Minimal solution of the crossing equation

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based on 0904.4929

We solve explicitly the crossing equation under sufficiently general assumptions on the structure of the dressing phase. We obtain the BES/BHL dressing phase as a minimal solution of the crossing equation.

Historical outlines:

• Bootstrap approach to the solution of the AdS/CFT spectral problem. Fixes Bethe equations up to a scalar (dressing) factor

M. Staudacher 0412188
N. Beisert and M. Staudacher 0504190

• Algebraic curve solution and nontrivial scalar factor

V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo 0402207
G. Arutyunov, S. Frolov and M. Staudacher 0406256
R. Hernandez and E. Lopez 0603204

• Crossing equation

R. A. Janik 0603038

• Strong coupling expansion

N. Beisert, R. Hernandez and E. Lopez 0609044

• Exact conjecture

N. Beisert, B. Eden and M. Staudacher 0610251

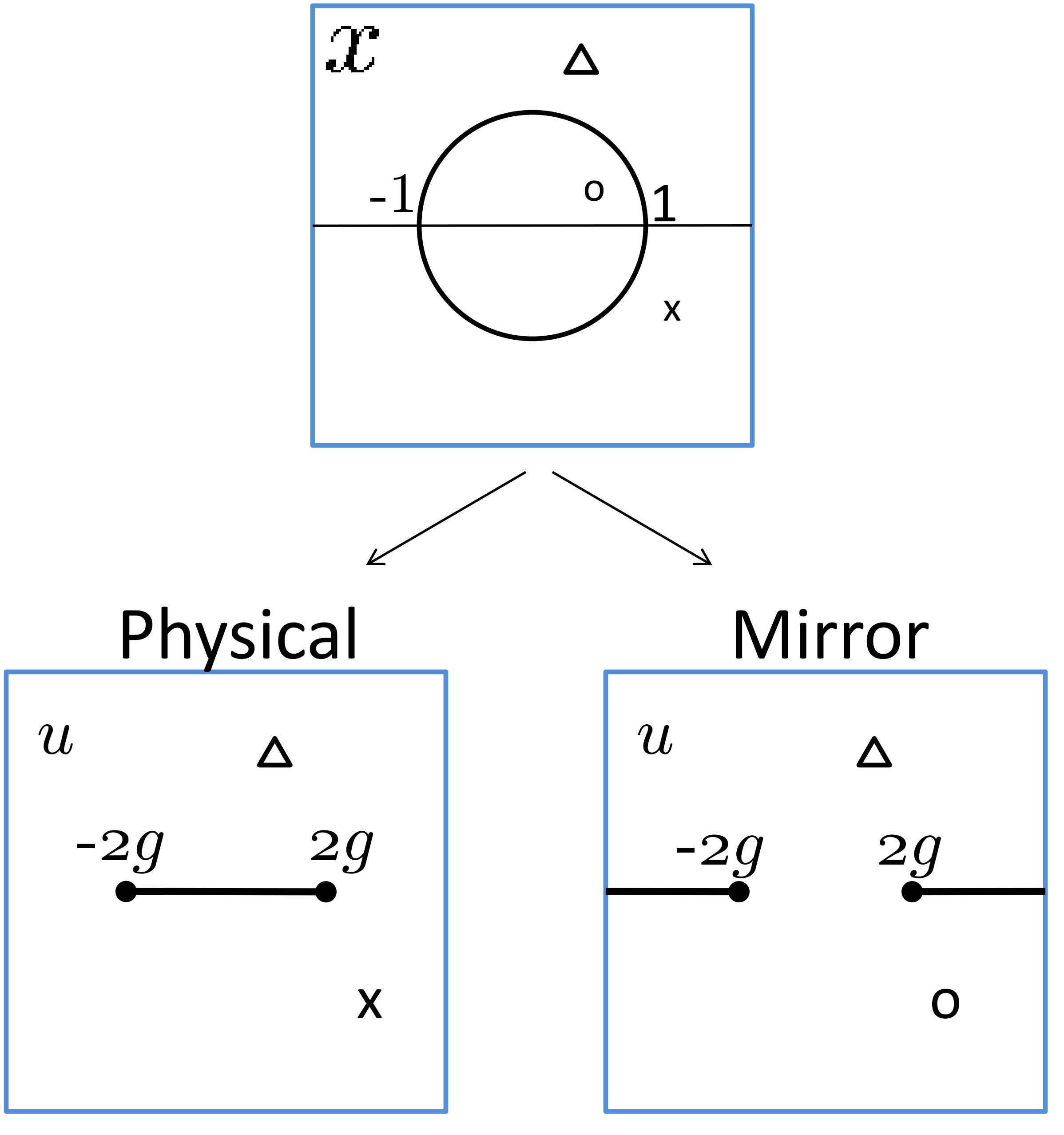
• Useful integral representations

I. Kostov, D. Serban and D. V. 0703031
N. Dorey, D. M. Hofman and J. M. Maldacena 0703104

For more references see **0904.4929**

Jukowsky map:

$$\frac{u}{g} = x + \frac{1}{x}$$

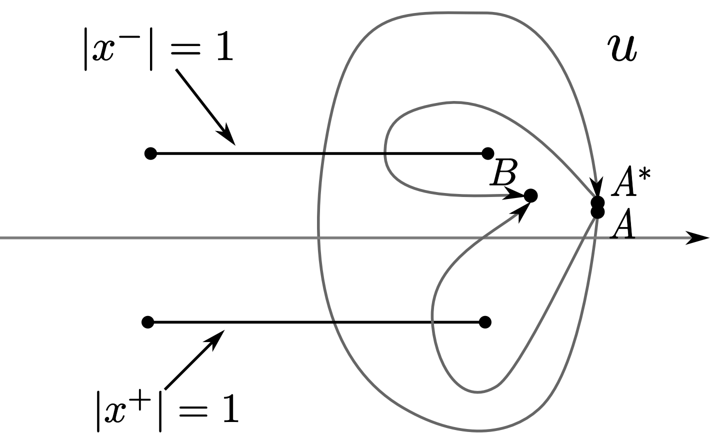


$$x = \frac{u}{2g}\left(1+\sqrt{1-\frac{4g^2}{u^2}}\right) \qquad x = \frac{1}{2g}\left(u + i\sqrt{4g^2-u^2}\right)$$

We also use notation: $x^{\pm} \equiv x[u \pm i/2]$

Crossing equation:

• For the case of the dressing phase in the physical theory we take the definition of the branch cut such that $|x|>1$



Crossing equation relates dressing factor and its analytic continuation.

$$\sigma[u,v]\sigma_{AA^*}[u,v]=\frac{y^-}{y^+}\frac{x^--y^+}{x^+-y^+}\frac{1-\frac{1}{x^-y^-}}{1-\frac{1}{x^+y^-}},$$

Assumptions on the structure of the dressing phase::

• Decomposition in terms of χ: $\sigma[u,v]=e^{i\theta[u,v]}$

$$\theta[u,v]=\chi[x^+,y^-]-\chi[x^-,y^-]-\chi[x^+,y^+]+\chi[x^-,y^+]$$

$$\chi[x,y]=-\chi[y,x]$$

- Comes from decomposition in terms of charges

We will use additional functions

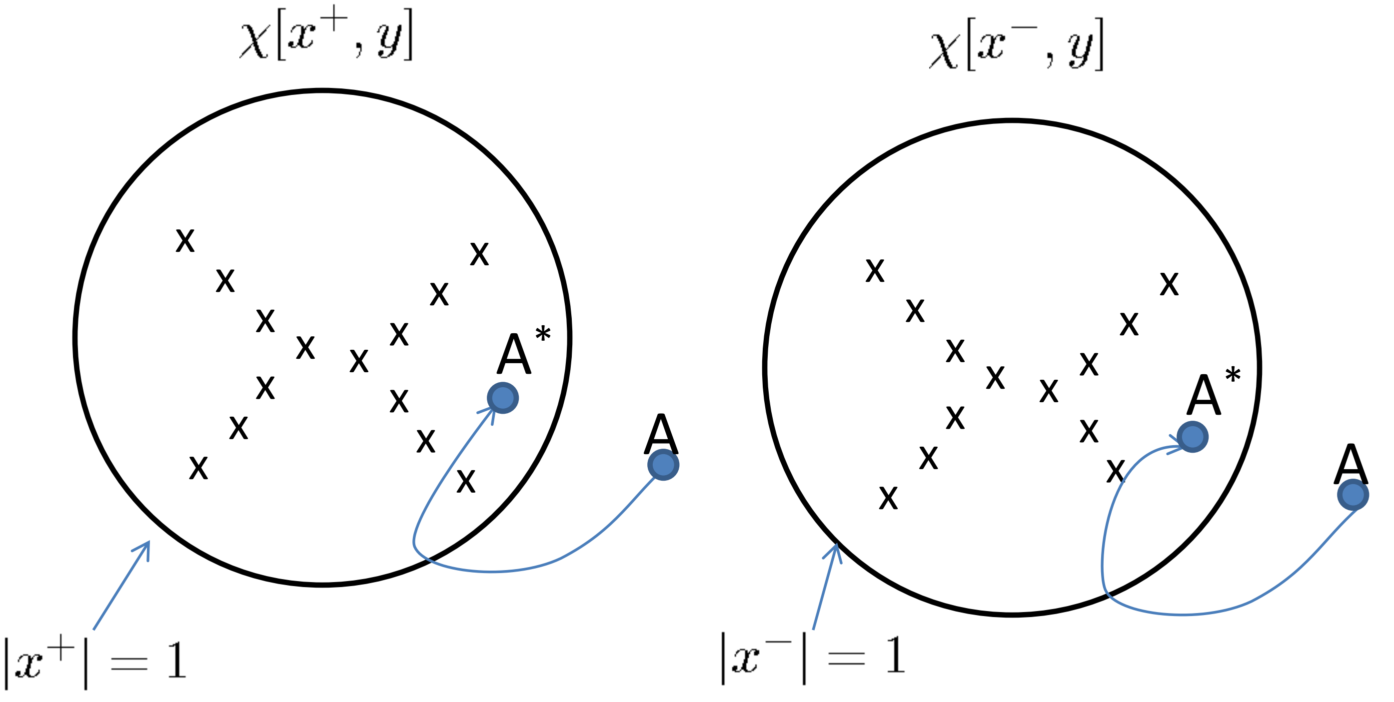
$$\sigma_1[x,v]=e^{i\chi[x,y^-]-i\chi[x,y^+]} \qquad \sigma_2[x,y]=e^{i\chi[x,y]}$$

• χ is analytic for |x|>1 (minimality condition).

• χ as a function of u does not have branch points except those that are explicitly required by the crossing equation. The required branch points are of the square root type. This is a condition of compatibility of the analytical structure of the dressing phase and analytical structure of the Bethe equations. It can be compared with demand for the S-matrix in relativistic theories to be meroporphic function of θ.

Passing to the point B:

Tricky point with the crossing equation:



The function χ[x,y] may have branch points inside the unit circle. Therefore the analytic continuation via the contour AA* may bring $\chi[x^+,y]$ and $\chi[x^-,y]$ (and it does actually) to different sheets. Dealing with the functions on the different sheets is not easy.

Simplification: analytically continue the equation to the point B.

$$\sigma_{A^*B}[u,v]\sigma_{AB}[u,v] = \frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^-y^-}}\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}}$$

Solution:

- Since at the point B the functions $\chi[x^+,y]$ and $\chi[x^-,y]$ are on the same sheet, one can write

$$\sigma_{A^*B} = \frac{\sigma_1[1/x^+, v]}{\sigma_1[x^-, v]} \qquad \sigma_{AB} = \frac{\sigma_1[x^+, v]}{\sigma_1[1/x^-, v]}.$$

- We introduce shift operator and a shorthand notation:

$$D \equiv e^{\pm \frac{i}{2}\partial_u} : f[u] \mapsto f[u \pm i/2]$$

$$f^{D^{\pm 1}} \equiv e^{D^{\pm 1} \log[f]}$$

- In this notation crossing equation is written as:

$$\left(\sigma_1[x, v]\sigma_1[1/x, v] \right)^{D-D^{-1}} = \left(\frac{x - \frac{1}{y^+}}{x - \frac{1}{y^-}} \right)^{D+D^{-1}} .$$

- And is solved by:

$$\sigma_1[x, v]\sigma_1[1/x, v] = \left(\frac{x - \frac{1}{y^+}}{x - \frac{1}{y^-}} \right)^{-\frac{D^2}{1-D^2} + \frac{D^{-2}}{1-D^{-2}}}$$

The solution was chosen to satisfy the demand that $\sigma[u,v]$ has at most square root branch points

Further simplification leads to

$$\sigma_2[x,y]\sigma_2[1/x,y] = \left(\frac{x-\frac{1}{y}}{\sqrt{x}}\right)^{-\frac{D^2}{1-D^2}+\frac{D^{-2}}{1-D^{-2}}}$$

$$\sigma_2[x,y]\sigma_2[1/x,y]\sigma_2[x,1/y]\sigma_2[1/x,1/y] =$$

$$= (u-v)^{-\frac{D^2}{1-D^2}+\frac{D^{-2}}{1-D^{-2}}} = \frac{\Gamma[1-i(u-v)]}{\Gamma[1+i(u-v)]}$$

Obtained Riemann-Hilbert problem is solved by

$$\chi[x,y] = -i\tilde{K}_u\tilde{K}_v \log\left[\frac{\Gamma[1-i(u-v)]}{\Gamma[1+i(u-v)]}\right]$$

$$(\tilde{K}\cdot F)[u] = \int_{-2g+i0}^{2g+i0} \frac{dw}{2\pi i}\,\frac{x-\frac{1}{x}}{z-\frac{1}{z}}\,\frac{1}{w-u}F[w]$$

This is a BES/BHL dressing phase

Important properties of the kernel K:

$$(\tilde{K} \cdot F)[u + i0] + (\tilde{K} \cdot F)[u - i0] = F[u], \; u^2 < 4g^2.$$

$$\tilde{K}_u \cdot \log[v - u] = \log\left[\frac{y - \frac{1}{x}}{\sqrt{y}}\right]$$

A nice way to write interaction terms in Bethe equations:

$$\prod_k \frac{1 - \frac{1}{xy_k^+}}{1 - \frac{1}{xy_k^-}} = \prod_k (u - v_k)^{-\tilde{K}_u (D - D^{-1})}$$

Relation to the mirror theory:

It is useful to introduce the “mirror” kernel, which is defined by choosing a complementary contour of integration

$$(\tilde{K}_m \cdot F)[u] = \int_{\mathbb{R}/[-2g,2g]} \frac{dw}{2\pi i} \, \frac{x-\frac{1}{x}}{z-\frac{1}{z}} \frac{1}{w-u} F[w]$$

If we put everywhere $\tilde{K}_m$ instead of $\tilde{K}$, we will get expressions for the mirror theory.

Relation between mirror and physical kernels is the following. Let us fix $Im[u]$>0. Then

$$(\tilde{K}_m \cdot F)[u] = (1-\tilde{K}) \cdot F,$$

if F has singularities only in the lower half plane.

$$(\tilde{K}_m \cdot F)[u] = \tilde{K} \cdot F,$$

if F has singularities only in the upper half plane

These properties allow us to establish a simple relation between physical and mirror dressing phases:

$$i\theta_{phys}[u,v] = -(D_u - D_u^{-1})(D_v - D_v^{-1}) K_{uu'} K_{vv'} \times$$

$$\times \left(\frac{D_{u'}^2}{1-D_{u'}^2} - \frac{D_{u'}^{-2}}{1-D_{u'}^{-2}} \right) \log[u'-v']$$

For Im[u]>0 and Im[v]>0 we have

$$i\theta_{phys}[u,v] = (D_u - D_u^{-1})(D_v - D_v^{-1})\tilde{K}_{m,uu'}\tilde{K}_{m,vv'}\times$$

$$\times\left(\frac{D_{u'}^2}{1-D_{u'}^2} - \frac{D_{u'}^{-2}}{1-D_{u'}^{-2}}\right)\log[u'-v'] - \log\left[\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^-y^+}}\right]$$

or

$$\sigma_{phys}^{-1} = \sigma_{mirror}\frac{1-\frac{1}{x^+y^-}}{1-\frac{1}{x^-y^+}}$$

[these relations appeared in works by and Gromov, Kazakov, Vieira and Arutyunov,Frolov]

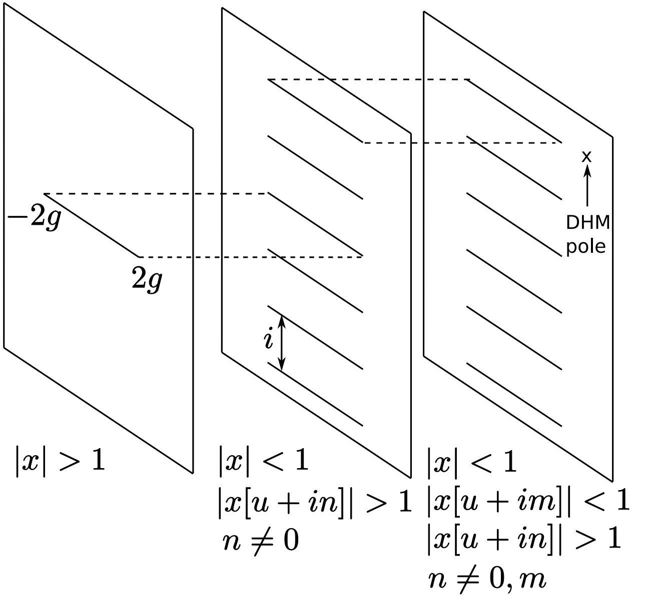
Analytical properties of the dressing phase:

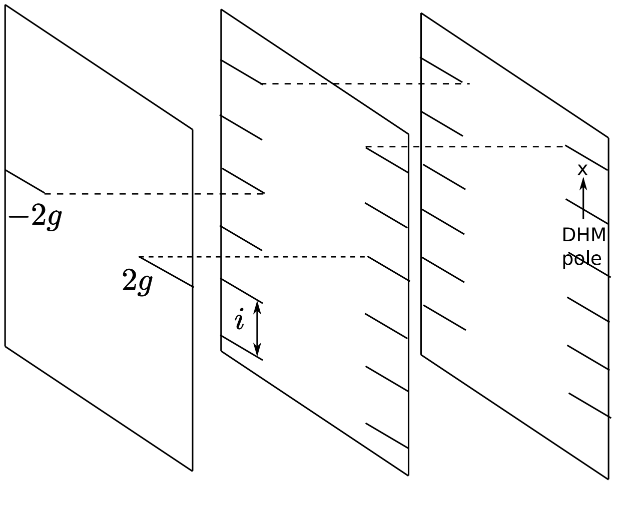
One can use the following formula to read analytical properties

$$\sigma_2[1/x,y] = \frac{1}{\sigma_2[x,y]} \left(\frac{x-\frac{1}{y}}{\sqrt{x}}\right)^{-\frac{D^2}{1-D^2}+\frac{D^{-2}}{1-D^{-2}}} =$$

$$= \frac{1}{\sigma_2[x,y]} \prod_{n=1}^{\infty}\left(\frac{x[u+in]-\frac{1}{y}}{\sqrt{x[u+in]}}\right)^{-1}\left(\frac{x[u-in]-\frac{1}{y}}{\sqrt{x[u-in]}}\right)$$

Analytical structure of σ_2





Analytical structure of $\sigma_{2,mirr}$