Conformal Models in Discrete Differential Geometry

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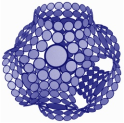


DFG Research Unit “Polyhedral Surfaces”

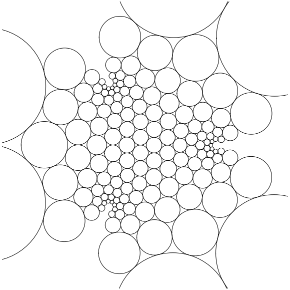
Discrete Differential Geometry

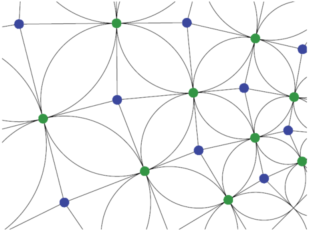
- **Aim:** Development of discrete equivalents of the geometric notions and methods of differential geometry. The latter appears then as a limit of refinements of the discretization.
- **Question:** Which discretization is the best one?
- **Main messages**:
  - Discretize the whole theory, not just the equations. The discrete geometric theory is as rich as the analogous theory for the smooth problem.
  - Differential geometry from incidence theorems of projective geometry
  - Existence theorems of classical theory can be made constructive when the discretization is proper
  - Important for applications: computer graphics, architecture
  - Same models in physics

- circle packings and circle patterns
- integrable circle patterns
- conformal equivalent simplicial metrics
- and hyperbolic geometry, ...

Circle Patterns as Discrete Complex Analysis

Circle packings - discrete analogs of conformal maps [Thurston '85]

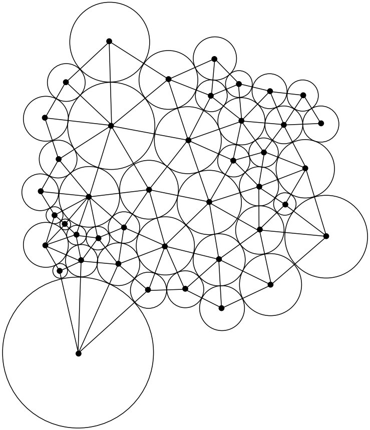




Conformal maps can be discretized as orthogonal circle patterns [Schramm '97]
$f : \mathbb{C} \to \mathbb{C}$ is a conformal map if $f_x \perp f_y$ and $|f_x| = |f_y|$

Circle Packings

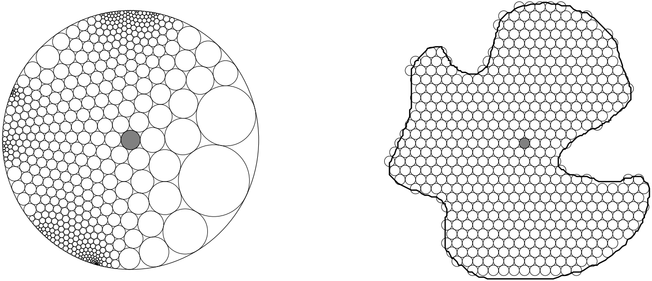
A circle packing is a configuration of circular disks (of different sizes) that may touch but not overlap.
Connect the centers of touching circles by lines to obtain the contact graph. Let the contact graph be a triangulation.



Riemann mapping theorem

Let $D \subsetneq \mathbb{C}$ be open, connected and simply connected.
Then there exists a conformal map f that maps D onto the unit disk.
The map f is unique up to post composition with Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ that map the unit disc onto itself.

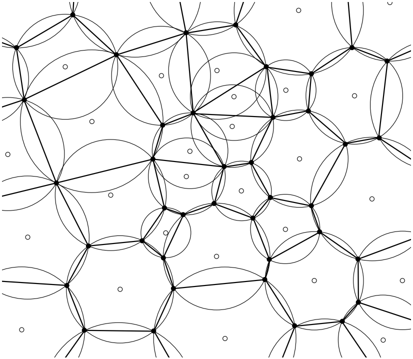
Discrete Riemann mapping



- Thurston's idea [~'85]: The Riemann mapping can be approximated using circle packings.
- the circle packing is unique up to a Möbius transformation that map the unit disc onto itself.
- Rodin & Sullivan ['87]: Proof of convergence.

Image by Ken Stephenson

Generalization: circle patterns

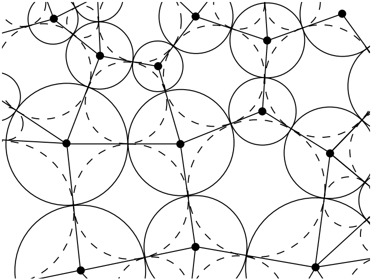


- A Delaunay decomposition of a surface is a cell decomposition such that the faces are polygons inscribed in circles and these circles contain no vertices in their interior.
- circle pattern

Orthogonal circle patterns

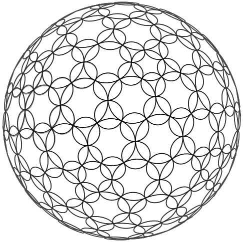
A pair of (orthogonal) circle packings such that

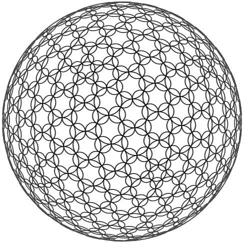
- The contact graphs of two circle packings are dual cell decompositions.
- The circles intersect orthogonally in their common touching points.
- In the case of a triangulation, the second packing exists automatically.



Orthogonal circle patterns

Theorem. [Andreev, Thurston, Schramm]
To every polytopal (strongly regular) cell decomposition of the sphere, there corresponds Möbius-uniquely a pair of orthogonally intersecting circle packings.

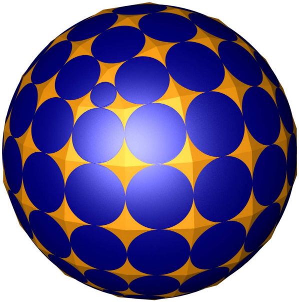




- strongly regular = no identifications on the boundary of the cells, and the boundaries of two cells may intersect at only one cell

Koebe polyhedra

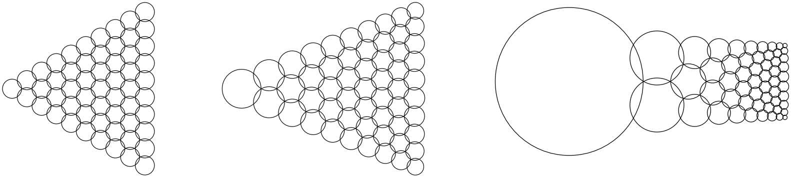
Theorem. [Andreev, Thurston, Schramm]
For every combinatorial type of convex polyhedra there exists a unique (up to a Möbius freedom) representative with edges tangent to the unit sphere.



- strongly regular = combinatorial types of convex polyhedra [Steinitz '22]

Discrete conformal maps

A discrete conformal map is a pair of circle patterns with the same combinatorics and intersection angles.



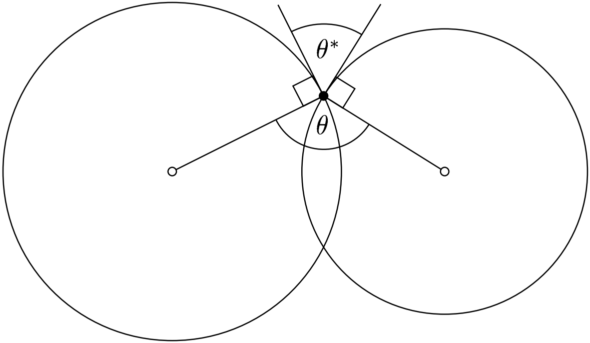
- At the centers: Length distortion does not depend on direction.
- At the intersection points: angles are preserved.

Circle pattern problem

Given

- a cell decomposition of a surface (combinatorially)
- an intersection angle $0 < \theta < \pi$ for each interior edge

Find a corresponding Delaunay decomposition with circles intersecting at the prescribed angles.



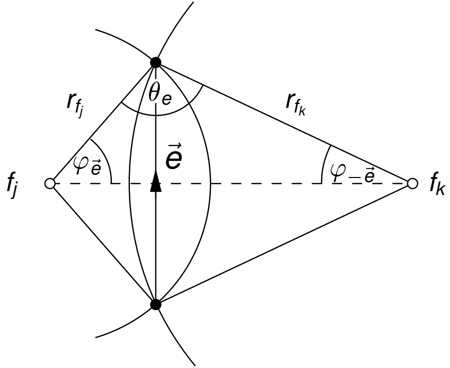
$\theta + \theta^* = \pi$

Analytic description of circle patterns

Given:
intersection angles θ_e

Variables:
one radius r_f per face f.

Equations:
one closure condition per face



$$\sum 2\varphi_{\vec{e}} = 2\pi$$

where

$$\varphi_{\vec{e}} = \frac{1}{2i} \log \frac{r_{f_j} - r_{f_k} e^{-i\theta_e}}{r_{f_j} - r_{f_k} e^{i\theta_e}}.$$

Variational principle [Bobenko, Springborn '04]

Logarithmic radii: $\rho = \log r$

Circle pattern functional:

$$S(\rho) = \sum_{f_j \circ \!\!\! | \!\!\! \circ f_k} \Big(\operatorname{Im} \operatorname{Li}_2 \left(e^{\rho_{f_k} - \rho_{f_j} + i\theta_e}\right) + \operatorname{Im} \operatorname{Li}_2 \left(e^{\rho_{f_j} - \rho_{f_k} + i\theta_e}\right) - (\pi - \theta_e)\left(\rho_{f_j} + \rho_{f_k}\right) \Big) + 2\pi \sum_{\circ f} \rho_f$$

Dilogarithm: $\operatorname{Li}_2(z) = \sum_1^\infty \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-\zeta)}{\zeta}\, d\zeta$

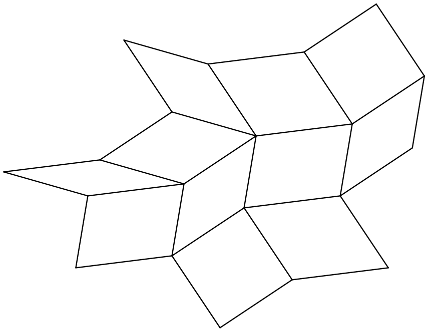
Partial derivatives: $\frac{\partial S}{\partial \rho_f} = -\sum_{f \circ \!\!\! | \!\!\! \circ f_k} 2\varphi_{\vec{e}} + 2\pi\,.$

Convexity

- the pattern can be reconstructed if we know the correct radii.
- The critical points of $S(\rho)$ are the solutions of the closure conditions.
- $S(\rho)$ is convex!
- The convexity of $S(\rho)$ implies uniqueness and existence (more tricky) of circle patterns.
- Other variational principles for circle packings and patterns (by Colin de Verdière, Brägger, Rivin, Leibon) can be derived from this.
- constructive method
- boundary conditions can be implemented

Rhombic quad-graphs

- Quad-graph = quadrilateral cell decomposition
- Rhombic quad-graph = there exists a rhombic representation in $\mathbb{R}^2$
- combinatorial characterization [Kenyon, Schlenker ’04]
  - no strip crosses itself or periodic
  - strips cross at most once



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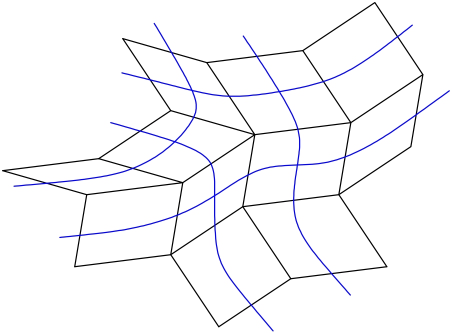

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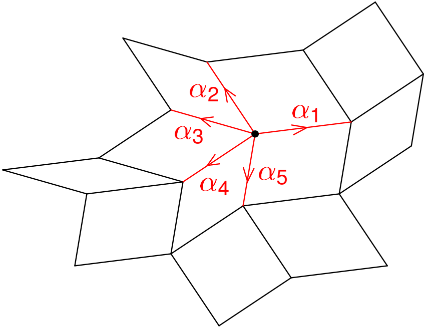
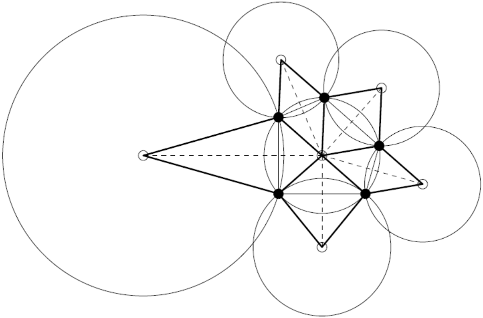


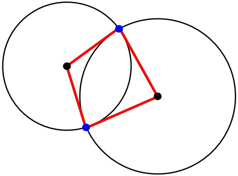
Integrable Circle Patterns

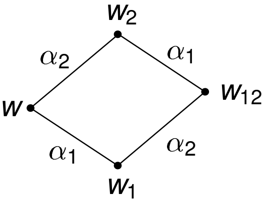
Circle patterns: combinatorial data G and intersection angles



- Combinatorial data and intersection angles belong to an integrable circle pattern iff they admit an isoradial realization. $\Rightarrow$ rhombic quad-graph, rhombic realization $\alpha_i \in \mathbb{C}$ unitary.

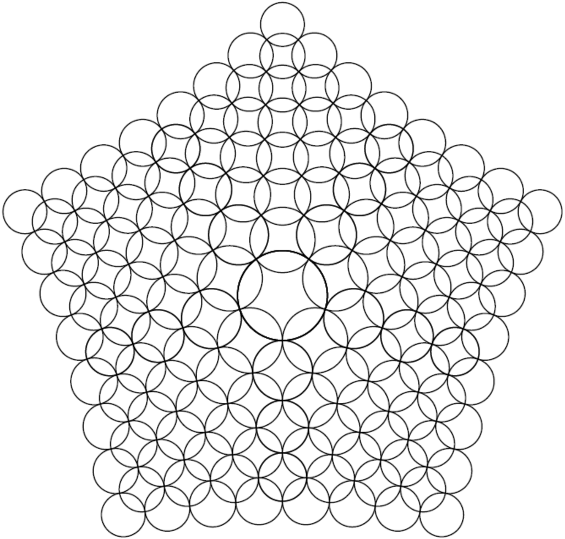
Integrable Circle Patterns





- $w(\bullet) \in \mathbb{R}_+$ - radii, $w(\color{blue}{\bullet})$ $\in S^1$ - rotation angles
- Hirota equation
  $\alpha_1(ww_1 - w_2 w_{12}) = \alpha_2(ww_2 - w_1 w_{12})$
- Quantization
  [Faddeev-Volkov '94],
  [Bazhanov-Mangazeev-Sergeev '08]

Z^a circle pattern



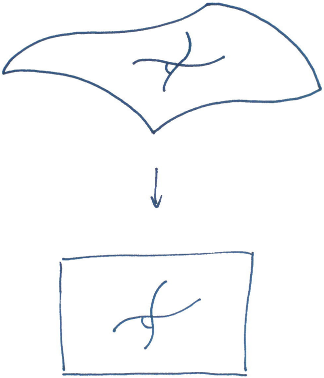
Conformal maps

- *conformal* means *angle preserving*
- infinitesimal lengths scaled by *conformal factor*

$$|df| = e^u |dx|$$

  independent of direction

- in the small like similarity transformations
- Problem:
  surface in space $\xrightarrow{\text{conformally}}$ plane



Smooth theory

Definition

Two Riemannian metrics $g, \tilde{g}$ on a smooth manifold M are called *conformally equivalent*, if

$$\tilde{g} = e^{2u}\, g$$

for some function $u : M \to \mathbb{R}$

- Gaussian curvatures

$$e^{2u}\, \tilde{K} = K + \Delta_g u$$

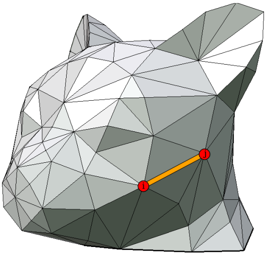
- mapping problem $\Leftrightarrow$

Given surface (M, g), find conformally equivalent flat metric $\tilde{g}$

- Poisson problem $\Delta_g u = -K$

Discrete

- abstract surface triangulation
  $M = (V, E, T)$



Definition

A *discrete metric* on M is a function

$$\ell : E \to \mathbb{R}_{>0}, \quad ij \mapsto \ell_{ij}$$

satifying all triangle inequalities:

$$\forall\, ijk \in T : \qquad \begin{aligned} \ell_{ij} &< \ell_{jk} + \ell_{ki} \\ \ell_{jk} &< \ell_{ki} + \ell_{ij} \\ \ell_{ki} &< \ell_{ij} + \ell_{jk} \end{aligned}$$

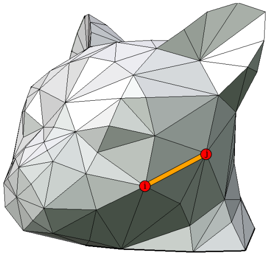
Discrete

Definition

Two discrete metrics ℓ, $\tilde{\ell}$ on M are *(discretely) conformally equivalent* if

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i+u_j)}\ell_{ij}$$

for some function $u : V \to \mathbb{R}$



- use $\lambda_{ij} = 2\log \ell_{ij}$

  so $\ell_{ij} = e^{\lambda_{ij}/2}$

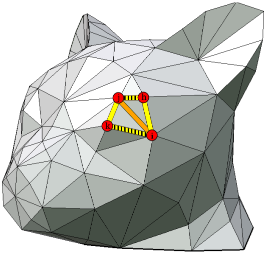
  and $\tilde{\lambda}_{ij} = \lambda_{ij} + u_i + u_j$

Length cross ratio

Definition

For interior edges *ij* define *length cross ratio*

$$\mathrm{lcr}_{ij} = \frac{\ell_{ih}\,\ell_{jk}}{\ell_{hj}\,\ell_{ki}}$$



ℓ, $\tilde{\ell}$ discretely conformally equivalent

$$\Updownarrow$$

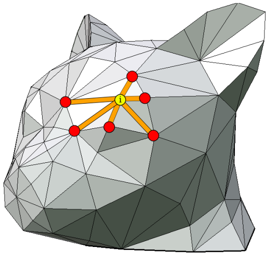
$$\widetilde{\mathrm{lcr}}_{ij} = \mathrm{lcr}_{ij}$$

Teichmüller space

- ∀ interior vertices i:

$$\prod_{ij \ni i} \mathrm{lcr}_{ij} = 1$$

- *discrete conformal structure* on M:
  equivalence class of discrete metrics



- M closed, compact, genus g:

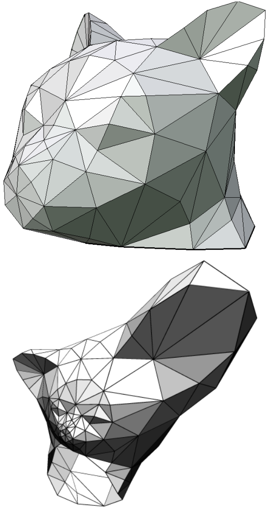
$$\dim\{\text{conformal structures}\} = |E| - |V| = 6g - 6 + 2|V| = \dim \mathcal{T}_{g,|V|}$$

$\mathcal{T}_{g,n}$: Teichmüller space for genus g with n punctures

Möbius invariance

- immersion $V \to \mathbb{R}^n$, $i \mapsto v_i$ induces discrete metric $\ell_{ij} = \|v_i - v_j\|$
- Möbius transformation: composition of inversions on spheres
- the only conformal transformations in $\mathbb{R}^n$ if $n \geq 3$

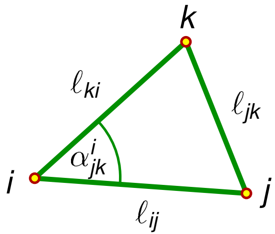
Möbius equivalent immersions induce conformally equivalent discrete metrics



Angles and curvatures

- lengths determine angles

$$\alpha^i_{jk} = 2\tan^{-1}\sqrt{\frac{(-\ell_{ij}+\ell_{jk}+\ell_{ki})(\ell_{ij}+\ell_{jk}-\ell_{ki})}{(\ell_{ij}-\ell_{jk}+\ell_{ki})(\ell_{ij}+\ell_{jk}+\ell_{ki})}}$$



- angles sum around vertex i

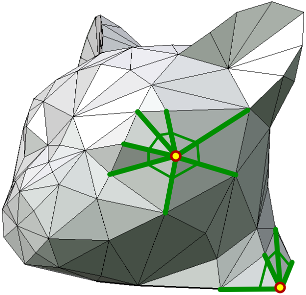
$$\Theta_i = \sum_{ijk \ni i} \alpha^i_{jk}$$

- curvature at interior vertex i

$$K_i = 2\pi - \Theta_i$$

- boundary curvature at boundary vertex

$$\kappa_i = \pi - \Theta_i$$



Mapping problem

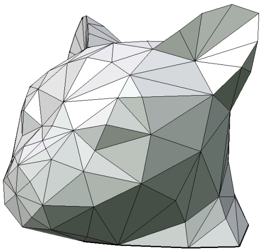
Discrete mapping problem

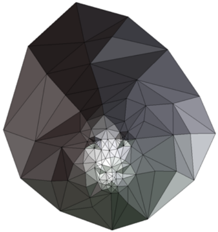
Given mesh M, metric $\ell_{ij} = e^{\frac{1}{2}\lambda_{ij}}$, and desired angle sums $\widehat{\Theta}_i$

Find conformally equivalent metric $\tilde{\ell}_{ij}$ with

$$\widetilde{\Theta}_i = \widehat{\Theta}_i$$

- $\widehat{\Theta}_i = 2\pi$ for interior vertices (except for cone-like singulatrities)
- non-linear equations for u_i

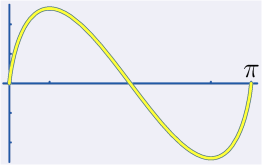




Variational principle [Springborn et al. '08]

- $$S(u) \stackrel{\text{def}}{=} \sum_{ijk \in T} \Big(\tilde\alpha_{ij}^k \tilde\lambda_{ij} + \tilde\alpha_{jk}^i \tilde\lambda_{jk} + \tilde\alpha_{ki}^j \tilde\lambda_{ki} - \pi(u_i + u_j + u_k) + 2\,Л(\tilde\alpha_{ij}^k) + 2\,Л(\tilde\alpha_{jk}^i) + 2\,Л(\tilde\alpha_{ki}^j) \Big) + \sum_{i \in V} \widehat\Theta_i u_i$$
- Milnor's Lobachevsky function

$$Л(\alpha) = -\int_0^\alpha \log|2\sin t|\, dt$$



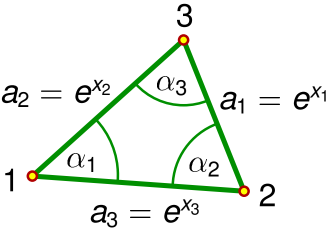
- $$\frac{\partial S}{\partial u_i} = \widehat\Theta_i - \widetilde\Theta_i$$

$\tilde\ell_{ij} = e^{\frac{1}{2}(\lambda_{ij}+u_i+u_j)}$ solves mapping problem

$\Updownarrow$

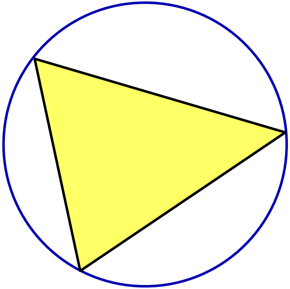
$u = (u_1, \dots, u_n)$ is critical point of $S(u)$

- $$S(u) = \sum_{ijk \in T} \left(2f\left(\frac{\tilde{\lambda}_{ij}}{2}, \frac{\tilde{\lambda}_{jk}}{2}, \frac{\tilde{\lambda}_{ki}}{2}\right) - \pi(u_i + u_j + u_k) \right) + \sum_{i \in V} \widehat{\Theta}_i u_i$$
- $$f(x_1, x_2, x_3) = \alpha_1\, x_1 + \alpha_2\, x_3 + \alpha_3\, x_3 + \text{Л}(\alpha_1) + \text{Л}(\alpha_2) + \text{Л}(\alpha_3)$$
- $S(u)$ strictly convex !
- domain not convex (due to triangle inequalities)
- solution is unique (if it exists)
- one finds it by minimizing $S(u)$



2D hyperbolic geometry

- circumcircle induces hyperbolic metric (Klein model)
- $\rightarrow$ hyperbolic metric on surface
- vertices at infinity (cusps)
- conformally equivalent discrete metrics induce same hyperbolic metric
- no wonder about $\dim \mathcal{T}_{g,n}$
- $\log \operatorname{lcr}_{ij}$ = Thurston Fock shear coordinates
  λ_{ij} = Penner coordinates



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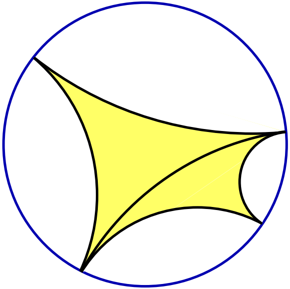

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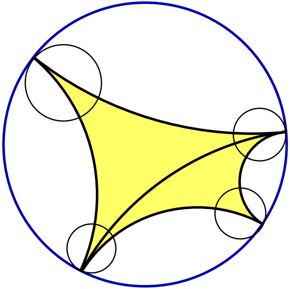

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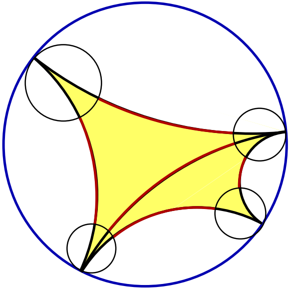

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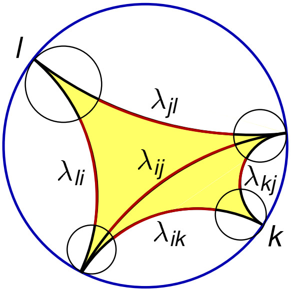
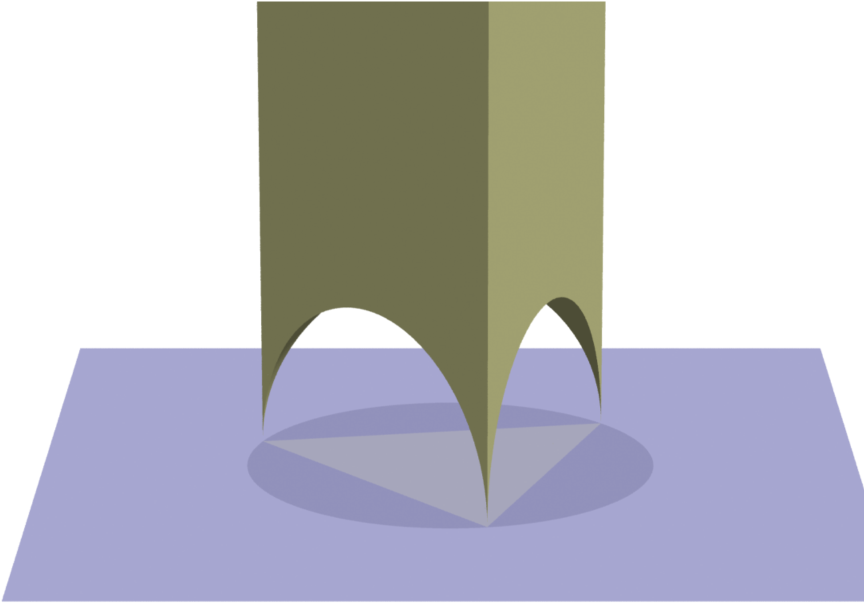


3D hyperbolic geometry. [with Springborn, Pinkall]



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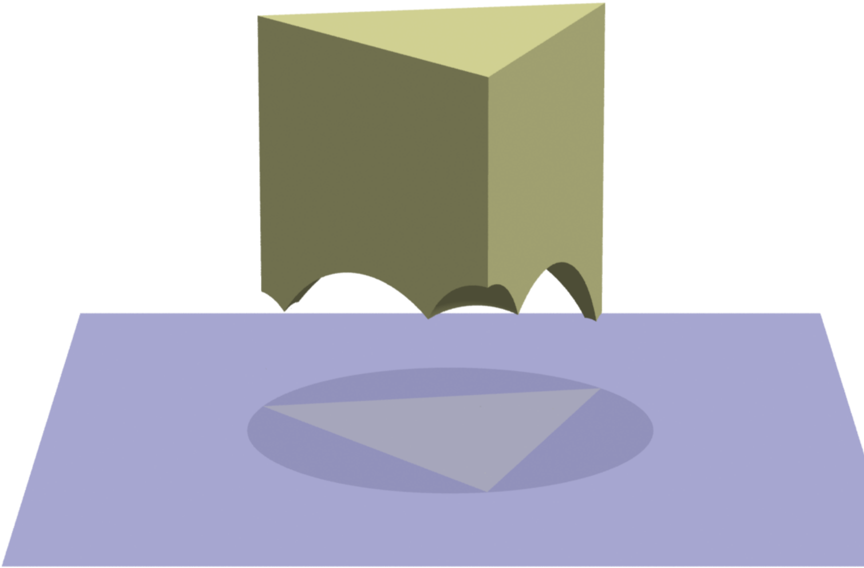

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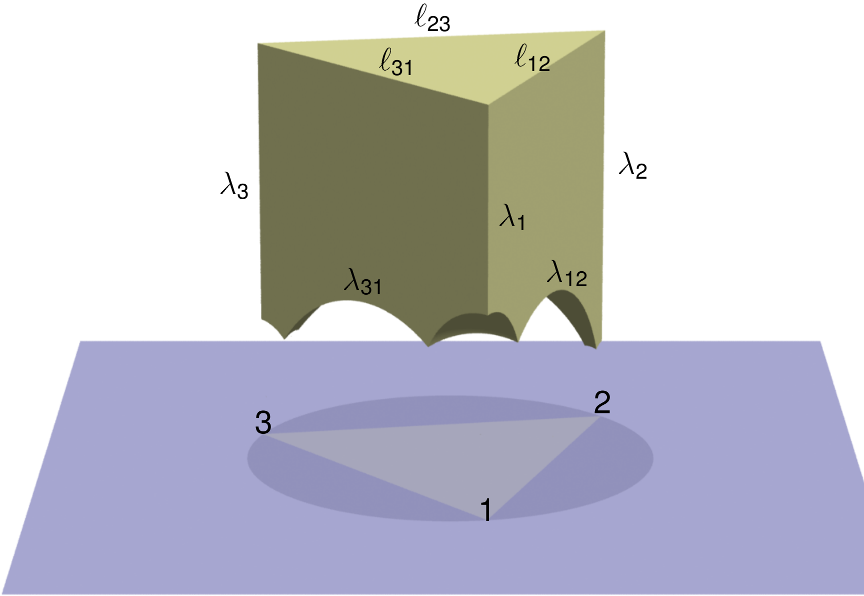
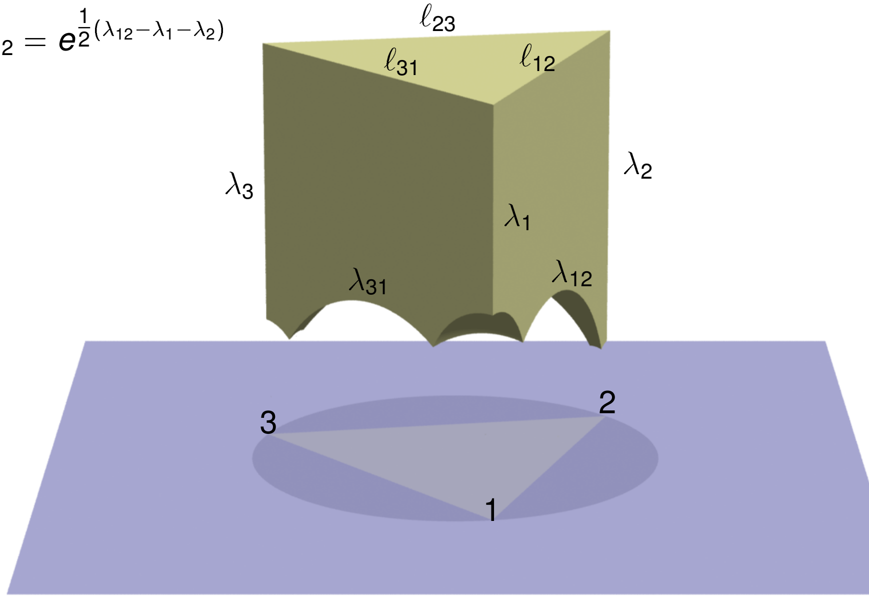


3D hyperbolic geometry. [with Springborn, Pinkall]

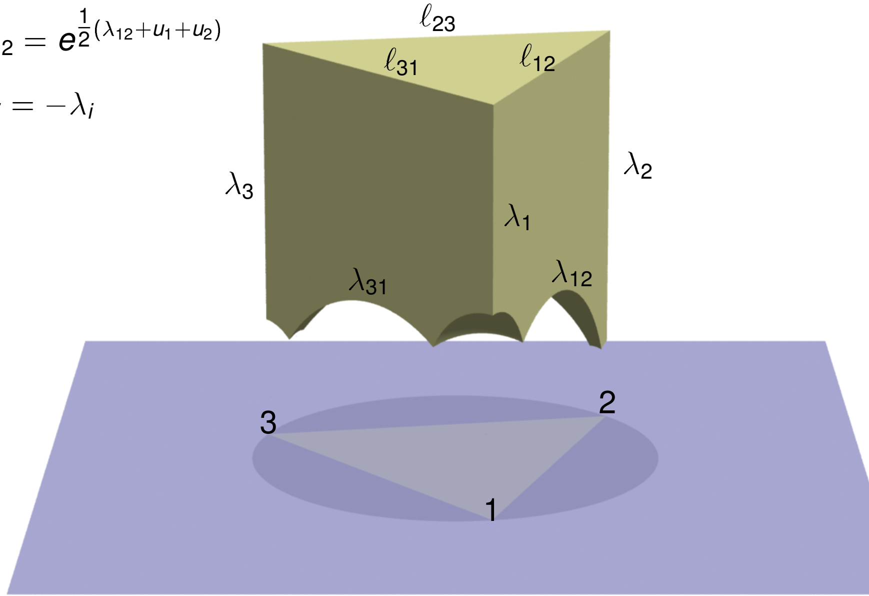
$$\ell_{12} = e^{\frac{1}{2}(\lambda_{12} - \lambda_1 - \lambda_2)}$$



3D hyperbolic geometry. [with Springborn, Pinkall]

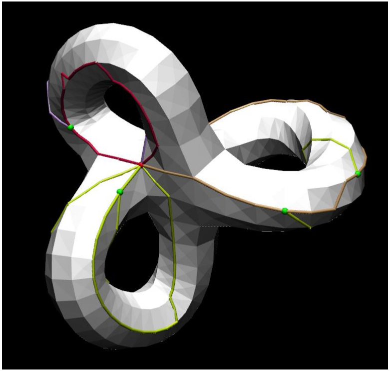
$$\ell_{12} = e^{\frac{1}{2}(\lambda_{12}+u_1+u_2)}$$

$$u_i = -\lambda_i$$



Discrete Uniformization. [with Springborn, Sechelmann]

- similar theory with finite hyperbolic triangles
- hyperbolic angles, $\sum \alpha^i_{jk} = 2\pi$ at interior vertex
- convex functional
- uniformization of surfaces of genus > 1.



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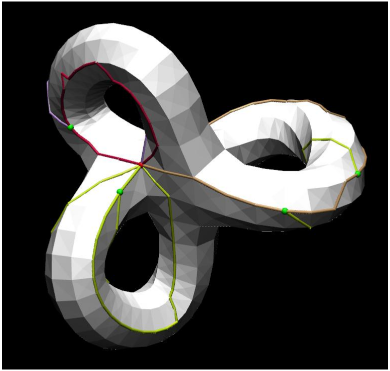

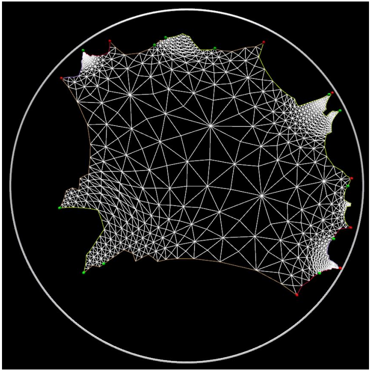

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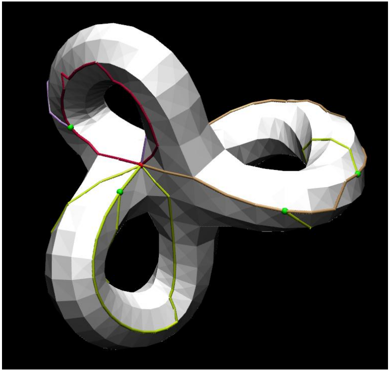

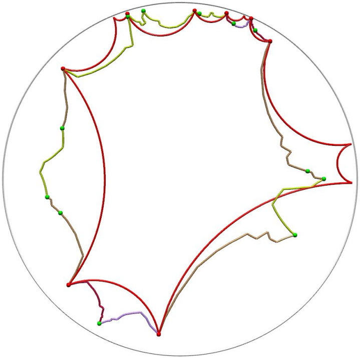

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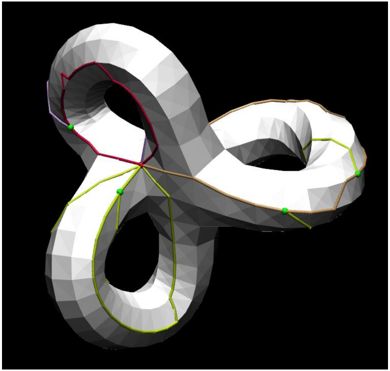

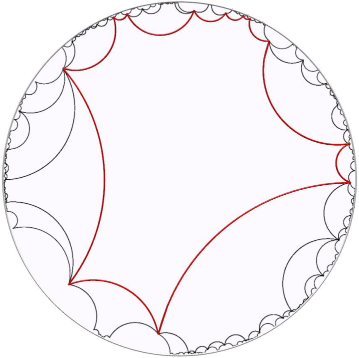

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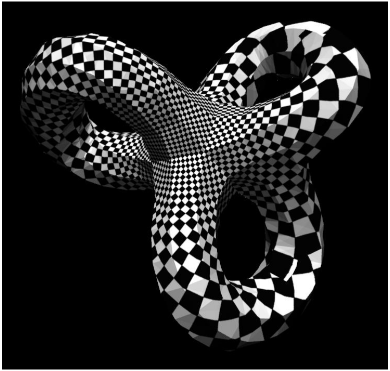

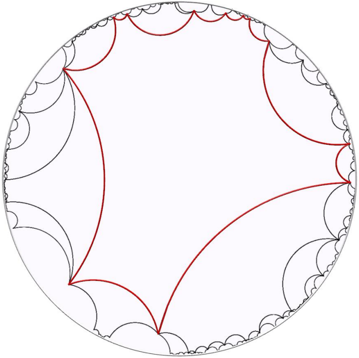

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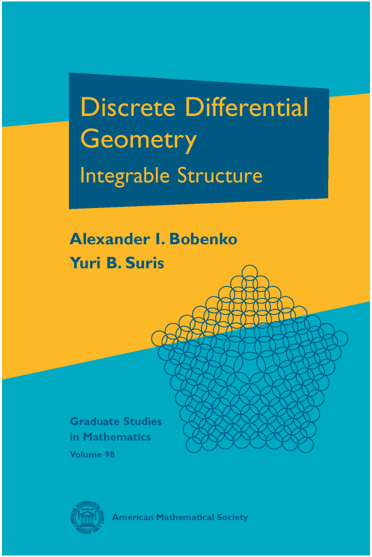
Book on DDG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