Finite-size Effects from Giant Magnons

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Integrability in Gauge and String theory

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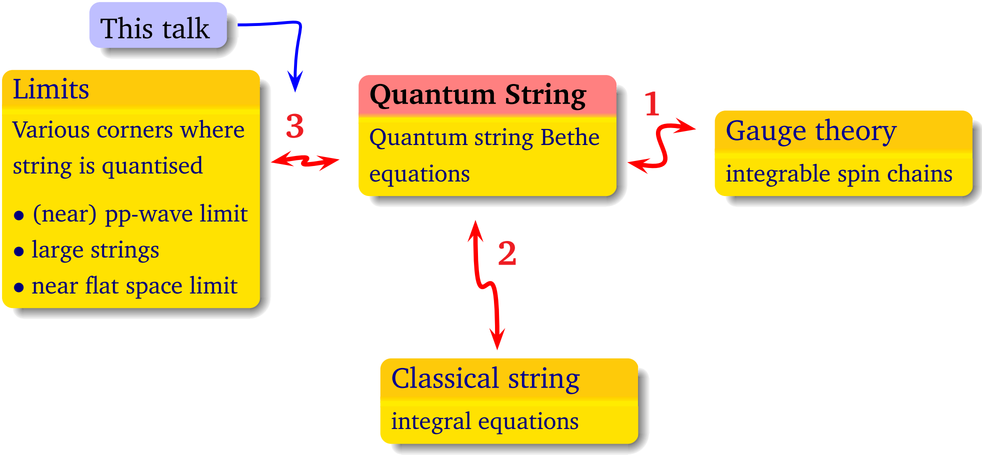
I Introduction

Central problem: How to quantise the string in $AdS_5 \times S^5$ background ?

It seems that this can be rephrased as the question of solving a particular set of algebraic equations — quantum string Bethe equations.



Basic ingredients for the Bethe equations:

- **Infinite volume:**
  - exist set of **elementary, asymptotic** (one particle) excitations **magnons** with **exact** dispersion relation $\mathcal{E}(p)$
  - multi-particle spectrum

$$\mathcal{E}_{tot} = \sum_i \mathcal{E}(p_k)$$

  - m-body interaction → 2-body S-matrices $S(p_k, p_j)$ and p_k, p_j do not change in scattering

- **Finite volume L:**
  - relevant parameters $1/L, \Delta_{\text{int}}/L$
  - leading $1/L$ effects from periodic boundary conditions on $\Psi(x_1, \ldots, x_N)$:

$$\Psi(x_1, \ldots, x_N) = \Psi(x_2, \ldots, x_N, x_1 + L) \Leftrightarrow \text{Bethe equations}$$

→ **constraints** on p_k

  - all the rest the same as in infinite volume
  - it may work exactly (e.g. XXX spin chain)

Bethe equations in string theory:

- significant simplifications in **infinite volumes**
- generic string state in **finite volumes** $\longrightarrow$ complicated situation $\longrightarrow$ not clear how much of Bethe will survive

- **Our goals**:
  - understand what is volume in string theory, how finite V affects quantum string Bethe "program"
  - understand better nature of elementary excitation in $V \leq \infty$, in particular the unusual dispersion relation

$$\epsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \, .$$

(I) Setting the stage: volume in string theory

Uniform gauge–conserved charges Q_a is uniformly distributed along the string world sheet $\longrightarrow$ $V \sim Q_a$

- Background with 2 isometries:

$$ds^2 = G_{tt}\, dt^2 \;+\; G_{\phi\phi}\, d\phi^2 \;+\; G_{ij}\, dx^i dx^j$$

isometries

instead of $(t, \phi) \quad \leftrightarrow \quad (x_+, x_-)$

$$x_- = \phi - t \;,\;\; x_+ = (1-a)\, t + a\, \phi$$

a–labels various "l.c." coordinates

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau\, \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N\, G_{MN}\;, \qquad \gamma^{\alpha\beta} \equiv \sqrt{-h}\, h^{\alpha\beta}$$

(I) Setting the stage: volume in string theory

- Imposing the gauge:

  (1) isometries → charges

  translation in (t, ϕ) → (E, J)

$$E = -\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\, p_t \;, \qquad J = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\, p_\phi \;.$$

  or translation in (x_-, x_+) → (P_-, P_+)

$$P_- = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\, p_- = J - E \;, \qquad P_+ = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\, p_+ = (1-a)\,J + a\,E \;.$$

  (2) fixing the gauge:

  (a) $x_+ = \tau + am\sigma$ $\qquad a = \frac{1}{2}$ light-cone gauge

  $\qquad a = 0$ temporal gauge

  (b) $p_+ = 1$ uniform → $\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_+ = P_+$ → $r = \frac{\pi}{\sqrt{\lambda}} P_+$

string "size" set by charge P_+

(proper size can be small, zero!)

(I) Setting the stage: volume in string theory

(3) first order form of action

$$S = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\mathrm{d}\tau \left(p_-\dot{x}_+ + p_+\dot{x}_- + p_i\dot{x}^i + \frac{\gamma^{01}}{\gamma^{00}}C_1 + \frac{1}{2\gamma^{00}}C_2\right)$$

$$C_1 = p_+x'_- + p_-x'_+ + p_ix'^i\,,$$

(4) solving constraints:

$$x'_- = -amp_- - p_ix'^i$$

$$\mathcal{H} = -p_-(x^i, x'^i, p_i) \quad \Rightarrow \quad H = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\,\mathcal{H} = E - J$$

Hamiltonian for physical d.o.f.

non-trivial equation for E

$$\left(r = \frac{\pi}{\sqrt{\lambda}}P_+ = \frac{\pi}{\sqrt{\lambda}}((1-a)J + E)\right)$$

gauge fixed action:

$$S = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\mathrm{d}\tau\,\left(p_i\dot{x}^i - \mathcal{H}\right)$$

(I) Setting the stage: how to isolate the magnon

- **world-sheet momentum**
  gauge-fixed S invariant under $\sigma \to \sigma + b$ with periodic boundary conditions for x_i, p_i →

$$p_{\rm ws} = -\int_{-r}^{r} d\sigma\, p_i x'^i = \int_{-r}^{r} x'_- \,,$$

- closed string: x_- periodic →

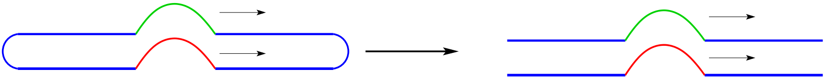
$$\Delta x_- = p_{\rm ws} = 0\,, \quad m = 0 \qquad \text{level-matching}$$

↑ total momentum of multi-magnon configuration

(I) Setting the stage: how to isolate the magnon

- Hence, if want **one-magnon only** ➜ need $p_{\rm ws} \neq 0$ ➜ $\Delta x_- \neq 0$!

  ➜ Solve closed string EOM dropping the level-matching condition:



  ➜ consider **OPEN string**

- Remarks:
  - this is NOT convenctional open string: x_i periodic, ($\Rightarrow \Delta x_- = const.$)
  - all this was valid for **ANY** P_+ (finite or infinite)
  - momentum does not vanish at string end points! Flows out and flows in . . .
  - if $r = \infty$ $(i.e.\ J = \infty)$ $\Rightarrow$ finiteness of the energy $\Rightarrow x_i' = 0$ Neumann b.c. no flow of momenta

(II) Constructing the Giant magnon

- consider action on $\mathbb{R} \times \mathrm{S}^2$

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int_{-r}^{r} \mathrm{d}\sigma \, \mathrm{d}\tau \, \gamma^{\alpha\beta} \left(-\partial_\alpha t \partial_\beta t + \partial_\alpha X_i \partial_\beta X_i \right) ,$$

$$X_1 + i X_2 = \sqrt{1 - z^2} e^{i\phi} , \quad X_3 = z , \quad -1 \leq z \leq 1$$

- solve constraints imposing uniform-LC gauge ➔ $S(z, z', \dot{z})$
- make ansatz

$$z = z(\sigma - v\tau)$$

- plug into S ➔ conserved charge $H_{red} = \frac{\omega - 1}{1 - a + a\,\omega}$
- solving H_{red} w.r.t. to z' get

$$z'^2 = \left(\frac{1 - z^2}{(1-a)\,(b^2 - z^2)} \right)^2 \frac{z^2 - z_{min}^2}{z_{max}^2 - z^2} ,$$

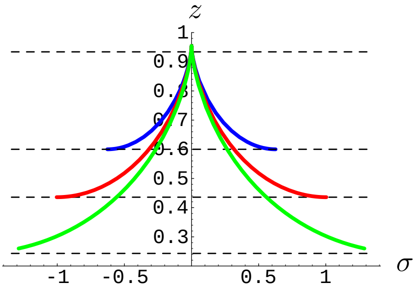
(II) Constructing the Giant magnon

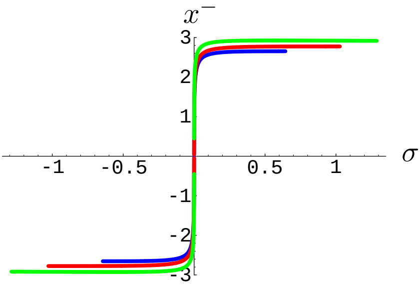
- Solution characterised by three parameters (a, ω, v)

$$0 \leq a \leq 1 \,, \quad \text{labels different gauges}$$

$$\left.\begin{array}{l} 1 \leq \omega < \infty \\ 0 \leq |v| \leq \dfrac{1}{\omega} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} z_{min}^2 = 1 - \dfrac{1}{\omega^2} \\ z_{max}^2 = 1 - v^2 \end{array}\right.$$

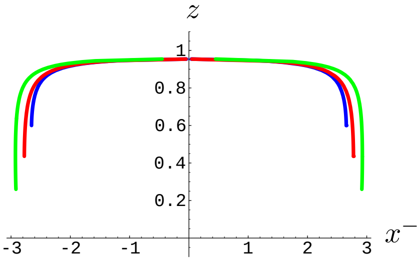
- integrate numerically





  - It's not smooth, but energy finite
  - here $J_1 < J_2 < J_3$

(II) Constructing the Giant magnon



Target space shape of the magnon

- to compute the dispersion relation

$$E - J = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} \mathrm{d}\sigma\,\mathcal{H} = \frac{\sqrt{\lambda}}{\pi}\int_{z_{min}}^{z_{max}} \mathrm{d}z\frac{\mathcal{H}}{|z'|}$$

$$r = \frac{\pi}{\sqrt{\lambda}}P_{+} = \int_{0}^{r} \mathrm{d}\sigma = \int_{z_{min}}^{z_{max}} \frac{\mathrm{d}z}{|z'|}$$

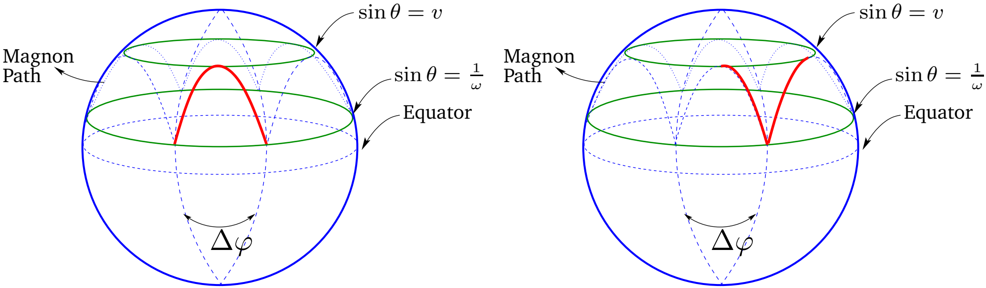
$$p_{\mathrm{ws}} = -\int_{-r}^{r} d\sigma p_z z' = 2\int_{z_{min}}^{z_{max}} \mathrm{d}z|p_z|$$

$$\longrightarrow$$

$$E - J = f(z_{min}, z_{max}; a) = f(p_{ws}, J; a)$$

Properties of the solution

$a = 0$ gauge: $x_+ = t = \tau \Rightarrow \Delta x_- = \Delta\varphi \quad (x_- = \varphi - t)$



movies: large v small v

Properties of the solution

Limit $J \to \infty$,i.e. infinitely long "string-chain"

- string becomes rigid, no wiggling!
- dispersion relation

$$E - J \equiv \epsilon = \frac{\sqrt{\lambda}}{\pi} \int_{z_{min}}^{z_{max}} \mathrm{d}z \frac{\mathcal{H}}{|z'|} = \frac{\sqrt{\lambda}}{\pi} \sqrt{1 - v^2}\,,$$

$$p_{\mathrm{ws}} = 2 \int_{z_{min}}^{z_{max}} \mathrm{d}z |p_z| = 2 \arccos v\,.$$

dispersion relation

$$\epsilon(p_{\mathrm{ws}}) = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p_{\mathrm{ws}}}{2} \right|$$

cf. $\epsilon(p_k) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p_k}{2}\right)} - 1$

see sine !

[Hofman, Maldacena]

Properties of the solution

Finite $J \sim$ finite length "string-chain"

- in general dispersion relation complicated
- look at large J limit:

dispersion relation

$$E - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_{\rm ws}}{2}\right|\Bigg[1 - \frac{4}{e^2}\sin^2\frac{p_{\rm ws}}{2}\, e^{-\mathcal{R}} - \frac{4}{e^4}\sin^2\frac{p_{\rm ws}}{2}\Big(\mathcal{R}^2(1+\cos p_{\rm ws}) + 2\mathcal{R}(2 + 3\cos p_{\rm ws} + ap\sin p_{\rm ws}) + 7 + 6\cos p_{\rm ws} + 6ap_{\rm ws}\sin p_{\rm ws} + a^2 p_{\rm ws}^2(1-\cos p_{\rm ws})\Big)e^{-2\mathcal{R}} + \cdots\Bigg].$$

$$\mathcal{R} = \frac{2\pi J}{\sqrt{\lambda}\left|\sin\frac{p_{\rm ws}}{2}\right|} + ap_{\rm ws}\cot\frac{p_{\rm ws}}{2}\,.$$

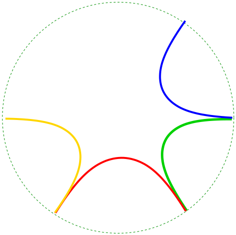
Properties of the solution

Comments:

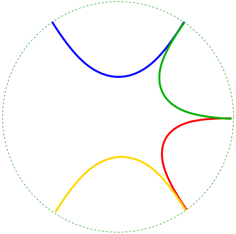
- dispersion relation depends on a – gauge parameter
- the dependence on a disappears in $J \to \infty$ limit
- if $a \neq 0 \Rightarrow E - J$ not periodic in $p_{ws} \Rightarrow a = 0$ seems preferred
- From $\mathcal{R} \sim \text{Vol}/\Delta_M$, $\text{Vol} = \frac{2\pi}{\sqrt{\lambda}} J$:
  - read-off "size" of magnon $\Delta_M \sim \sin(p/2)$
  - agrees with Hubbard [Rej,Serban,Staudacher; Minahan's talk]
  - BMN: Vol-finite and $p \sim 1/\sqrt{\lambda} \Rightarrow \Delta_M \to 0 \Rightarrow \mathcal{R} \to \infty$

Properties of the solution

- Reconstructing closed string – multi soliton configuration
  - in general non-trivial
  - there still exists a simple superposition (cf. $J = \infty$)

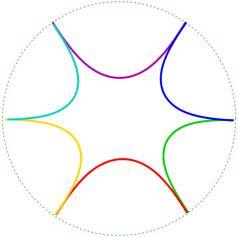


$t = 0$



$t = T$

multi-magnon open, non-rigid string



all t

multi-magnon closed, rigid string !

Symmetry algebra at finite J

- key step: **drop level matching condition**
- consider simpler example:
  flat space L.C. gauge, dynamical generators of the Lorentz algebra

$$J^{i-} = \int_0^{2\pi} d\sigma \, (x_i \dot{x}_- - x_- \dot{x}_i) \, .$$

(non)-conservation

$$\dot{J}^{i-} = \int_0^{2\pi} d\sigma \, (x_i \ddot{x}_- - \ddot{x}_i x_-) = -{x_i}'(0)\Big(x_-(2\pi) - x_-(0)\Big)$$

conserved for: (a) Neumann open string or

(b) closed string

- if $J = \infty$ have Neumann b.c. → **all generators conserved**
- if J–finite → **dynamical generators broken!**

Symmetry algebra at finite J

- strings in uniform a-gauge on $\mathbb{R} \times S^2$

$$j_{MN} = -\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma \partial_\sigma \left(\gamma^{\sigma\alpha}\partial_\alpha x_{[M} x_{N]}\right) = -\frac{\sqrt{\lambda}}{2\pi}\left(\gamma^{\sigma\alpha}\partial_\alpha x_{[M} x_{N]}\right)\Big|_{\sigma=-r}^{\sigma=r}$$

  get that:

  $J_{12} \leftrightarrow \phi$ is conserved

  J_{13}, J_{23} not conserved since x_- not periodic (i.e. $\phi = \tau + (1-a)x_-$)

- curiously all conserved when $a = 1$ (i.e. $\phi = \tau$)

- **N.B**
  For full model (in arbitrary "l.c." gauge)
  - For $J = \infty$ $\longrightarrow$ **all generators conserved**
  - If relax level-matching, by explicit computation, one recovers centrally extended $\mathfrak{su}(2|2) \times \mathfrak{su}(2|2)$ algebra

[G.Arutyunov, S.Frolov, J.Plefka and M.Zamaklar, to appear]

Finite J Giant magnon in conformal gauge

- conformal gauge

$$\gamma_{\mu\nu} = \text{diag}(-1, 1)$$

and the condition $t = \tau$ (close to $a = 0$ L.C. gauge)

- motivated by L.C. analysis, impose boundary conditions

$$z(r,\tau) - z(-r,\tau) = 0\,, \quad \Delta\phi = \phi(r,\tau) - \phi(-r,\tau) = p = const.\,,$$

i.e. open string with fixed separation of end-points

- make ansatz

$$\varphi = \phi - \omega t\,, \quad \varphi = \varphi(\sigma - v\omega\tau)\,, \quad z = z(\sigma - v\omega\tau)\,.$$

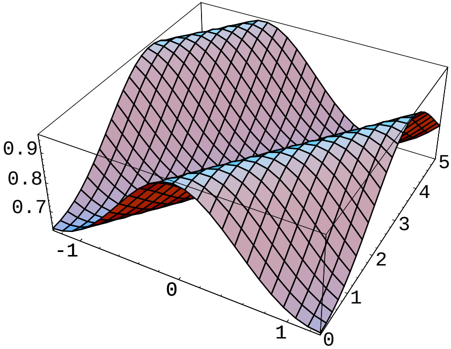
- integrate analytically

$$z = \frac{\sqrt{1-v^2}}{\omega\sqrt{\eta}}\text{dn}\left(\frac{1}{\sqrt{\eta}}\frac{\sigma - v\tau}{\sqrt{1-v^2}}, \eta\right),$$

$$\eta = \frac{1-\omega^2 v^2}{\omega^2(1-v^2)}\,.$$

Finite J Giant magnon in conformal gauge

- world-sheet solution smooth, unlike in L.C. gauges



- target space picture agrees with $a = 0$ gauge
- dispersion relation etc. the same as for $a = 0$ gauge
- see that period goes to infinity as $J \to \infty$

Summary/outlook

Giant magnons — good laboratory for studying properties of isolated magnon:

- seen sin: "lattice" $\leftrightarrow$ compactness of S^2
- new prediction for the dispersion relation
- size of magnon, and structure of exp corrections agrees with Hubbard
- algebra broken at finite J

Summary/outlook

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Questions:

- Is Bethe going to survive finite J?
- Implication of the reduced algebra?
- gauge dependence at finite $J \Rightarrow a = 0$ preferred?
  (i.e. is finite J Bethe/i.e. Hubbard possible only in particular gauge?)