Bethe ansatz in Sigma Models

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T. Klose, K. Z., hep-th/0603039

S. Schäfer-Nameki, M. Zamaklar, K.Z., in progress

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“Integrability in Gauge and String Theory”, Potsdam, 24.07.2006

Topological expansion of gauge theory

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String theory

Early examples:

- 2d QCD ‘t Hooft’74
- Matrix models Brezin,Itzykson,Parisi,Zuber’78

4d gauge/string duality:

- AdS/CFT correspondence Maldacena’97

a (theoretically) testable prediction of string theory

AdS/CFT correspondence

Maldacena'97

$\mathcal{N}=4$ SYM	Strings on $AdS_5\times S^5$
't Hooft coupling: $\lambda = g_{YM}^2 N$	String tension: $T=\frac{\sqrt{\lambda}}{2\pi}$
Number of colors: N	String coupling: $g_s = \frac{\lambda}{4\pi N}$
Large-N limit	Free strings
Strong coupling	Classical strings
Local operators Scaling dimension: Δ	String states Energy: E

Gubser,Klebanov,Polyako
Witten'98

Strings in $AdS_5 x S^5$

Green-Schwarz-type coset sigma model

on SU(2,2|4)/SO(4,1)xSO(5).

Metsaev,Tseytlin'98

Conformal gauge is problematic:

no kinetic term for fermions, no holomorphic factorization for currents, …

Light-cone gauge is OK.

The action is complicated, but the model is integrable!

Bena,Polchinski,Roiban'03

Arutyunov,Frolov'04

Spectrum $\Longleftrightarrow$ S-matrix

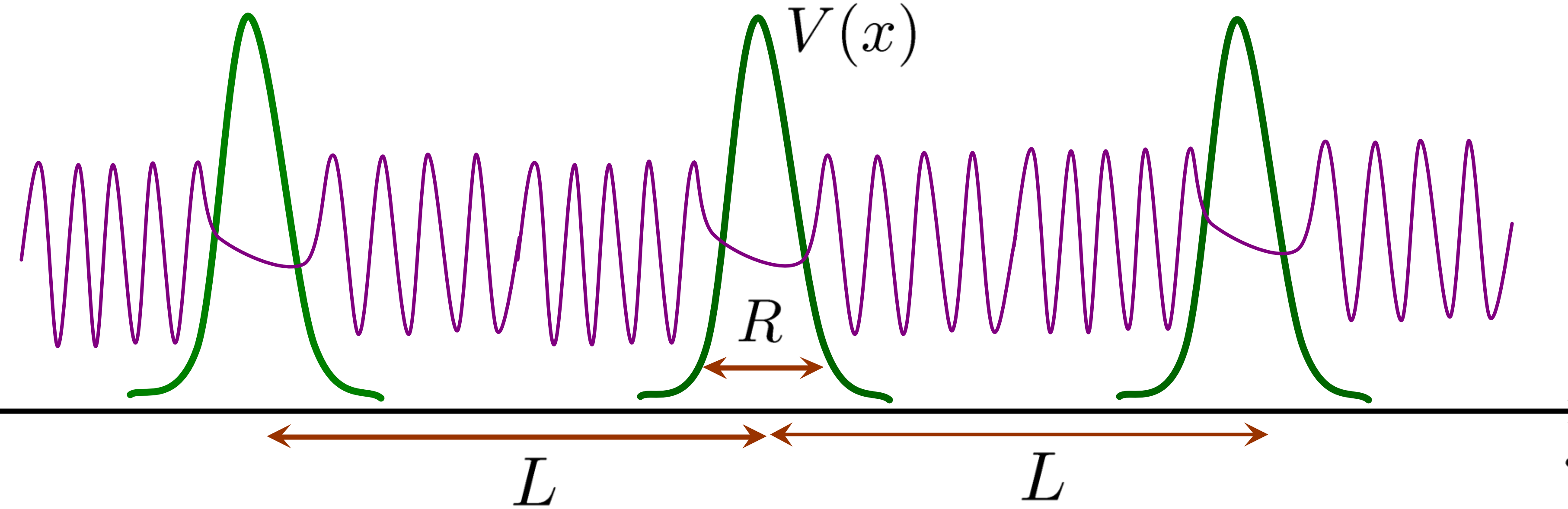
$$0 \leq x < L$$

(string is closed)

?

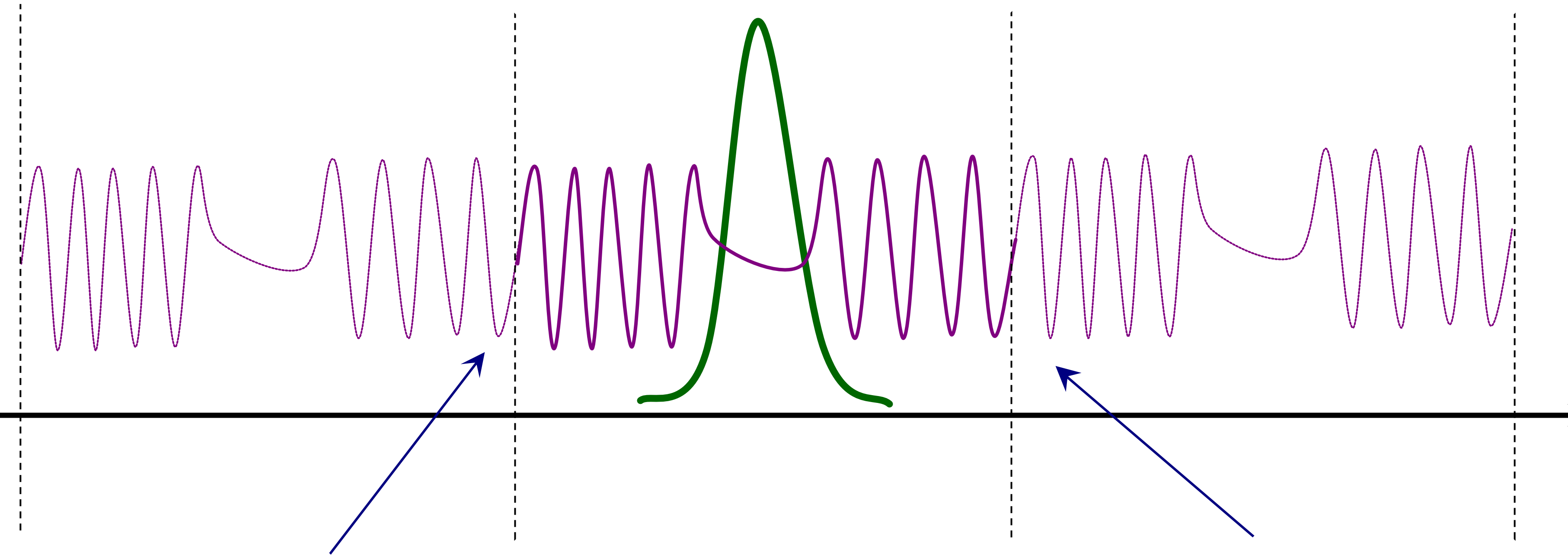
$$-\infty < x < +\infty$$

(asymptotic states)



$$\sqrt{2mE_n} = \frac{2\pi n}{L} - \frac{\Delta(E_n)}{L}$$

- exact only for V(x) = g d(x)



$$\psi(x) \approx A\cos(px + \varphi_0)$$

$$\psi(x) \approx A\cos(px + \varphi_0 + \Delta$$

Continuity of periodized wave function

$$pL + \Delta = 2\pi n$$

$$\left(p = \sqrt{2mE}\right)$$

$$e^{ipL} = S^{-1}(p)$$

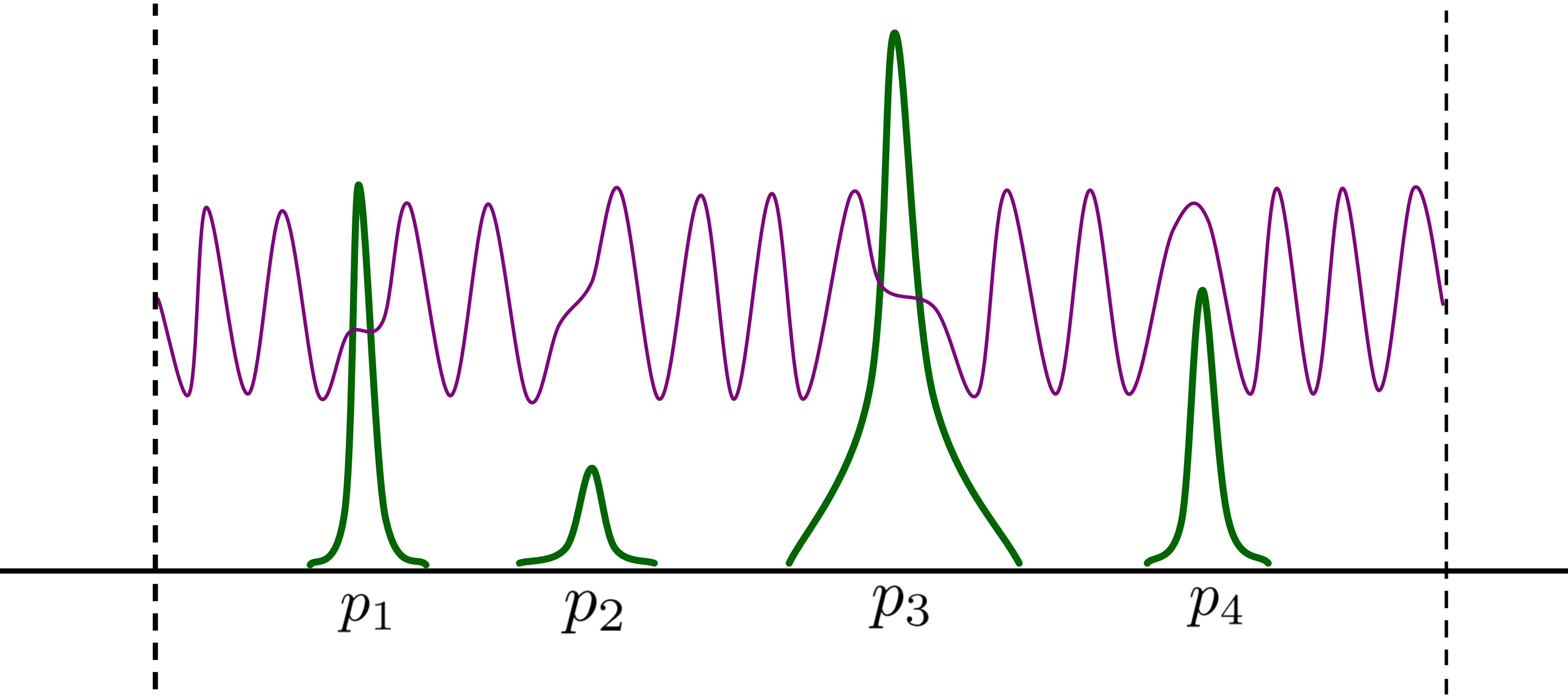
where

$$S(p) = e^{i\Delta(p)}$$

is (eigenvalue of) the S-matrix

- correct up to $O(e^{-L/R})$
- works even for bound states via analytic continuation to complex momenta

Multy-particle states



$$\mathrm{e}^{\,ip_k L} = \prod_{j\neq k} S(p_j, p_k)$$

Bethe equations

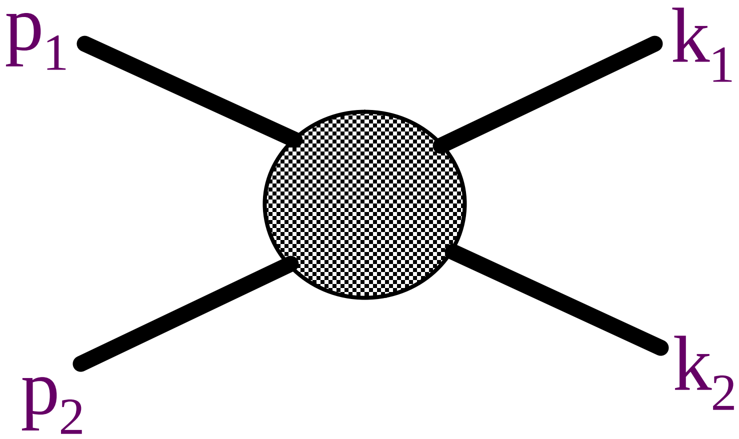
$$e^{ip_k L} = \prod_{j \neq k} S(p_j, p_k)$$

$$E = \sum_k \varepsilon(p_k)$$

Assumptions:

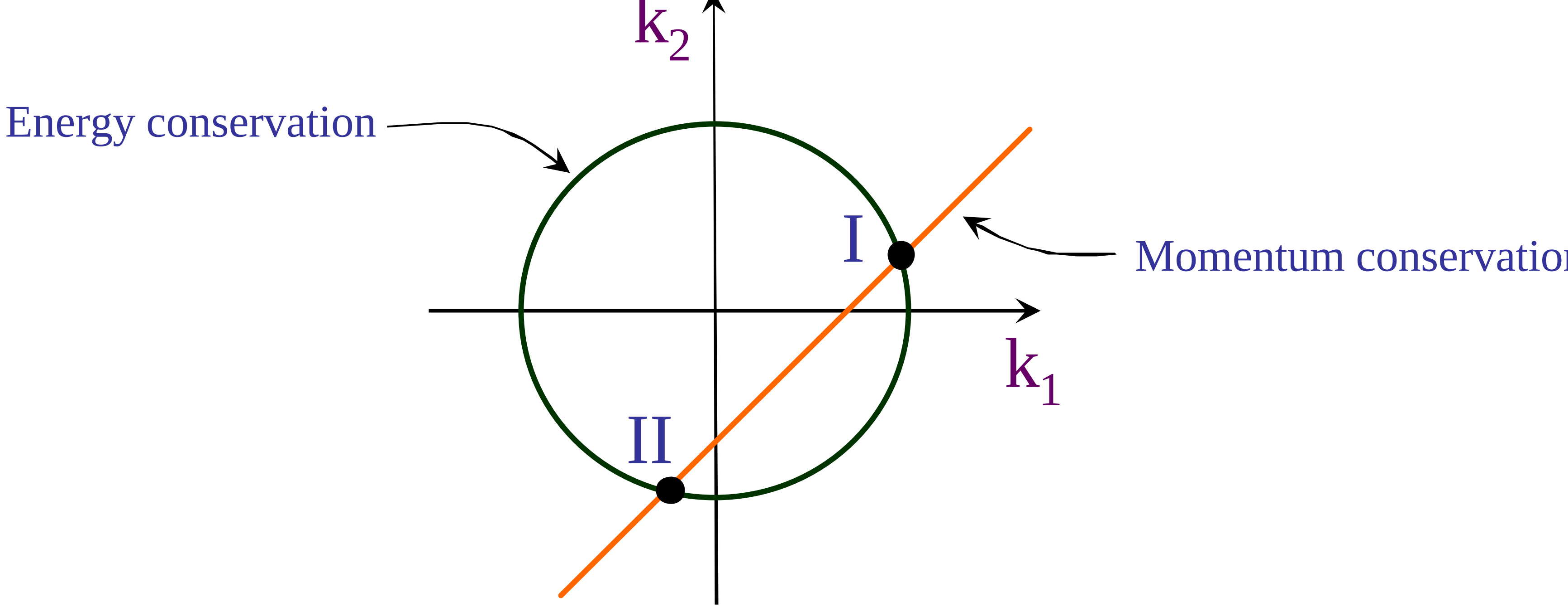
- R<<L
- scattering does not affect momenta of the particles
- no inelastic processes

2? 2 scattering in 2d



Energy and momentum conservation:

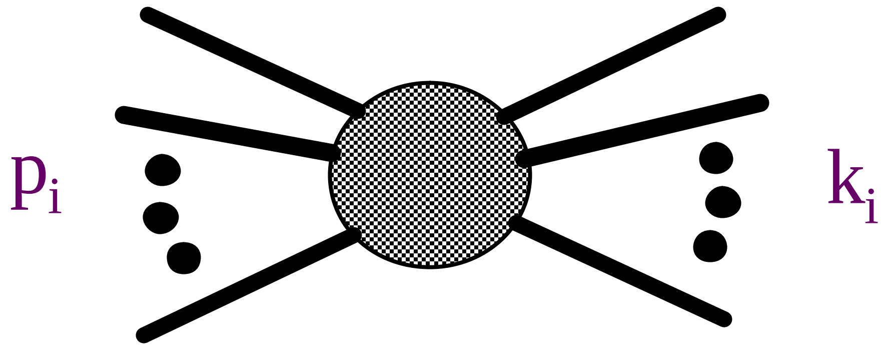
$$\begin{aligned} k_1 + k_2 &= p_1 + p_2 \\ \varepsilon(k_1) + \varepsilon(k_2) &= \varepsilon(p_1) + \varepsilon(p_2) \end{aligned}$$



I: $k_1=p_1$, $k_2=p_2$ (transition)

II: $k_1=p_2$, $k_2=p_1$ (reflection)

n? n scattering



$$
\begin{aligned}
k_1 + \ldots + k_n &= p_1 + \ldots + p_n \\
\varepsilon(k_1) + \ldots + \varepsilon(k_n) &= \varepsilon(p_1) + \ldots + \varepsilon(p_n)
\end{aligned}
$$

2 equations for n unknowns

⇓

(n-2)-dimensional phase space

Unless there are extra conservation laws!

Integrability:

$$Q_I(k_1) + \ldots + Q_I(k_n) = Q_I(p_1) + \ldots + Q_I(p_n)$$
$$I = 1 \ldots \infty$$

- No phase space: $k_i = p_{\sigma(i)}, \ \sigma \in S_n$
- No particle production (all 2? many processes are kinematically forbidden)

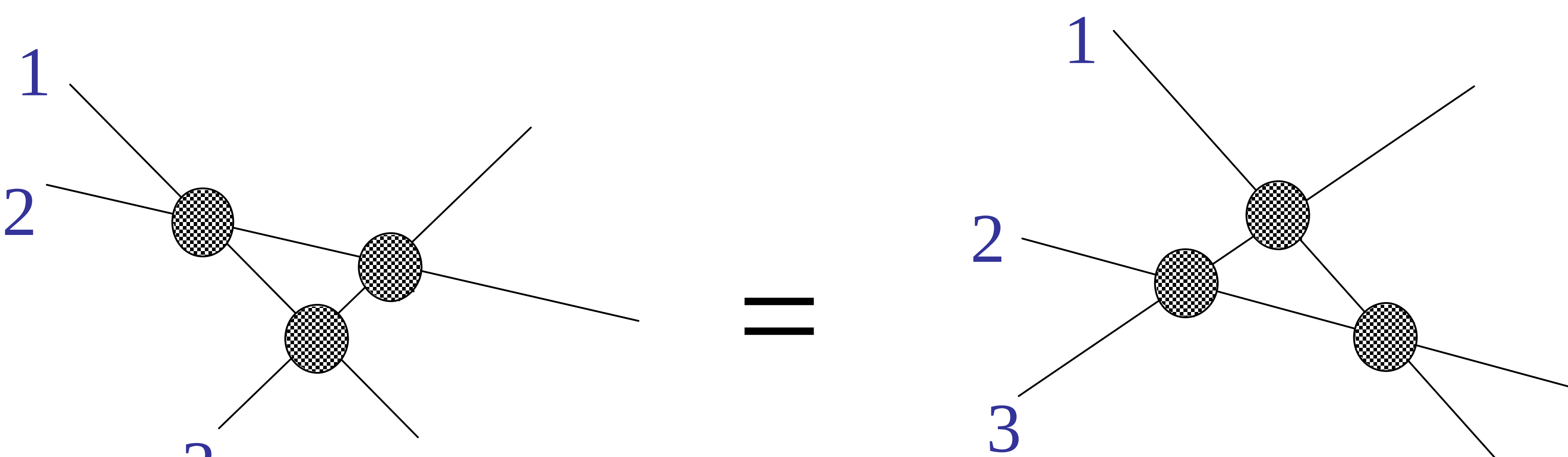
$$\text{permutation} = \prod (\text{transpositions})$$

Factorization:

$$S_{n\to n} = S_{2\to 2}(i_1, i_1+1)\ldots S_{2\to 2}(i_l, i_l+1)$$

Consistency condition (Yang-Baxter equation):

$$S(1,2)S(1,3)S(2,3) = S(2,3)S(1,3)S(1,2)$$



Integrability + Locality ⇒ Bethe ansatz

Strategy:

find the dispersion relation (solve the one-body problem):

$$\varepsilon = \varepsilon(p)$$

find the S-matrix (solve the two-body problem):

$$S(p, p') = e^{\,i\Delta(p,p')}$$

Bethe equations ⇒ full spectrum

find the true ground state

Non-linear Schrödinger model

Second quantized:

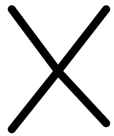
$$\mathcal{L} = i\phi^\dagger \overleftrightarrow{\partial}_t \phi - |\partial_x \phi|^2 - c\,|\phi|^4$$

First quantized:

$$|x_1 \ldots x_n\rangle = \phi^\dagger(x_1)\ldots\phi^\dagger(x_n)\,|0\rangle$$

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} + c \sum_{i \neq j} \delta(x_j - x_j)$$

S-matrix

S = 1 + 

$$S = 1 + \frac{2ic}{p - p'}$$

Bethe equations:

$$e^{ip_k L} = \prod_{j \neq k} \frac{p_k - p_j - ic}{p_k - p_j + ic}$$

Lieb,Liniger'

Landau-Lifshitz sigma-model

$$\mathcal{L} = -\frac{1}{2}\int_0^1 d\xi\, \mathbf{n}\cdot[\partial_\xi \mathbf{n} \times \partial_t \mathbf{n}] - \frac{1}{4}\left(\partial_x \mathbf{n}\right)^2, \qquad \mathbf{n}^2 = 1$$

WZ term

Low-energy effective theory of Heisenberg ferromagnet

Describes one-loop anomalous dimensions of operators $\mathrm{tr}\, Z^{L-M} W^M$ in N=4 SYM in the limit $L, M \to \infty,\ \frac{M}{L} - \text{fixed}$

Minahan,Z.'0

Decribes fast-moving strings on $S^3 x R^1$ (SU(2) sector of string theory on $AdS_5 x S^5$)

Kruczenski'03

Exact S-matrix (of magnons):

$$S(p,p') = \frac{p' - p + ipp'}{p' - p - ipp'}$$

Klose,Z.'06

Bethe equations:

$$\mathrm{e}^{\,ip_j L} = \prod_{k\neq j} \frac{p_k - p_j + ip_k p_j}{p_k - p_j - ip_k p_j}$$

The vacuum turns out to be unstable

– there are bound states with E<0.

AAF model

$$= \int d^2x \left[\bar{\psi}\left(i\gamma^a\partial_a - m\right)\psi + \frac{g}{4m^2}\,\varepsilon^{ab}\left(\bar{\psi}\partial_a\psi\,\bar{\psi}\gamma^3\partial_b\psi - \partial_a\bar{\psi}\psi\,\partial_b\bar{\psi}\gamma\right.\right.$$

$$\left. - \frac{g^2}{8m^4}\,\varepsilon^{ab}\left(\bar{\psi}\psi\right)^2\partial_a\bar{\psi}\gamma^3\partial_b\psi \right]$$

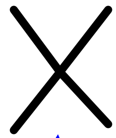
Alday,Arutyunov,Frolov'05

su(1|1) subsector of AdS_5xS^5 string theory:

2 world-sheet fermions,

other d.o.f. are gauge-fixed or frozen

First order of perturbation theory

S = 1 +  + ...

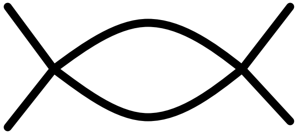
Born approximation
for Dirac equation

$$S = 1 + 2ig\sinh(\theta - \theta') + \ldots$$

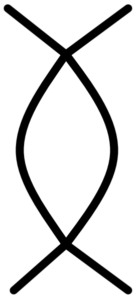
$$p = m\sinh\theta$$
$$\varepsilon = m\cosh\theta$$

θ - rapidity

One loop

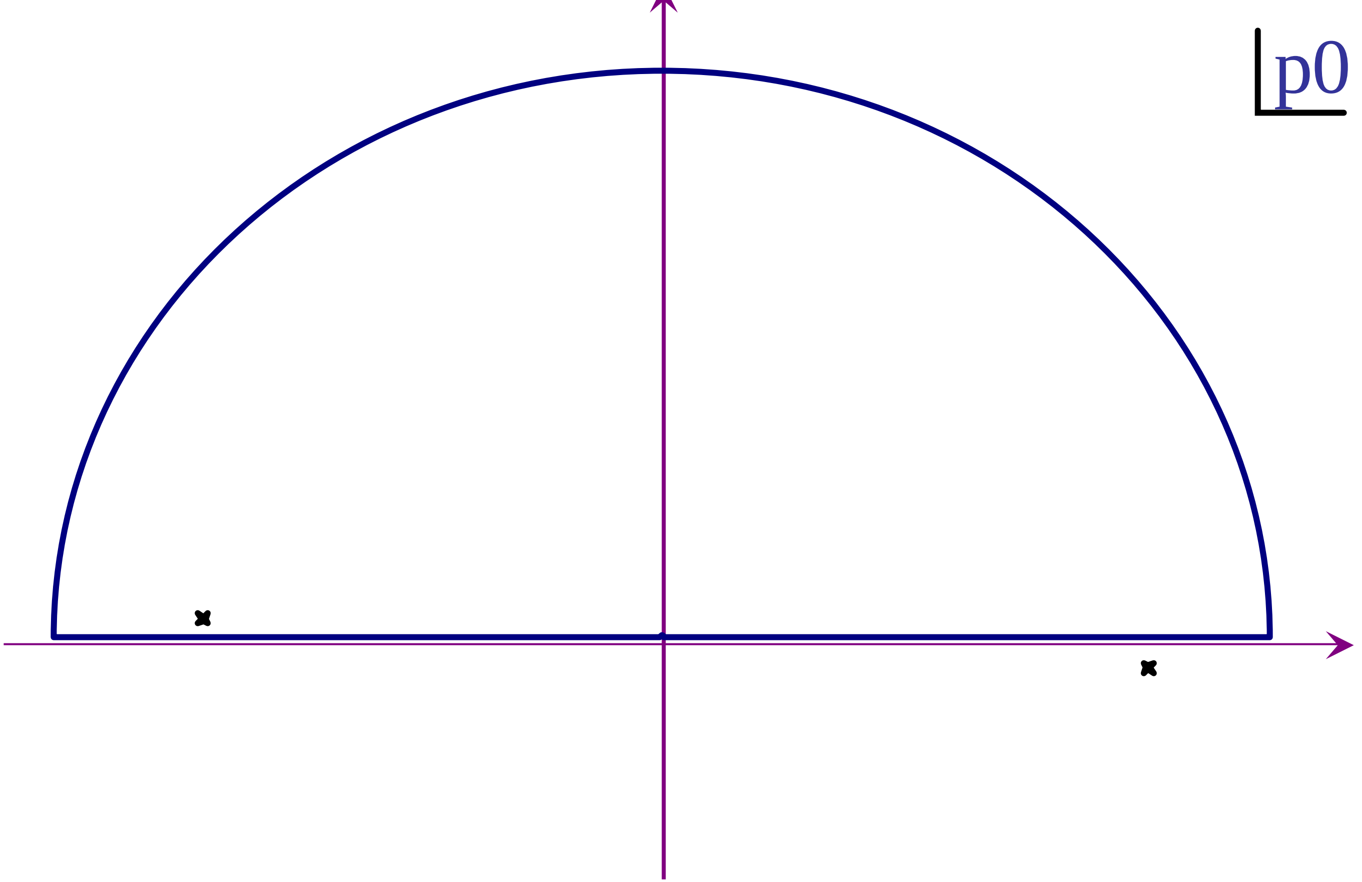


first iteration of Lippmann-Schwinger equation

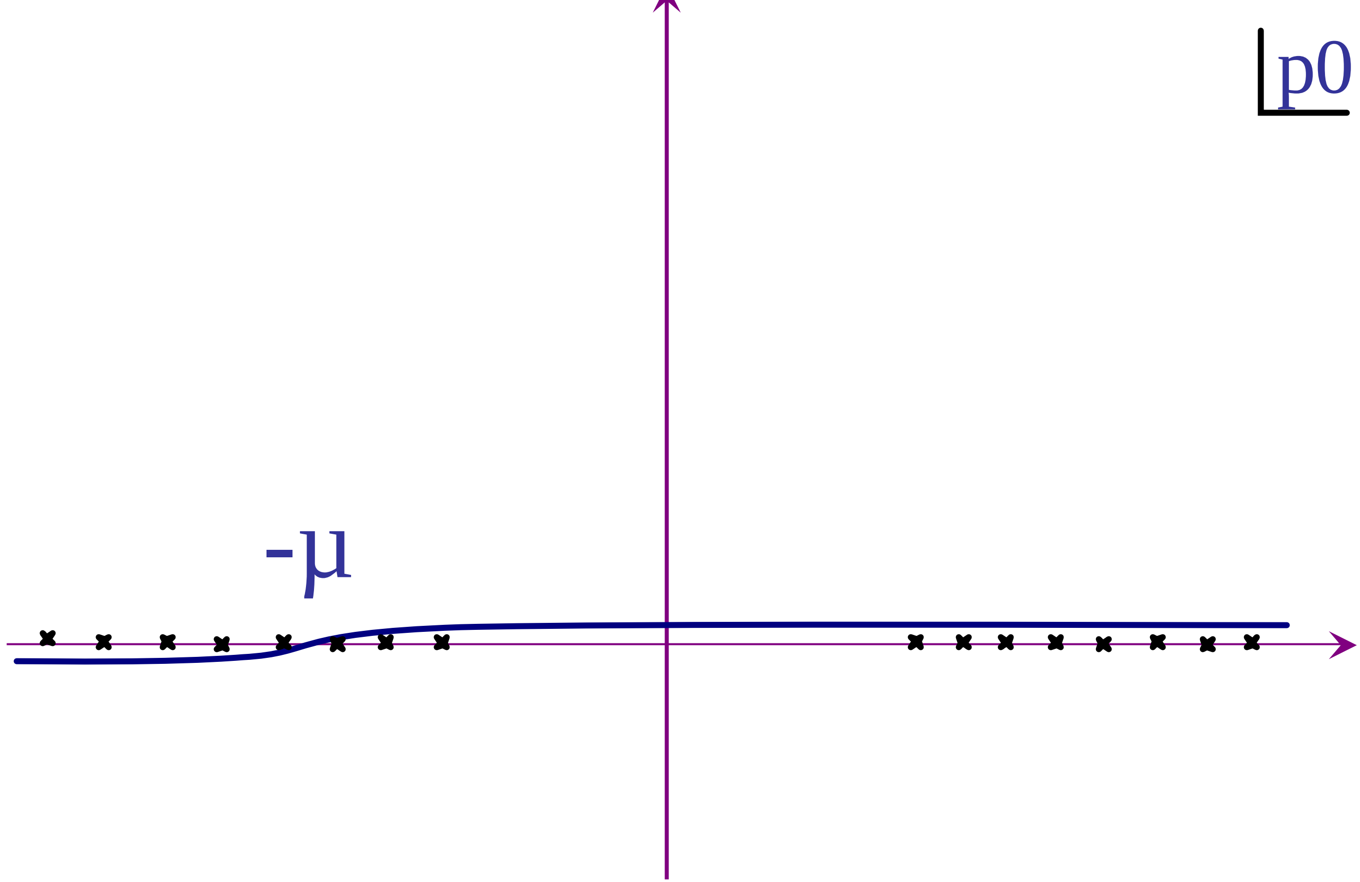


correction to the potential due to vacuum polarization

$$V_{1-\text{loop}}(x) \sim g^2 \mathrm{e}^{-2m|x|}$$



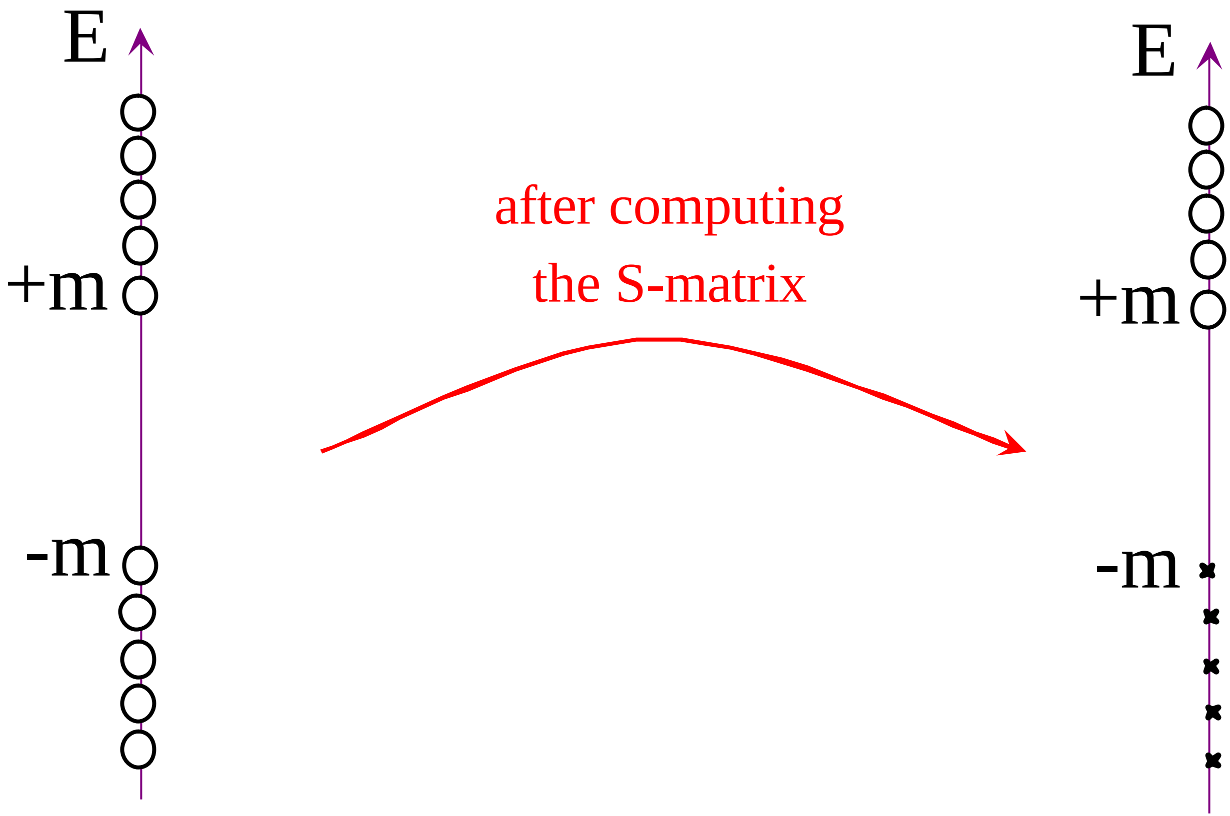
$$S(p) = \frac{\not{p} + m}{(p_0 + i\epsilon)^2 - p_1^2 - m^2}$$



μ – chemical potential

μ? -8 ⇒ All poles are below the real axis.

Empty Fermi sea: Physical vacuum:

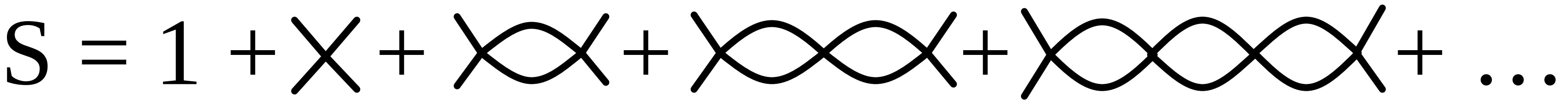


$$\psi(x)\,|0\rangle = 0$$

Berezin,Sushko’65; Bergknoff,Thaker’79; Korepin’79

Empty Dirac sea:

1. the potential remains local
2. S-matrix is the sum of bubble diagrams



$$S(\theta,\theta') = \frac{1 + ig\,\sinh(\theta-\theta')}{1 - ig\,\sinh(\theta-\theta')}$$

Klose, Z.'06

coincides with the S-matrix of breathers in SG

Bethe ansatz

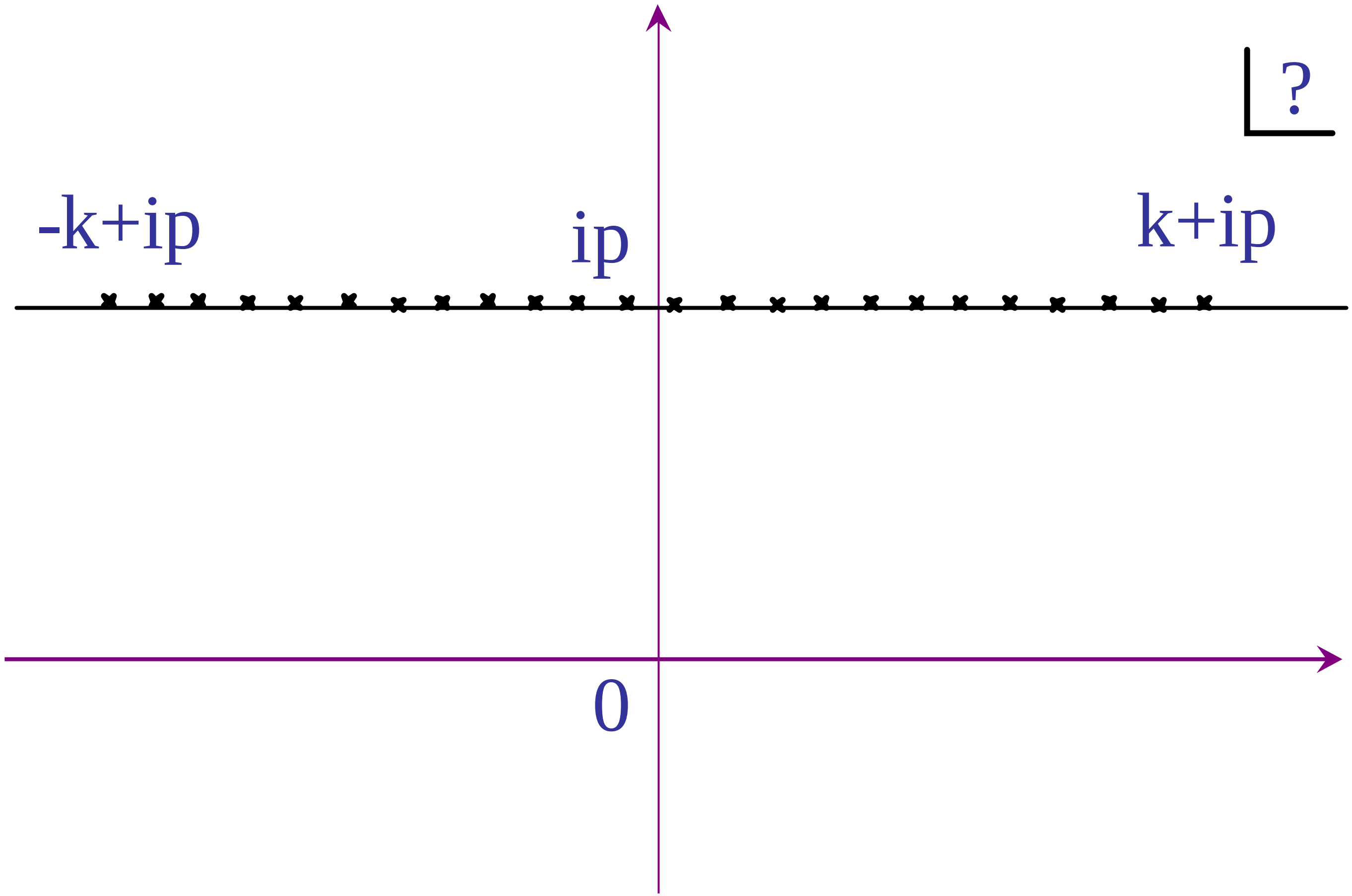
$$e^{imL\sinh\theta_j} = \prod_{k\neq j} \frac{1 + ig\sinh(\theta_j - \theta_k)}{1 - ig\sinh(\theta_j - \theta_k)}$$

$$E = m\sum_j \cosh\theta_j$$

Im $?_j$=0: positive-energy states

Im $?_j$=p: negative-energy states

Ground state



$\Lambda = m\,e^{k}$ - UV cutoff

Mass renormalization: $M = m\Lambda^{\nu(g)}$

Ground state in thermodynamic limit

$$m \cosh \alpha = 2\pi \rho(\alpha) + \int_{-\infty}^{+\infty} d\bar{\alpha}\, \rho(\bar{\alpha})\, \frac{2g \cosh(\alpha - \bar{\alpha})}{1 + g^2 \sinh^2(\alpha - \bar{\alpha})}$$

$$E_{\rm vac} = -L \int_{-k}^{k} d\alpha\, \rho(\alpha)\, \cosh \alpha$$

- mass renormaliztion
- physical spectrum
- physical S-matrix

- Weak-coupling $(-1<g<1)$:

  Non-renormalizable

  (anomalous dimension of mass is complex)

- Strong attraction $(g>1)$

  Unstable

  (Energy unbounded below)

- Strong repulsion $(g<-1)$:

  Spectrum consists of fermions and anti-fermions

  with non-trivial scattering

AdS/CFT

- S-matrix is virtually known Beisert'05; Janik'06; …
- Bethe equations are nearly known

  Kazakov,Marshakov,Minahan,Z.'04; Arutyunov,Frolov,Staudacher'04;

  Beisert,Kazakov,Sakai,Z.'05; Beisert,Staudacher'05;

  Beisert,Tseytlin'05; Hernadez,Lopez'06; …

- Elementary excitations are solitons* (giant magnons) Hofman,Maldacena'06; …
- A lot of evidenence from SYM

  Minahan,Z.'02; Beisert,Kristjansen,Staudacher'03;
  Beisert,Staudacher'03; Beisert,Dippel,Staudacher'04; Staudacher'04;
  Rej,Serban,Staudacher'05

* which means that there are finite-size correction to the

$AdS_3 x S^1$

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\theta^2 + d\phi^2$$

t - global time

ρ - radial coordinate in AdS

θ - angle in AdS

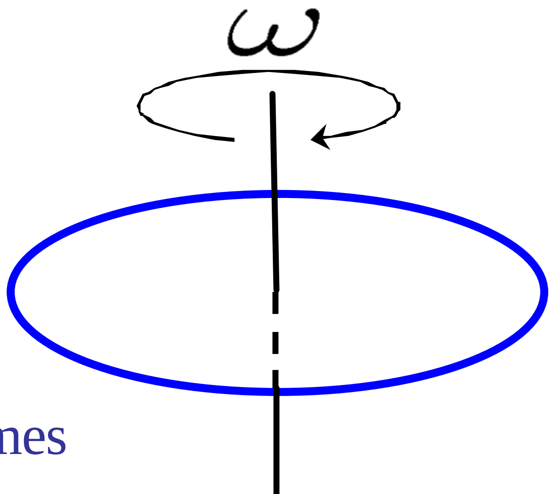
ϕ - angle on S^5

Rigid string solution

$$\begin{aligned} \rho &= \text{const} \\ t &= \kappa\tau \\ \theta &= \omega\tau + k\sigma \\ \phi &= \omega\tau + m\sigma \end{aligned}$$

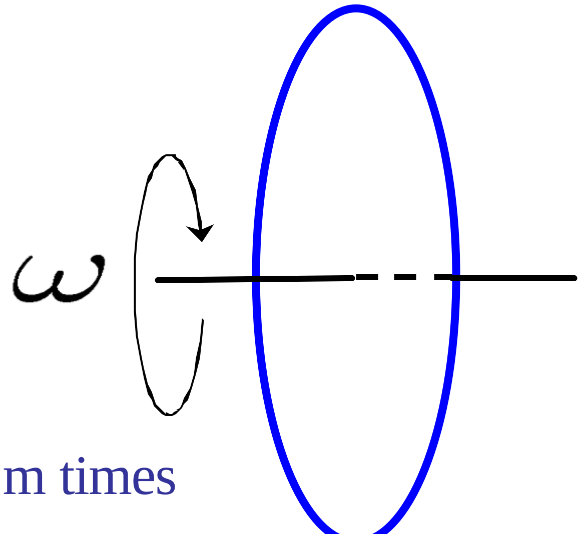
Arutyunov,Russo,Tseytlin'03

AdS_5

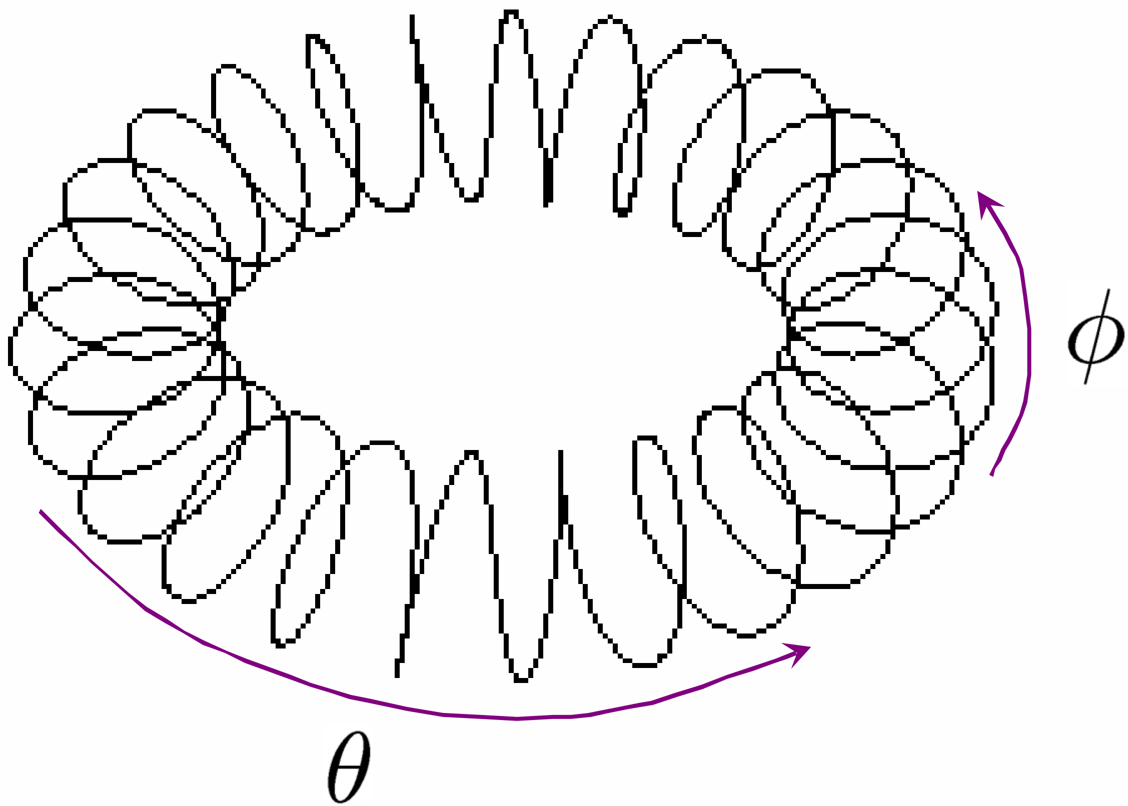


winds k times

S^5



winds m times



ϕ
θ

Quantum numbers

$$kS + mJ = 0$$

$$2\kappa \frac{E}{\sqrt{\lambda}} - \kappa^2 = 2\sqrt{\kappa^2 + k^2}\, \frac{S}{\sqrt{\lambda}} + \frac{J^2}{\lambda} + m^2$$

$$\frac{E}{\sqrt{\lambda}} = \frac{\kappa}{\sqrt{\kappa^2 + k^2}}\, \frac{S}{\sqrt{\lambda}} + \kappa$$

E - energy

S - AdS spin

J - angular momentum on S^5

Quantum corrections

$$= \sqrt{\lambda} E_0 + E_1 + O\left(\lambda^{-1/2}\right) \qquad E_1 = \frac{1}{2}\sum_n (-1)^{F_n}\omega_n$$

string fluctuation frequencies

Frolov,Tseytlin'03

xplicitly,

$$= \frac{1}{2\kappa}\left[4\nu + 2\kappa + 2\sqrt{\kappa^2 + (1+r_1^2)k^2} - 8\sqrt{c^2 + a^2}\right]$$

$$+\frac{1}{\kappa}\sum_{n=1}^{\infty}\left[4\sqrt{n^2+\nu^2} + 2\sqrt{n^2+\kappa^2} - 4\sqrt{(n+\gamma)^2+\alpha^2} - 4\sqrt{(n-\gamma)^2+\alpha^2} + \frac{1}{2}\sum_{I=1}^{4}\operatorname{sgn} C_I^{(n)} \omega\right.$$

$$= \sqrt{\frac{J^2}{\lambda} - m^2} \qquad (\omega_{I,n}^2 - n^2)^2 + 4r_1^2\kappa^2\omega_{I,n}^2 - 4(1+r_1^2)\left(\sqrt{\kappa^2+k^2}\,\omega_{I,n} - kn\right)^2 =$$

$$= \sqrt{\frac{\kappa^2+\nu^2}{2}} \qquad C_I^{(n)} = (\omega_I^2 - n^2)\prod_{J\neq I}(\omega_I - \omega_J)$$

$$= \frac{\kappa^2 - 2m^2 - \nu^2}{2k^2}$$

$$= \frac{1}{2}\kappa\left[1 + \frac{2k^2(1+r_1^2)}{\kappa^2-\nu^2}\right]\sqrt{\frac{\kappa^2-\nu^2-2k^2r_1^2}{2(\kappa^2+k^2)}}$$

Park,Tirziu,Tseytlin'05

Bethe equations

azakov,Marshakov,Minahan,Z.'04; Kazakov,Z.'04

Classical Bethe equation

$$2\fint dy\, \frac{\rho(y)}{x-y} - 2\pi k + 2\pi\left(\frac{\frac{J}{\sqrt{\lambda}}+m}{x-1}+\frac{\frac{J}{\sqrt{\lambda}}-m}{x+1}\right)$$

$$= \frac{4\pi^2}{\sqrt{\lambda}}\,\frac{x^2\,\rho'(x)\coth\pi\rho(x)}{x^2-1} + \frac{1}{2\sqrt{\lambda}}\int\frac{dy\,\rho(y)\theta(x,y)}{x-y}$$

Anomaly

Quantum correction to scattering phas

Kazakov'04;Beisert,Kazakov,Sakai,Z.'05

Beisert,Tseytlin'05

Beisert,Tseytlin,Z.'05; Schäfer-Nameki,Zamaklar,Z.'05

$$,y) = \log\frac{y-1}{y+1}\log\frac{x-1/y}{x-y} + \mathrm{Li}_2\frac{\sqrt{y}-1/\sqrt{y}}{\sqrt{y}-\sqrt{x}} - \mathrm{Li}_2\frac{1/\sqrt{y}+\sqrt{y}}{\sqrt{y}-\sqrt{x}} + \mathrm{Li}_2\frac{\sqrt{y}-1/\sqrt{y}}{\sqrt{y}+\sqrt{x}} - \mathrm{Li}_2\frac{\sqrt{y}+1/\sqrt{}}{\sqrt{y}+\sqrt{x}}$$

Hernandez,Lopez'06; Arutyunov,Frolov'0

Internal length of the string is $\dfrac{2\pi J}{\sqrt{\lambda}}$

Perturbative SYM regime:

$$\frac{\lambda}{J^2} \ll 1$$

(string is very long)

String and Bethe calculations agree to all orders in $\dfrac{\sqrt{\lambda}}{J}$

Schäfer-Nameki,Zamaklar,Z.'05; Beisert,Tseytlin'05; Hernandez,Lopez'06

Decompatification limit

$$\frac{J}{\sqrt{\lambda}} \to \infty, \qquad k \to \infty, \qquad \frac{J}{k\sqrt{\lambda}} - \text{finite}$$

Minahan,Tirziu,Tseytlin'06

string becomes infinitely long

$$\begin{aligned} E_1^{\text{string}} &= 0 \\ E_1^{\text{Bethe}} &= 0 \end{aligned}$$

agree

Large winding limit

$$k \to \infty, \qquad \frac{J}{\sqrt{\lambda}} - \text{finite}$$

Schäfer-Nameki,Zamaklar,Z.'05

string stays finite

$$E_1^{\text{string}} = \frac{2F\left(0, \sqrt{\mathcal{J}^2 - m^2}\right) + 2F\left(0, \mathcal{J} + m\right) - 4F\left(\left\{\frac{|k|}{2}\right\}, \sqrt{\mathcal{J}(\mathcal{J}+m)}\right)}{\mathcal{J} + m}$$
$$+\sqrt{m\mathcal{J}} + (\mathcal{J}+m)\ln\frac{\sqrt{\mathcal{J}+m}}{\sqrt{\mathcal{J}} + \sqrt{m}} - m$$

disagree

$$E_1^{\text{Bethe}} = \frac{\mathcal{J}+m}{2}\ln\frac{\sqrt{\mathcal{J}}+\sqrt{m}}{\sqrt{\mathcal{J}}-\sqrt{m}} - \sqrt{m\mathcal{J}}$$

$$\mathcal{J} = \frac{J}{\sqrt{\lambda}} \qquad F(\beta,\alpha) \equiv \sqrt{\alpha^2+\beta^2} - \beta^2 + \alpha^2 \int_0^\infty \frac{d\xi}{\mathrm{e}^{\,\xi}-1}\left(\frac{2J_1(\alpha\xi)}{\alpha\xi}\cosh\beta\xi - 1\right)$$

Spectrum of AdS/CFT:

$$e^{ip_k L} = \prod_{j \neq k} S(p_j, p_k)$$

Remaining questions:

What is $S(p, p')$?

What is e^{ipL} ?

Why are they equal??