of θ_0 is that it seems to be tied to a characteristic line C_+ on a string worldsheet.
But I will now explain how to rewrite it in terms of an arbitrary contour on the worldsheet.



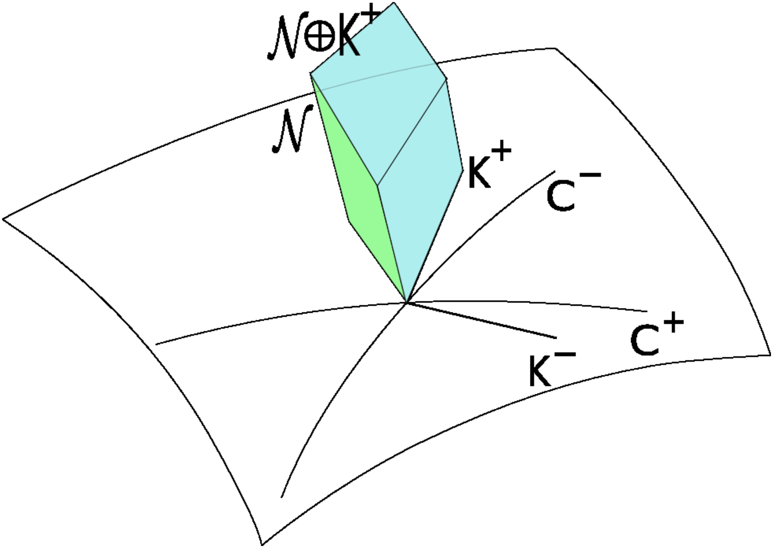
Through every point on the string worldsheet Σ go two lightlike curves. They form a “light cone” on the string worldsheet. Let C_+ and C_- be the projections of these directions to S^N (=characteristics).



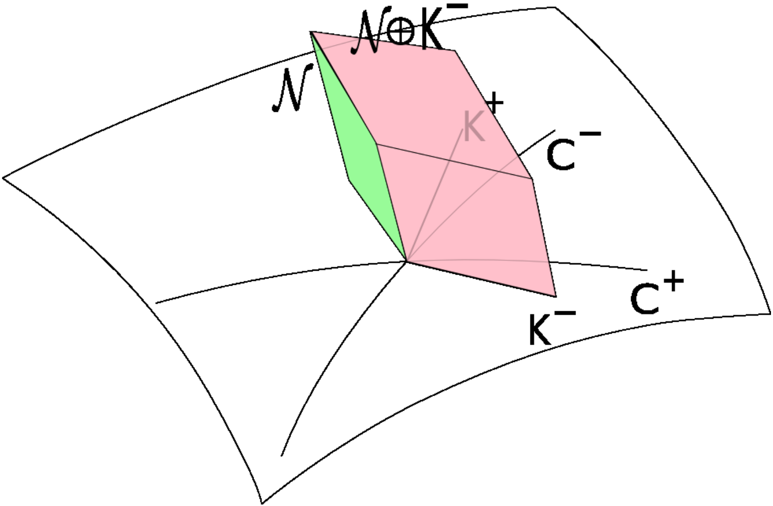
It is useful also to consider the directions in $T\Sigma$ orthogonal to C^+ and C^-. We will call them K^+ and K^-.



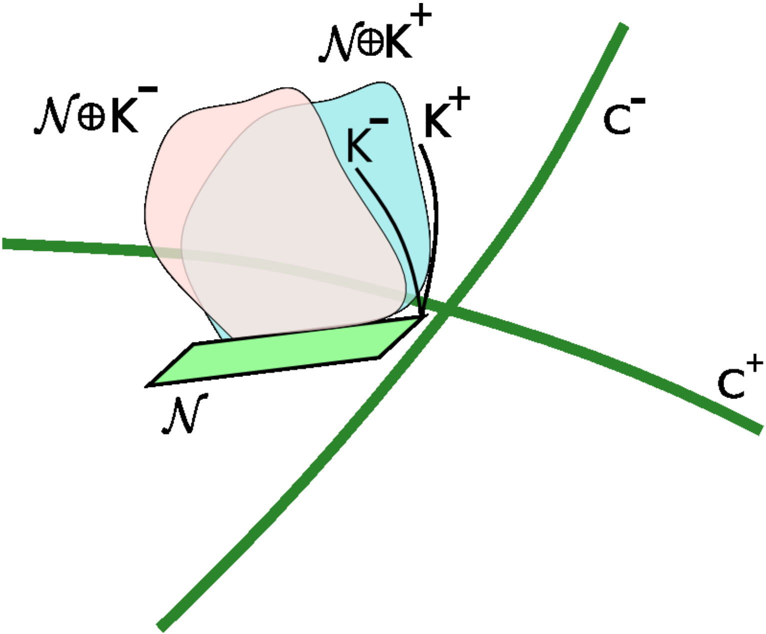
There is also a normal bundle to Σ which we call $\mathcal{N}$. It is formed by those vectors of TS^N which are orthogonal to $T\Sigma$.

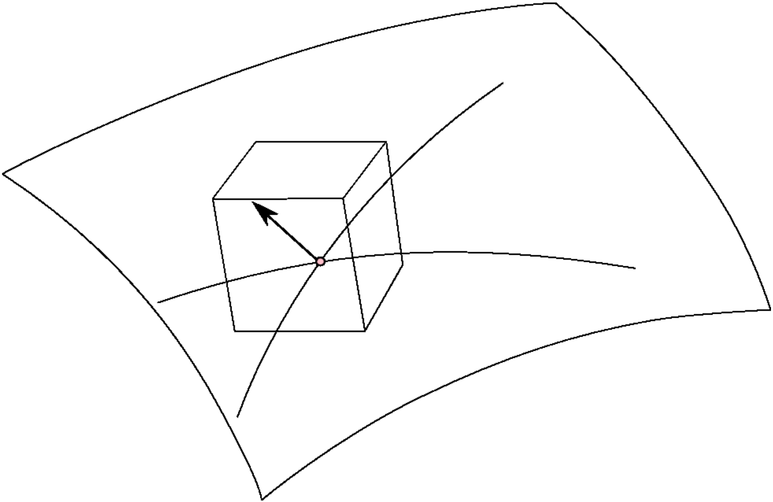


Notice that $\mathcal{N}$ has dimension $N - 2$. Let us consider an $N - 1$ dimensional bundle $\mathcal{N} \oplus K_+$.

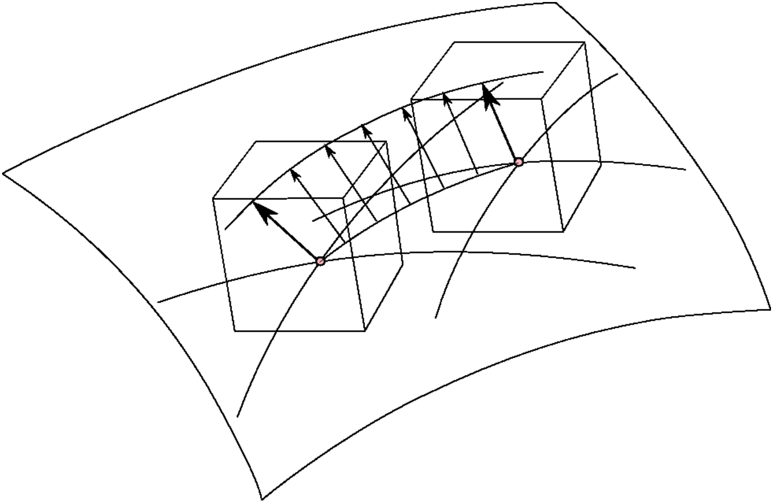


We will also need and $N-1$ dimensional vector bundle $\mathcal{N} \oplus K_-$.


$\mathcal{N}\oplus K^+$
$\mathcal{N}\oplus K^-$
K^+
K^-
C^-
C^+
$\mathcal{N}$



Let $i : \Sigma \to S^N$ denote the embedding map. The tangent bundle TS^N can be restricted to Σ, and the restricted bundle is formally called $i^* TS^N$.

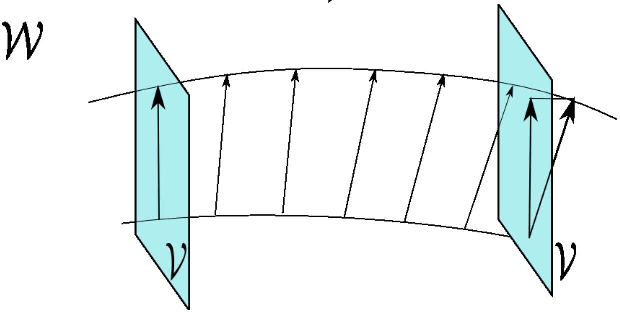


The Levi-Civita connection in TS^N can be restricted to $i^* TS^N$, so we can parallel transport the vectors tangent to S^N along the curves in Σ. We call the corresponding covariant derivative $D_{\bar{0}}$.

Remember that a connection can be projected from a bundle to a subbundle. Suppose that we have an orthogonal vector bundle $\mathcal{W}$ with the connection $\nabla_{\mathcal{W}}$. Consider a subbundle $\mathcal{V} \subset \mathcal{W}$. For any vector $v \in \mathcal{V}$ we define

$$\nabla_{\mathcal{V}} v = P_V \nabla_{\mathcal{W}} v$$

This is the definition of $\nabla_{\mathcal{V}}$.



Let us use the notation: $\nabla_{\mathcal{W}}|_{\mathcal{V}}$ for this induced connection $\nabla_{\mathcal{V}}$.

Let us restrict $D_{\bar{0}}$ on $\mathcal{N} \oplus K_+$ and $\mathcal{N} \oplus K_-$ and denote the resulting connections ∇^L and ∇^R:

$$\begin{aligned} \nabla^L &= D_{\bar{0}}|_{\mathcal{N}\oplus K_+} \\ \nabla^R &= D_{\bar{0}}|_{\mathcal{N}\oplus K_-} \end{aligned} \qquad (16)$$

It turns out that ∇^L and ∇^R are both flat:

$$[\nabla^L_+, \nabla^L_-] = 0, \quad [\nabla^R_+, \nabla^R_-] = 0$$

This follows from the string equations of motion.

Let us introduce some trivialization of $\mathcal{N}$. A trivialization of $\mathcal{N}$ is a choice of $N-2$ sections $\mathbf{e}_1, \ldots \mathbf{e}_{N-2}$ of $\mathcal{N}$ which form an orthonormal system:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$$

Notice that the trivialization of $\mathcal{N}$ defines the trivializations of both $\mathcal{N} \oplus K_+$ and $\mathcal{N} \oplus K_-$. Indeed, to get an orthonormal system in $\mathcal{N} \oplus K_+$ we just add to $\mathbf{e}_1, \ldots \mathbf{e}_{N-2}$ the unit vector in K_+. (A different choice of the trivialization of $\mathcal{N}$ will give the same answer for the symplectic form.)

Now, having the trivializations $\mathcal{N} \oplus K_+ \simeq \mathbf{R}^{N-1}$ and $\mathcal{N} \oplus K_- \simeq \mathbf{R}^{N-1}$ we can consider the monodromies of the *flat* connections ∇^L and ∇^R. The monodromies are just the matrices g^L and g^R such that:

$$\nabla^L g^L = 0, \quad \nabla^R g^R = 0$$

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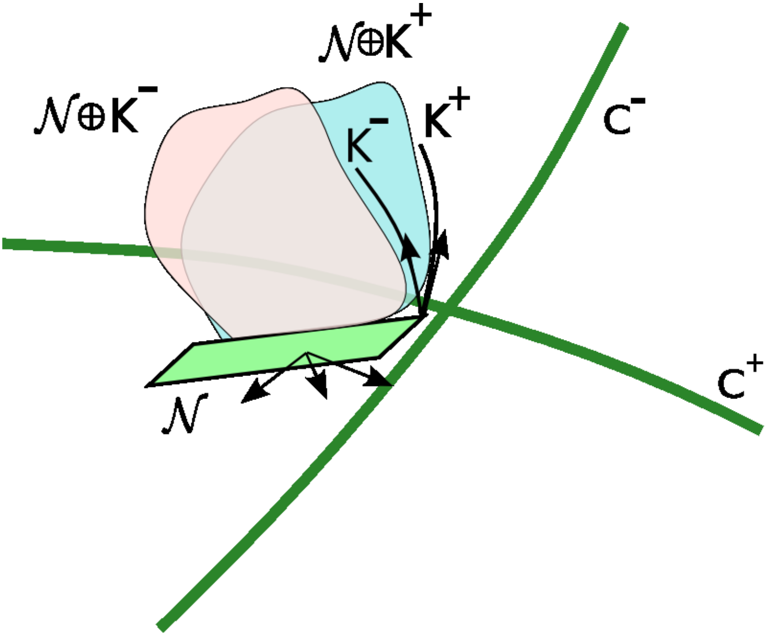
$\mathcal{N} \oplus K^+$
$\mathcal{N} \oplus K^-$
K^-
K^+
C^-
C^+
$\mathcal{N}$

The symplectic structure of the generalized sine-Gordon

$$\Omega = \oint [8\delta\varphi * d\delta\varphi + \\ + \mathrm{tr}\left((\delta g_L g_L^{-1})\delta(d g_L g_L^{-1})\right) - \mathrm{tr}\left((\delta g_R g_R^{-1})\delta(d g_R g_R^{-1})\right)] \tag{17}$$

where 2φ is the angle between C^+ and C^-.

In this form it is relatively easy to prove that Ω does not depend on the choice of the contour. (A calculation in coordinates.)

To prove that Eq. (17) is equivalent to Eq. (15) we notice that the monodromy matrix g in the normal frame is related to g_L and g_R in the following way:

$$G = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & g_R^{-1} \end{bmatrix} \begin{bmatrix} \cos 2\varphi & -\sin 2\varphi & \mathbf{0} \\ \sin 2\varphi & \cos 2\varphi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & g_L \end{bmatrix} \qquad (18)$$

This formula allows us to prove that Eq. (15) is equal to Eq. (17) using the Polyakov-Wiegmann type of identities and the fact that dGG^{-1} is of the form

$$dGG^{-1} = \begin{bmatrix} 0 & * & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that in our approach we constructed both the canonical symplectic form Eq. (4) and the non-standard form Eq. (17) in terms of the geometry of the string worldsheet. (The generalized sine-Gordon was actully used only to prove the compatibility.)

The non-standard symplectic form Ω_0 is local only if we add the additional fields g_L and g_R. It would be interesting to see if they have any meaning in string theory.

Open questions

- Generalize the construction of Ω_0 to the full superstring $AdS_5 \times S^5$.
- Are bihamiltonian structures useful in the quantum theory of integrable models?
- Suppose that we can quantize the vector mKdV with the boost invariant Poisson bracket θ_0. Can we then translate the result of the quantization to the quantization of θ_{str}?