Potsdam 7/06

Calogero Fermions & $\mathcal{N}=4$ SYM

Work with A. Agarwal

- $\mathcal{N}=4$ SYM in $d=4$: $\mathcal{O} = \mathrm{tr}(\Phi_1 \cdots \Phi_n)$ mix under renormalization → "evolve" as states

- Bare dimensions in general renormalize to quantum spectrum of anomalous dimensions

- Simplest states: $\mathrm{tr}(Z^{n_1}) \cdots \mathrm{tr}(Z^{n_k})$ are ½BPS: anomalous dimension = bare dimension
  $D = \sum_{i=1}^{k} n_i$: degeneracy = partitions (D)

- By standard bosonization (→counting) arguments, spectrum maps to (N) free bosons or free fermions

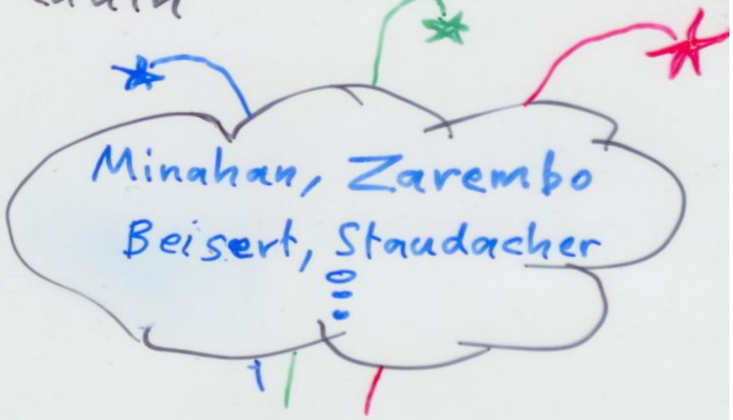
★ The "free fermion" picture realizes open/close string duality: $\mathrm{tr}(Z^n) \to$ string mode n
fermions → D branes

- Also useful for AdS/CFT correspondence:
  fermion "droplets" → string geometries
  (deep) hole excitations → "giant gravitons" etc.

  (Berenstein,
  –"–, Maldacena, Nastase
  Lin, –"–, Lunin
  Corley, Jevicki, Ramgoolam etc.)

What about general operators ?

- In general non-BPS → mix nontrivially
- Mixing (dilatation) hamiltonian maps to integrable spin chain
- Powerful technique that allows computation of anomalous dimensions and checks of AdS/CFT conjecture!



* Is there a "fermion" picture?
Not in general. However...

- Limit $g_{YM} \to 0$: no mixing, still nontrivial states → "string" picture?
- Many operators are BPS in the large-N limit
- What would be the fermionic / D-brane description of such states?

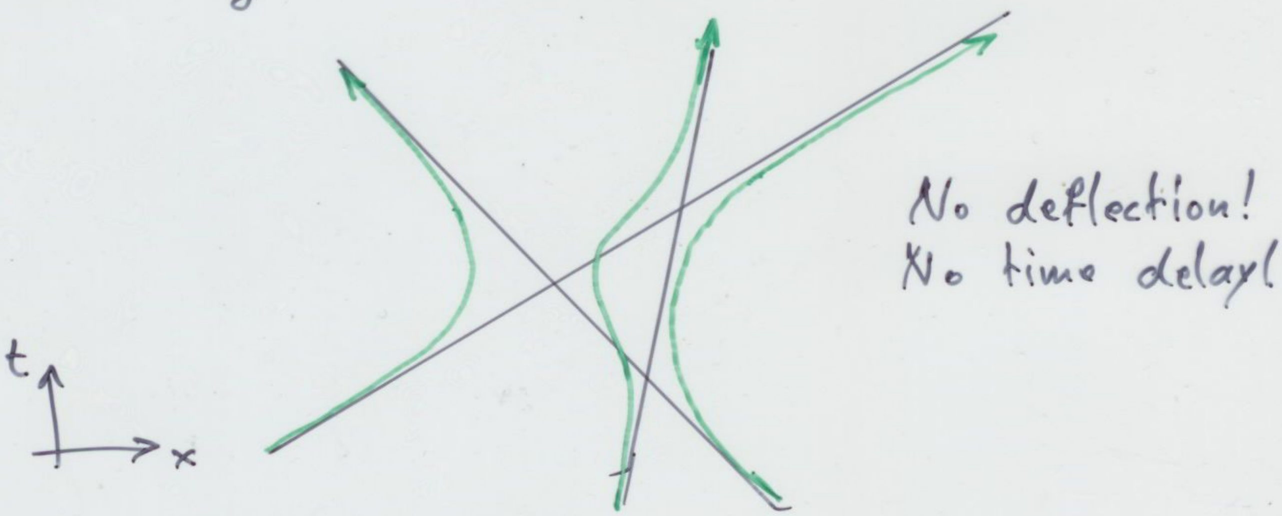
↪ Emergence of the celebrated Calogero rank of models...

Brief trip to Calogeroland

(Calogero '69, '71
Sutherland '71
Moser '75
⋮)

$$H = \sum_i \left(\tfrac{1}{2} p_i^2 + \tfrac{1}{2} x_i^2\right) + \frac{1}{2}\sum_{i\neq j} \frac{\ell(\ell \pm \vec{S}_i \cdot \vec{S}_j)}{(x_i - x_j)^2}$$

- Model is integrable & solvable
- $\frac{1}{\sin^2}$, $\frac{1}{\sinh^2}$, Weierstrass generalizations
- Scattering (no oscillators):



- QM: $\theta_{sc} = \frac{N(N-1)}{2} \ell \pi$
- Realizes particles with generalized statistics
- "Freezing trick": spin chains

* Can be analyzed by exchange operators or by matrix models ⟶ our connection

The (hermitian) matrix model - Calogero

- $L = \mathrm{tr}\left(\frac{1}{2}\dot{M}^2 - \frac{1}{2}M^2\right)$ $\quad$ M: $N\times N$ herm.

- $H = \mathrm{tr}\left(\frac{1}{2}\Pi^2 + \frac{1}{2}M^2\right) = \frac{1}{2}\mathrm{tr}\left(A^+ A\right)$

  $A = \Pi + iM$

- N^2 decoupled harmonic oscillators $\rightarrow$ solvable.

* $SU(N)$ symmetry: $\quad M \rightarrow V^{-1} M V$ $N\times N$ unitary

$\hookrightarrow J = i\,[M, \dot{M}] = \frac{1}{2}\,[A, A^+]$

$$\{J_{ij}, J_{kl}\} = \delta_{il}\, J_{kj} - \delta_{kj}\, J_{il} \qquad (SU(N))$$

* Go to eigenvalues & angular variables

$$\left.\begin{aligned} M &\equiv U^{-1} x\, U \\ \Pi &\equiv U^{-1}(p + A)\, U \end{aligned}\right| \quad J \equiv U^{-1} K\, U$$

$$K_{ij} = i\,[x, p + A]_{ij} = i\,(x_i - x_j)\, A_{ij}$$

$\hookrightarrow$ $$H = \sum_i \left(\frac{1}{2}p_i^2 + \frac{1}{2}x_i^2\right) + \frac{1}{2}\sum_{i\neq j} \frac{K_{ij}\, K_{ji}}{(x_i - x_j)^2}$$

- J generates $U \rightarrow UV$
  K generates $U \rightarrow VU$

K is also an $SU(N)$ algebra

- Diagonal part of K leaves M invariant
  $(U^{-1}V^{-1}xVU = U^{-1}xU)$
  $\hookrightarrow$ it is a gauge symmetry of the (x, U) description

$$\to K_{ii} = 0 \quad (\text{"Gauss' law"})$$
(no sum)

- Q.M: states in the angular variables U transform under some irrep of $SU(N)$ under K (# boxes $= nN$)

- K can be realized in a Jordan-Wigner way:

$$K_{ij} = \underbrace{\sum_{a=1}^{m}}_{m \text{ "flavors"}} \underbrace{a_i^{a\dagger} a_j^a}_{\text{boson or fermion oscillators}} - \frac{1}{N}\delta_{ij}\sum_{i,a} a_i^{a\dagger} a_i^a$$

$$\rightsquigarrow K_{ii} = 0 \Rightarrow \sum_a a_i^{a\dagger} a_i^a = n \quad (\text{fixed})$$

$S_i^{ab} = a_i^{a\dagger} a_i^b$ is an $SU(m)$ n-symmetric operator

$$\to K_{ij}K_{ji} = a_i^{a\dagger} a_j^a \, a_j^{b\dagger} a_i^b = C_{2m} \pm S_i^{ab} S_j^{ba}$$

* Interaction $\frac{K_{ij}K_{ji}}{(x_i - x_j)^2}$ became an (anti)ferro. spin-coupling for particles (with fixed strength)

(work of Joe + A.)

A new possibility:

$$K_{ij} = \underbrace{b_i^+}_{\text{boson}} b_j + \underbrace{f_i^+ f_j}_{\text{fermion}} - \delta_{ij}(\dots)$$

- $K_{ii}|\psi\rangle = 0 \Rightarrow$

$$b_i^+ b_i + f_i^+ f_i = m$$
$$B_i + F_i = m$$

Either $B_i = m$, $F_i = 0$ or $B_i = m-1$, $F_i = 1$

↪ each particle comes in either a bosonic or a fermionic flavor

$$K_{ij} K_{ji} = m(m + \Pi_{ij})$$

- Π_{ij} is graded permutation operator

$$\Pi_{ij} |F, F\rangle = -|F, F\rangle$$

↪ an exchange-Calogero model with particles exchanging their F or B quantum number $\equiv$ SUSY Calogero model

- Still fully solvable

... back to SYM:

Look at operators of the form:

$$\mathrm{tr}(Z^{n_1})\ldots\mathrm{tr}(Z^{n_k})\cdot\mathrm{tr}(Z^{m_1}\Psi)\ldots\mathrm{tr}(Z^{m_\ell}\Psi)$$

(a Dabholkar-like truncation)

Z : a complex scalar
Ψ: a fermi field

- Protected (non-renorm.) at large N, a consistent truncation of full set of ops.
- We may consider Z as the "string" field and Ψ as an "impurity".
- Anomalous dimension is given by hamiltonian

$$H = \mathrm{tr}(A^+A) + \frac{3}{2}\mathrm{tr}(\Psi^+\Psi)$$

A: a matrix operator annihilating Z
A^+: — " — creating Z

$(A^+ = Z, \quad A = \frac{\partial}{\partial Z})$

→ We have a standard matrix model with an extra fermionic matrix

- "States" are invariant under $U(N)$ gauge transformations

$$\rightarrow \quad J = J_A + J_\Psi = 0$$

For a D-brane picture, go to eigenvalues-angles parametrization for $M = (Z + Z^+)/\sqrt{2}$

As before:

$$H = \sum_i \left(\tfrac{1}{2} p_i^2 + \tfrac{1}{2} x_i^2 \right) + \frac{1}{2} \sum_{i \neq j} \frac{K_{ij} K_{ji}}{(x_i - x_j)^2}$$
$$+ \frac{3}{2} \, tr \, (\psi^+ \psi)$$

where $\Psi = U^{-1} \psi U$

- On gauge invariant states:

$$J = J_A + J_\Psi = 0 \;\Rightarrow\; K + K_\psi = 0$$

So the interaction strength $K_{ij} K_{ji}$ can be written entirely in terms of ψ.

☆ Final step: For D-like states with at most one Ψ in each trace:

$$K_{ij} = \underbrace{b_i^+ b_j}_{\text{bosons}} - \underbrace{f_j^+ f_i}_{\text{fermions}}$$

$\Big($Note: $b_i^+ b_j - f_j^+ f_i = b_i^+ b_j + f_i f_j^+ = b_i^+ b_j + \bar{f}_i^+ \bar{f}_j$ $\Big]$

same as before

Why? Hilbert spaces are isomorphic

- $K_{ii} = 0 \;\rightarrow\; b_i^+ b_i = f_i^+ f_i$

  ↪ equal # of b^+ and f^+ per state

- States can only be of form

  $\cdots tr(A^{+n}) \, |0\rangle$ or

  $\cdots f^+ A^{+n} b^+ \, |0\rangle$

  where $A|0\rangle = b|0\rangle = f|0\rangle = 0$

  and $f^+ = (f_1^+ \cdots f_N^+)$, $b^+ = \begin{pmatrix} b_1^+ \\ \vdots \\ b_N^+ \end{pmatrix}$

- Writing $\Psi = b^+ f^+$ (a square Fermi matrix)

  $$f^+ A^{+n} b^+ = tr(A^{+n} \Psi) \sim tr(Z^n \Psi)$$

- Traces with more than on Ψ do not arise, since

  $$Tr(A^{+n} \Psi A^{+m} \Psi) = Tr(A^{+n} \Psi) \, Tr(A^{+m} \Psi)$$

  for $\Psi = b^+ f^+$ Q.E.D.

Net result: $\mathcal{N}=4$ SYM operators with at most one fermionic "impurity" per trace

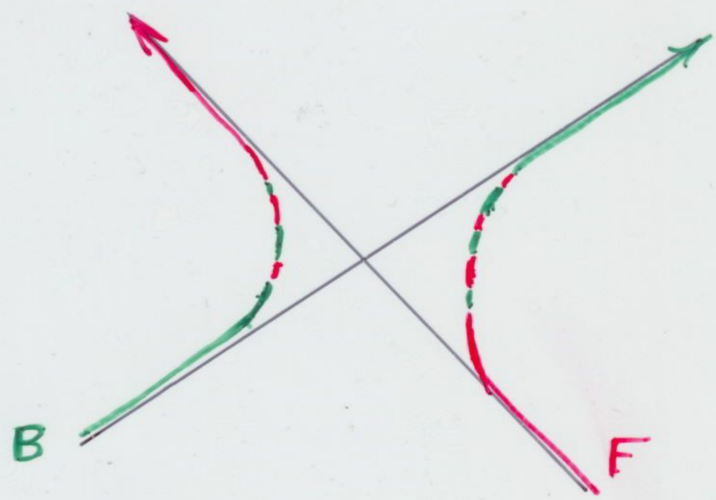
$$\Vert$$

Super-Calogero model (of strength $\ell=1$)

- "Minimal" deviation from free fermion model
- Free fermions obtained as the bosonic sector of S-Calogero, since

$$1-\Pi_{ij} = 1-1=0 \quad \text{on such states}$$

- Superparticles scatter by "going through"



↳ allows a consistent "two-fluid" phase space description

- Mapping to wavefunctions, Yangians etc.

CONCLUSIONS - OUTLOOK

- SuperCalogero model emerges as the open string dual of $su(1|1)$ subsector of $\mathcal{N}=4$ SYM
- Exchange-type interaction allows particles to go through $\rightarrow$ holographic description of phase space motion.
- Find similar descriptions for other sectors
- Explore phase space "droplet" description & mapping to sugra states à la LLM. (Supergiant gravitons?)
- For non-BPS states, does any Calogero description survive? (Interacting...)
- Do other Calogero systems & related spin chain reductions play any role?