Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ mirror model

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Outline

$AdS_5 \times S^5$ superstring in the light-cone gauge

- It is a model on a cylinder of circumference $P_+ = J$, where J is an angular momentum of string rotating around S^5
- When $J \to \infty$ the cylinder $\Longrightarrow$ a plane. Integrability implies factorized scattering. Find the S-matrix and compare with the spin chain one — Staudacher '04
- In the limit $J \to \infty$ the symmetry algebra of the l.c. model

$$\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \in \mathfrak{psu}(2,2|4)$$

is extended by two central charges depending on the world-sheet momentum P — Beisert '05; Arutyunov, Frolov, Plefka, Zamaklar '06

- The world-sheet S-matrix factorises

$$\mathcal{S}(p_1,p_2) = S_0 \cdot S(p_1,p_2) \otimes S(p_1,p_2)$$

each 16×16-matrix S is $\mathfrak{psu}(2|2)_{c.e.}$-invariant — Beisert '05; Arutyunov, Frolov, Zamaklar '06

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Dispersion relation and rapidity torus

- The dispersion relation (Beisert, Dippel, Staudacher '04)
  follows from the symmetry algebra (Beisert '05)

$$H^2 = 1 + 4g^2 \sin^2 \frac{p}{2}$$

  and can be uniformized on a torus (Janik '06)

$$p = 2\,\mathrm{am}\, z\,, \qquad \sin\frac{p}{2} = \mathrm{sn}\,(z,k)\,, \qquad H = \mathrm{dn}\,(z,k)$$

  - elliptic modulus: $k = -4g^2 = -\lambda/\pi^2$
  - torus real and imaginary periods: $2\omega_1(k)$ and $2\omega_2(k)$

- Constrained parameters $x^\pm$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}$$

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S-matrix for fundamental particles

$$
\begin{aligned}
S(p_1,p_2) = {} & \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\,\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\left(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\right)\\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\,\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\left(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\right)\\
& - \left(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\right)\\
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& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde\eta_1\tilde\eta_2}\left(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\right)\\
& + \frac{x_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}\left(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\right)\\
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\end{aligned}
$$

$$
\eta_1 = \eta(p_1)\exp(\tfrac{i}{2}p_2)\,,\quad \eta_2 = \eta(p_2)\,,\quad \tilde\eta_1 = \eta(p_1)\,,\quad \tilde\eta_2 = \eta(p_2)\exp(\tfrac{i}{2}p_1)\,,\qquad \eta(p) = \exp(\tfrac{i}{4}p)\sqrt{ix^- - ix^+}
$$

Spectrum on a large circle

- Bethe-Yang equations

Beisert, Staudacher '05

$$\text{“}e^{ip_k J} \prod_{j \neq k} \mathcal{S}(p_k, p_j) = 1\text{”}$$

(+ additional equations with auxiliary roots encoding non-diagonal structure of $\mathcal{S}$)

- Given $\{p_i\}_{i=1}^{M}$, the energy (dimension) is given by

$$E = \sum_{i=1}^{M} \sqrt{1 + 4g^2 \sin^2 \frac{p_i}{2}} = E(g, J)$$

- This is NOT the correct answer for finite J!

Wrapping interactions (distinguished Feynman graphs), finite-size corrections to classical string energies, BFKL analysis, all points to this...

- Lüscher's formulae and TBA ideas were successfully used to explain exponential corrections in string theory and wrapping effects in fields theory

Ambjorn, Janik, Kristjansen '05

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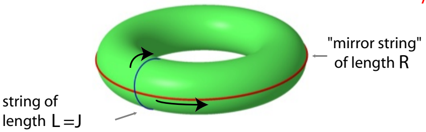
TBA and mirror theory

Follow the TBA approach for relativistic models (Zamolodchikov ’90)

Arutyunov, Frolov ’07



- One Euclidean theory – two Minkowski theories. One is related to the other by the double Wick rotation:

$$\tilde{\sigma} = -i\tau\,, \qquad \tilde{\tau} = i\sigma$$

The Hamiltonian $\tilde{\mathrm{H}}$ w.r.t. $\tilde{\tau}$ defines the *mirror theory*.

- Ground state energy $(R \to \infty)$ is related to the free energy of its mirror

$$E(L) = L\mathcal{F}(L)$$

Free energy $\mathcal{F}$ can be found from the Bethe ansatz for the mirror model because $R \to \infty$

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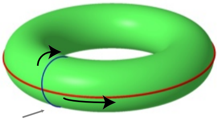

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Free energy $\mathcal{F}$ can be found from the Bethe ansatz for the mirror model because $R \to \infty$

Boundary conditions for fermions

Periodicity of fermions

- *Fermions of the string model: periodic or anti-periodic in the space direction, anti-periodic in time*
- *Fermions of the mirror model: anti-periodic in the space direction, periodic or anti-periodic in time*

Ground state energy for periodic fermions is related to Witten's index of the mirror theory:

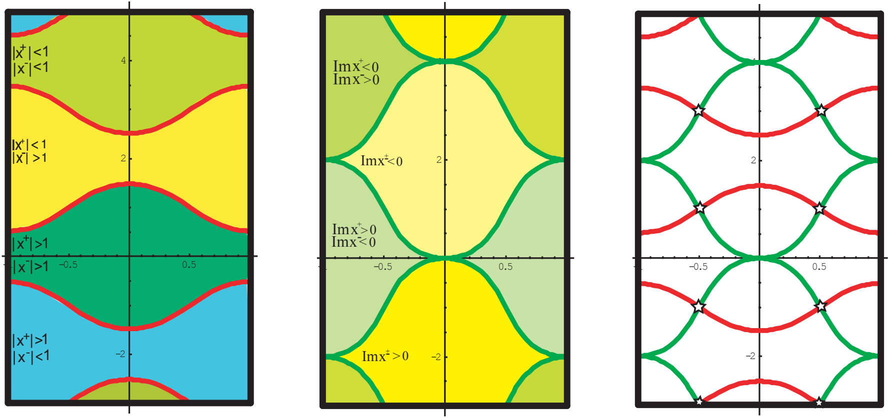
$$\mathrm{Tr}\left((-1)^{\mathrm{F}}\mathrm{e}^{-\beta\tilde{\mathrm{H}}}\right)$$

Comparison chart

Arutyunov, Frolov '07

	Strings	Mirrors
Dispersion relation	$\mathcal{E} = \sqrt{Q^2 + 4g^2 \sin^2 \frac{p}{2}}$	$\widetilde{\mathcal{E}} = 2\,\mathrm{arcsinh}\left(\frac{1}{2g}\sqrt{Q^2 + \widetilde{p}^2}\right)$
Momentum	$-\pi \leq p < \pi$	$-\infty < \widetilde{p} < \infty$
Type of theory	Lattice model	Continuum model
Giant magnon	Soliton in $\mathrm{R} \times \mathrm{S}^5$	Soliton in AdS_5
S – matrix	$S(z_1, z_2)$	$S(z_1 + \frac{\omega_2}{2}, z_2 + \frac{\omega_2}{2})$
Dressing factor	$\sigma(1,2)^* \,\sigma(1,2) = 1$	$\sigma(1,2)^* \,\sigma(1,2) = \frac{x_1^+}{x_1^-}\frac{x_2^-}{x_2^+}$
Bethe – Yang eqs	BS; $\mathrm{P} = 0$	extra $\sqrt{x^+/x^-}$
Bound states	Symmetric irrep su(2) sector	Antisymmetric irrep sl(2) sector
Physical region	"Fish" (?)	"Leaf" (?)

z-torus



$$x = \text{Re}(\frac{2}{\omega_1}z)\,, \quad y = \text{Re}(\frac{4}{\omega_2}z)$$

Bethe-Yang equations for the mirror model

Arutyunov, Frolov '07

$$1 = e^{i\widetilde{p}_k R} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm I}} S^{11}_{\mathfrak{sl}(2)}(x_k, x_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}}$$

$$-1 = \prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\rm III}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}$$

$$1 = \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm III}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}$$

where the S-matrix of the $\mathfrak{sl}(2)$-sector enters

$$S^{11}_{\mathfrak{sl}(2)}(x_1, x_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \sigma_{12}^{-2}, \qquad v = y + \frac{1}{y}$$

Bound states of the mirror model

The $\mathfrak{sl}(2)$ S-matrix

$$S^{11}_{\mathfrak{sl}(2)}(x_1, x_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \sigma_{12}^{-2}$$

exhibits a pole for complex values of momenta

$$\tilde{p}_1 = \frac{p}{2} + iq\,, \qquad \tilde{p}_2 = \frac{p}{2} - iq\,, \qquad \mathrm{Re}\, q > 0$$

for which $x^-(\tilde{p}_1) - x^+(\tilde{p}_2) = 0 \quad \Longrightarrow \quad q = q(p)$

This pole leads to the existence of a Q-particle bound state

$$x_1^- = x_2^+\,, \quad x_2^- = x_3^+\,, \quad \ldots\,, \quad x_{Q-1}^- = x_Q^+$$

The mirror asymptotic spectrum contains fundamental particles and their bound states. Mirror bound states transform in the atypical anti-symmetric irreps of $\mathfrak{su}(2|2)_{c.e.}$

Arutyunov, Frolov '07

Bethe-Yang for mirror particles and their bound states

The Bethe-Yang equations for bound states are obtained by fusing the equations for the constituent fundamental particles:

$$
\begin{aligned}
1 &= e^{i\widetilde{p}_k R} \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{I}}} S_{\mathfrak{sl}(2)}^{Q_k Q_l}(x_k, x_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
-1 &= \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}} \\
1 &= \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}} .
\end{aligned}
$$

$S_{\mathfrak{sl}(2)}^{Q_k Q_l}$ is obtained by fusing the fundamental constituents $S_{\mathfrak{sl}(2)}^{11}$

String hypothesis

The main issue is to understand the structure of solutions to the BY equations in the thermodynamic limit:

$$R \to \infty, \quad K^{I}/R = \text{fixed}, \quad K^{II}_{(\alpha)}/R = \text{fixed}, \quad K^{III}_{(\alpha)}/R = \text{fixed}$$

This is done by formulating the corresponding

string hypothesis

Arutyunov, Frolov '09(a)

TBA equations are derived from it following a textbook route!

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Root structure

Consider a generic term in the first BY equation

$$1 = e^{i\tilde{p}_k R} \dots \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \dots$$

For physical mirror particles $x^{\pm *} = 1/x^{\mp}$, therefore,

$$1 = e^{-i\tilde{p}_k R} \dots \frac{\frac{1}{x_k^+} - y_l^{(\alpha)*}}{\frac{1}{x_k^-} - y_l^{(\alpha)*}} \sqrt{\frac{x_k^+}{x_k^-}} \dots \implies 1 = e^{i\tilde{p}_k R} \dots \frac{x_k^- - \frac{1}{y_l^{(\alpha)*}}}{x_k^+ - \frac{1}{y_l^{(\alpha)*}}} \sqrt{\frac{x_k^+}{x_k^-}} \dots$$

- A single y-root must be on the unit circle:

$$|y| = 1 \implies -2 \le v = y + 1/y \le 2$$

- y-roots which are not on the circle come in pairs $(y_1, y_2 = 1/y_1^*)$, and lead to the vw-string configurations

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String hypothesis for the mirror model

Arutyunov, Frolov '09(a)

In the thermodynamic limit $R, K^{\rm I}, K^{\rm II}_{(\alpha)}, K^{\rm III}_{(\alpha)} \to \infty$ with $K^{\rm I}/R$ and so on fixed solutions arrange themselves into seven different classes of Bethe strings

1. A single Q-particle with real momentum $\widetilde{p}_k$
2. A single $y^{(\alpha)}$-particle corresponding to a root $y^{(\alpha)}$ with $|y^{(\alpha)}| = 1$
3. $2M$ roots $y^{(\alpha)}$ and M roots $w^{(\alpha)}$ combining into a $M|vw^{(\alpha)}$-string

$$v_j^{(\alpha)} = v^{(\alpha)} + (M+2-2j)\frac{i}{g}\,, \quad v_{-j}^{(\alpha)} = v^{(\alpha)} - (M+2-2j)\frac{i}{g}\,,$$

$$w_j^{(\alpha)} = v^{(\alpha)} + (M+1-2j)\frac{i}{g}\,, \quad j = 1, \dots, M\,, \quad v \in \mathbf{R}\,.$$

4. N roots $w^{(\alpha)}$ combining into a single $N|w^{(\alpha)}$-string

$$w_j^{(\alpha)} = w^{(\alpha)} + \frac{i}{g}(N+1-2j)\,, \quad j = 1, \dots, N\,, \quad w \in \mathbf{R}$$

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String hypothesis for the l.c. STRING MODEL

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Function x(u)

Introduce the function

$$x(u) = \frac{1}{2}\left(u - i\sqrt{4-u^2}\right)\,, \quad \mathrm{Im}(x(u)) < 0 \ \text{ for any } u \in \mathbb{C}$$

mapping the u-plane onto the physical region of mirror model.
The cuts in the u-plane run from $\pm\infty$ to ± 2 along the real lines.

$$|x(u)| = 1 \quad \text{for } -2 \le u \le 2$$

Compare with

$$x_s(u) = \frac{1}{2}\left(u + \sqrt{u^2-4}\right)\,, \quad |x_s(u)| > 1 \ \text{ for any } u \in \mathbb{C}$$

that maps the u-plane onto the physical region of the string model.
The cut in the u-plane is $[-2, 2]$.

Function x(u)

Introduce the function

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Thermodynamic limit

Densities $\rho(u)$ of particles, and $\bar{\rho}(u)$ of holes; $u \in \mathbf{R}$, $\alpha = 1, 2$.

1. $\rho_Q(u)$ of Q-particles, $-\infty \le u \le \infty$, $Q = 1, \dots, \infty$
2. $\rho^{(\alpha)}_{y^-}(u)$ of y-particles with $\text{Im}(y) < 0$, $-2 \le u \le 2$.
   The y-coordinate is expressed in terms of u as $y = x(u)$
3. $\rho^{(\alpha)}_{y^+}(u)$ of y-particles with $\text{Im}(y) > 0$, $-2 \le u \le 2$.
   The y-coordinate is expressed in terms of u as $y = \frac{1}{x(u)}$
4. $\rho^{(\alpha)}_{M|vw}(u)$ of $M|vw$-strings, $-\infty \le u \le \infty$, $M = 1, \dots, \infty$
5. $\rho^{(\alpha)}_{M|w}(u)$ of $M|w$-strings, $-\infty \le u \le \infty$, $M = 1, \dots, \infty$,

and the corresponding densities of holes.

Thermodynamic limit

Integral eqs in the thermodynamic limit

$$\rho_i(u) + \bar{\rho}_i(u) = \frac{R}{2\pi}\frac{d\widetilde{p}_i}{du} + K_{ij} \star \rho_j(u)$$

where $\widetilde{p}_i$ does not vanish only for Q-particles.

- Star operation is defined as

$$K_{ij} \star \rho_j(u) = \int \mathrm{d}u' \, K_{ij}(u,u')\rho_j(u')$$

- Kernels $K's$ are expressed via the corresponding S-matrices as

$$K_{ij}(u,v) = \frac{1}{2\pi i}\frac{d}{du}\log S_{ij}(u,v)$$

The right action is defined as

$$\rho_j \star K_{ji}(u) = \int \mathrm{d}u' \, \rho_j(u')K_{ji}(u',u)$$

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Free energy and equations for pseudo-energies

To describe both sectors, we consider generalized free energy

Cecotti, Fendley, Intriligator, Vafa '92

$$\mathcal{F}_\gamma(L) = \mathcal{E} - \frac{1}{L} S + \frac{i\gamma}{L}(N_F^{(1)} - N_F^{(2)})\,,$$

- $\mathcal{E}$ is the energy per unit length carried by Q-particles

$$\mathcal{E} = \int \mathrm{d}u \sum_{Q=1}^{\infty} \widetilde{\mathcal{E}}^Q(u)\rho_Q(u)\,, \quad \widetilde{\mathcal{E}}^Q(u) \text{ is } Q\text{-particle energy}$$

- S is the total entropy
- $i\gamma/L$ plays the role of a chemical potential
- $N_F^{(\alpha)}$ is the fermion number which counts the number of $y^{(\alpha)}$-particles

$$N_F^{(1)} - N_F^{(2)} = \int \mathrm{d}u\,(\rho_{y^-}^{(1)}(u) + \rho_{y^+}^{(1)}(u) - \rho_{y^-}^{(2)}(u) - \rho_{y^+}^{(2)}(u))$$

- Minus sign between $N_F^{(1)}$ and $N_F^{(2)}$ is needed for the reality of $\mathcal{F}_\gamma(L)$
- $\gamma = \pi \implies$ Witten's index. $\gamma = 0 \implies$ the usual free energy.

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Free energy and equations for pseudo-energies

Free energy: $\mathcal{F}_\gamma(L) = \int \mathrm{d}u \sum_k \left[\widetilde{\mathcal{E}}_k\, \rho_k - \frac{i\gamma_k}{L}\, \rho_k - \frac{1}{L}\mathfrak{s}(\rho_k)\right]$

Variations of the densities of particles and holes are subject to

$$\delta\rho_k(u) + \delta\bar{\rho}_k(u) = K_{kj} \star \delta\rho_j\,.$$

Using the extremum condition $\delta\mathcal{F}_\gamma(L) = 0$, one derives the TBA eqs

$$\epsilon_k = L\,\widetilde{\mathcal{E}}_k - \log\left(1 + e^{i\gamma_j - \epsilon_j}\right) \star K_{jk}\,,$$

where the pseudo-energies ϵ_k are $e^{i\gamma_k - \epsilon_k} = \frac{\rho_k}{\bar{\rho}_k}\,,$

At the extremum $\mathcal{F}_\gamma(L) = -\frac{R}{L}\int \mathrm{d}u \sum_k \frac{1}{2\pi}\frac{d\widetilde{p}_k}{du}\log\left(1 + e^{i\gamma_k - \epsilon_k}\right)$

The energy of the ground state of the l.c. string theory

$$E_\gamma(L) = \lim_{R\to\infty}\frac{L}{R}\mathcal{F}_\gamma(L) = -\int \mathrm{d}u \sum_{Q=1}^{\infty}\frac{1}{2\pi}\frac{d\widetilde{p}^Q}{du}\log\left(1 + e^{-\epsilon_Q}\right)$$

TBA equations for pseudo-energies of mirror particles

Arutyunov, Frolov '09(b)

- Q-particles
$$\epsilon_Q = L\,\widetilde{\mathcal{E}}_Q - \log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'Q}_{\mathfrak{sl}(2)} - \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K^{M'Q}_{vwx} - \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}\right)\star K^{yQ}_{-} - \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}\right)\star K^{yQ}_{+}$$
- y-particles
$$\epsilon^{(\alpha)}_{y\pm} = -\log\left(1+e^{-\epsilon_Q}\right)\star K^{Qy}_{\pm} + \log\frac{1+e^{-\epsilon^{(\alpha)}_{M|vw}}}{1+e^{-\epsilon^{(\alpha)}_{M|w}}}\star K_M$$
- $M|vw$-strings
$$\epsilon^{(\alpha)}_{M|vw} = -\log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'M}_{xv} + \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}}\star K_M$$
- $M|w$-strings
$$\epsilon^{(\alpha)}_{M|w} = \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|w}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}}\star K_M$$
- The ground state energy
$$E(L) = -\int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi}\frac{d\widetilde{p}^Q}{du}\log\left(1+e^{-\epsilon_Q}\right)$$

Equivalent TBA eqs
and also for $\mathfrak{sl}(2)$ sector

Bombardelli, Fioravanti, Tateo '09

Gromov, Kazakov, Kozak, Vieira '09

Dressing Factor for the Mirror Model

Arutyunov, Frolov '09(c)

$$K^{QQ'}_{\mathfrak{sl}(2)}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log S^{QQ'}_{\mathfrak{sl}(2)}(u,u')\,,$$

$$S^{QQ'}_{\mathfrak{sl}(2)}(u,u') = S_{QQ'}(u-u')^{-1}\,\Sigma_{QQ'}(u,u')^{-2}\,,$$

$$S_{QQ'}(u-u') = \frac{u-u'-\frac{i}{g}(Q+Q')}{u-u'+\frac{i}{g}(Q+Q')}\,\frac{u-u'-\frac{i}{g}(Q'-Q)}{u-u'+\frac{i}{g}(Q'-Q)}\prod_{j=1}^{Q-1}\left(\frac{u-u'-\frac{i}{g}(Q'-Q+2j)}{u-u'+\frac{i}{g}(Q'-Q+2j)}\right)^2$$

Here $\Sigma_{QQ'}(u,u')$ is related to the analytically continued BES dressing factor as

Beisert, Eden, Staudacher '06

$$\Sigma_{QQ'}(u,u') = \prod_{j=1}^{Q}\prod_{k=1}^{Q'}\sigma(x_j^{\pm}(u),x_k^{\pm}(u'))\,\frac{1-\frac{1}{x_j^+(u)x_k^-(u')}}{1-\frac{1}{x_j^-(u)x_k^+(u')}}\,,$$

$$x_j^+(u) = x(u+\frac{i}{g}(Q+2-2j))\,,\; x_j^-(u) = x(u+\frac{i}{g}(Q-2j))\,,\quad x_j^- = x_{j+1}^+$$

$\Sigma_{QQ'}(u,u')$ is holomorphic in the physical region of the mirror model.

Simplified TBA equations

Introduce the Y-functions

$$Y_Q = e^{-\epsilon_Q}\,, \quad Y^{(\alpha)}_{M|vw} = e^{\epsilon^{(\alpha)}_{M|vw}}\,, \quad Y^{(\alpha)}_{M|w} = e^{\epsilon^{(\alpha)}_{M|w}}\,, \quad Y^{(\alpha)}_{\pm} = e^{\epsilon^{(\alpha)}_{y^\pm}}$$

and the kernels

$$K_M(u) = \frac{1}{\pi}\,\frac{gM}{M^2+g^2u^2}\,, \quad s(u) = \frac{g}{4\cosh\frac{g\pi u}{2}}\,,$$

$$K_{Qy}(u,v) = K(u-\frac{i}{g}Q,v) - K(u+\frac{i}{g}Q,v)\,, \quad K(u,v) = \frac{1}{2\pi i}\,\frac{\sqrt{4-v^2}}{\sqrt{4-u^2}}\,\frac{1}{u-v}$$

Simplified TBA equations

Arutyunov, Frolov '09(b)

- $M|w$-strings: $\log Y^{(\alpha)}_{M|w} = \log(1+Y^{(\alpha)}_{M-1|w})(1+Y^{(\alpha)}_{M+1|w}) \star s + \delta_{M1} \log \frac{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}} \star s$
- $M|vw$-strings:

$$\log Y^{(\alpha)}_{M|vw} = \log \frac{(1+Y^{(\alpha)}_{M-1|vw})(1+Y^{(\alpha)}_{M+1|vw})}{1+Y_{M+1}} \star s + \delta_{M1} \log \frac{1-e^{-ih_\alpha}\, Y^{(\alpha)}_-}{1-e^{-ih_\alpha}\, Y^{(\alpha)}_+} \star s$$

- y-particles: $\log \frac{Y^{(\alpha)}_+}{Y^{(\alpha)}_-} = \log(1+Y_Q) \star K_{Qy}$,

$$\log Y^{(\alpha)}_+ Y^{(\alpha)}_- = -\log(1+Y_Q) \star K_Q + 2\log \frac{1+\frac{1}{Y^{(\alpha)}_{M|vw}}}{1+\frac{1}{Y^{(\alpha)}_{M|w}}} \star K_M$$

- Q-particles for $Q \geq 2$: $\log Y_Q = \log \frac{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)}{(1+\frac{1}{Y_{Q-1}})(1+\frac{1}{Y_{Q+1}})} \star s$
- $Q=1$-particle: $\log Y_1 = \log \frac{\left(1-\frac{e^{ih_1}}{Y^{(1)}_-}\right)\left(1-\frac{e^{ih_2}}{Y^{(2)}_-}\right)}{1+\frac{1}{Y_2}} \star s - \Delta \star s$

Simplified TBA equations

Arutyunov, Frolov '09(b)

$Q = $ 1-particle: $\log Y_1 = \log \frac{\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)}{1+\frac{1}{Y_2}} \star s - \Delta \star s$

$$\begin{aligned}\Delta(u) &= \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\left(\theta(-u-2)+\theta(u-2)\right)\\ &+ L\check{\mathcal{E}} - \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\left(1-\frac{e^{ih_1}}{Y_+^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_+^{(2)}}\right)\star\check{K}\\ &- \log\left(1+\frac{1}{Y_{M|vw}^{(1)}}\right)\left(1+\frac{1}{Y_{M|vw}^{(2)}}\right)\star\check{K}_M + 2\log\left(1+Y_Q\right)\star\check{K}_Q^{\Sigma}\,,\end{aligned}$$

$\Delta(u)$ determines analytic properties of the Y-system on the u-plane

Gromov, Kazakov, Vieira '09(a)

TBA and Y-equations for *w*-strings

$$\log Y^{(\alpha)}_{M|w} = \log(1+Y^{(\alpha)}_{M-1|w})(1+Y^{(\alpha)}_{M+1|w})\star s + \delta_{M1}\,\log\frac{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}}\star s$$

Since the functions $Y^\alpha_\pm$ are defined on the interval $-2<u<2$, the integral in the last term is taken along $[-2,2]$.

Define $(f\star s^{-1})(u) = \lim_{\epsilon\to 0^+}\left[f(u+\frac{i}{g}-i\epsilon)+f(u-\frac{i}{g}+i\epsilon)\right]$.

It satisfies $(s\star s^{-1})(u)=\delta(u)$. In general $f\star s^{-1}\star s\neq f$.

Introduce the notation $f^\pm(u)\equiv f(u\pm\frac{i}{g}\mp i0)$, and get the Y-equations

$$Y^{(\alpha)+}_{M|w}\,Y^{(\alpha)-}_{M|w} = \left(1+Y^{(\alpha)}_{M-1|w}\right)\left(1+Y^{(\alpha)}_{M+1|w}\right) \quad \text{if } M\geq 2\,,$$

$$Y^{(\alpha)+}_{1|w}\,Y^{(\alpha)-}_{1|w} = \left(1+Y^{(\alpha)}_{2|w}\right)\frac{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}}\,, \qquad |u|\leq 2\,,$$

$$Y^{(\alpha)+}_{1|w}\,Y^{(\alpha)-}_{1|w} = 1+Y^{(\alpha)}_{2|w}\,, \qquad |u|>2\,.$$

Y-system requires $Y^{(\alpha)}_+ = Y^{(\alpha)}_-$ for $|u|>2$.

TBA and Y-equations for w-strings

$$\log Y^{(\alpha)}_{M|w} = \log(1 + Y^{(\alpha)}_{M-1|w})(1 + Y^{(\alpha)}_{M+1|w}) \star s + \delta_{M1}\, \log \frac{1 - \frac{e^{ih_\alpha}}{Y^{(\alpha)}_{-}}}{1 - \frac{e^{ih_\alpha}}{Y^{(\alpha)}_{+}}} \star s$$

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Define $(f \star s^{-1})(u) = \lim_{\epsilon \to 0^+} \left[f(u + \frac{i}{g} - i\epsilon) + f(u - \frac{i}{g} + i\epsilon) \right]$.

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$$Y^{(\alpha)+}_{1|w}\, Y^{(\alpha)-}_{1|w} = 1 + Y^{(\alpha)}_{2|w}, \qquad |u| > 2.$$

Y-system requires $Y^{(\alpha)}_{+} = Y^{(\alpha)}_{-}$ for $|u| > 2$.

TBA and Y-equations for w-strings

$$\log Y^{(\alpha)}_{M|w} = \log(1 + Y^{(\alpha)}_{M-1|w})(1 + Y^{(\alpha)}_{M+1|w}) \star s + \delta_{M1} \log \frac{1 - \frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1 - \frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}} \star s$$

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$$Y^{(\alpha)+}_{1|w}\, Y^{(\alpha)-}_{1|w} = 1 + Y^{(\alpha)}_{2|w}\,, \qquad |u| > 2\,.$$

Y-system requires $Y^{(\alpha)}_+ = Y^{(\alpha)}_-$ for $|u| > 2$.

Ground state energy: any L, small h

Frolov, Suzuki '09

Naively, for $h=0$ the TBA equations are solved by

$$Y_Q = 0, \qquad Y_+^{(\alpha)} = Y_-^{(\alpha)} = 1, \qquad Y_{M|vw}^{(\alpha)} = Y_{M|w}^{(\alpha)} \neq 0, \qquad e^{ih_\alpha} = 1\,.$$

A subtle point is that the TBA equation for Q-particles is singular at $Y_Q = 0$

$$-\log Y_Q = L\widetilde{\mathcal{E}}_Q - \log\left(1+Y_{Q'}\right)\star K_{\mathfrak{sl}(2)}^{Q'Q} - \log\left(1+\frac{1}{Y_{M|vw}^{(\alpha)}}\right)\star K_{vwx}^{MQ}$$

$$-\frac{1}{2}\log\frac{1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}}\star K_Q - \frac{1}{2}\log\left(1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}\right)\left(1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}\right)\star K_{yQ}\,.$$

Consider $h\neq 0$ and take $h\to 0$. For small h, the functions $Y_\pm^{(\alpha)}$ have expansion

$$Y_\pm^{(\alpha)} = 1 + hA_\pm^{(\alpha)} + \cdots\,.$$

The last term behaves as $\log h$, and we get

$$-\log Y_Q = -2\log h \star K_{yQ} + \text{finite terms}\,.$$

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$$-\frac{1}{2}\log\frac{1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}}\star K_Q-\frac{1}{2}\log\left(1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}\right)\left(1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}\right)\star K_{yQ}\,.$$

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Ground state energy: any L, small h

$$-\log Y_Q = -2 \log h \star K_{yQ} + \text{finite terms}\,.$$

Taking into account that $1 \star K_{yQ} = 1$, we conclude

$$Y_Q = h^2 B_Q + \cdots \,,$$

and the ground state energy expands as

$$E_h(L) = -h^2 \int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\widetilde{p}^Q}{du} B_Q + \cdots \,.$$

The leading order solution of the TBA eqs

$$Y_Q = 4h^2\, Q^2\, e^{-L\widetilde{\mathcal{E}}_Q}\,, \quad Y_{\pm}^{(\alpha)} = 1 + \mathcal{O}(h^2)\,, \quad Y_{M-1|vw}^{(\alpha)} = Y_{M-1|w}^{(\alpha)} = M^2 - 1$$

Ground state energy: any L, small h

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Ground state energy: any L, small h

Since $\widetilde{\mathcal{E}}_Q = \log \frac{x^{Q-}}{x^{Q+}}$, $x^{Q\pm}(u) = x(u \pm \frac{i}{g}\, Q)$, the Y_Q-functions acquire the form

$$Y_Q = 4\, h^2\, Q^2 \left(\frac{x^{Q+}}{x^{Q-}}\right)^L + \mathcal{O}(h^3)\,.$$

Y_Q-functions are NOT analytic on the u-plane.

In terms of the z-torus rapidity variable $x^{Q+}/x^{Q-} = (\mathrm{cn}\; z + i\,\mathrm{sn}\; z)^2$.
Thus, the Y_Q-functions ARE meromorphic on the z-torus if

$$L = \frac{1}{T} \quad \text{is integer or half-integer.}$$

L is quantized!!! if Y_Q is analytic on z-torus.

The ground state energy at the leading order in h and arbitrary L is given by

$$E_h(L) = -h^2 \int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\widetilde{p}^Q}{du}\, 4\, Q^2\, e^{-L\widetilde{\mathcal{E}}_Q} = -h^2 \sum_{Q=1}^{\infty} \int \frac{d\widetilde{p}^Q}{2\pi}\, 4\, Q^2\, e^{-L\widetilde{\mathcal{E}}_Q}\,.$$

For $L = 2$ the series in Q diverges?! as $\frac{1}{Q}$

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For $L = 2$ the series in Q diverges?! as $\frac{1}{Q}$

Ground state energy: any L, small h

Since $\widetilde{\mathcal{E}}_Q = \log \frac{x^{Q-}}{x^{Q+}}$, $x^{Q\pm}(u) = x(u \pm \frac{i}{g}\, Q)$, the Y_Q-functions acquire the form

$$Y_Q = 4\, h^2\, Q^2 \left(\frac{x^{Q+}}{x^{Q-}} \right)^L + \mathcal{O}(h^3)\,.$$

Y_Q-functions are NOT analytic on the u-plane.

In terms of the z-torus rapidity variable $x^{Q+}/x^{Q-} = (\mathrm{cn}\ z + i\, \mathrm{sn}\ z)^2$.
Thus, the Y_Q-functions ARE meromorphic on the z-torus if

$$L = \frac{1}{T} \quad \text{is integer or half-integer.}$$

L is quantized!!! if Y_Q is analytic on z-torus.

The ground state energy at the leading order in h and arbitrary L is given by

$$E_h(L) = -h^2 \int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\widetilde{p}^Q}{du}\, 4\, Q^2\, e^{-L\widetilde{\mathcal{E}}_Q} = -h^2 \sum_{Q=1}^{\infty} \int \frac{d\widetilde{p}^Q}{2\pi}\, 4\, Q^2\, e^{-L\widetilde{\mathcal{E}}_Q}\,.$$

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Ground state energy: any h, large L

Generalized Lüscher formula — Janik, Lukowski '07

$$E_{\rm gL}(L) = -\int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\tilde{p}^Q}{du}\, e^{-L\,\tilde{\mathcal{E}}_Q}\, {\rm tr}_Q e^{i(\pi+h)F} + \cdots .$$

The trace runs through all $16\,Q^2$ polarizations of a Q-particle state. We obtain

$$E_{\rm gL}(L) = -\int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\tilde{p}^Q}{du}\, 16\,Q^2 \sin^2\frac{h}{2}\, e^{-L\,\tilde{\mathcal{E}}_Q} + \cdots .$$

At small values of h it agrees with the previous one.

Expansion of Y-functions in terms of $e^{-L\,\tilde{\mathcal{E}}_Q}$ is similar to the small h one

$$Y_Q \approx 16\,Q^2 \sin^2\frac{h}{2}\, e^{-L\,\tilde{\mathcal{E}}_Q}\,, \quad Y_\pm^{(\alpha)} \approx 1\,, \quad Y_{M|w}^{(\alpha)} \approx Y_{M|vw}^{(\alpha)} \approx M(M+2)\,,$$

and the energy of the ground state agrees with the Lüscher formula.

For $h=\pi$ it should give the energy of the non-BPS ground state in the sector with anti-periodic fermions.

Ground state energy: any h, large L

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Analyticity of Y-system

Frolov, Suzuki '09

$Q = $ 1-particle: $\log Y_1 = \log \frac{\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)}{1+\frac{1}{Y_2}} \star s - \Delta \star s$

$$\begin{aligned}\Delta(u) &= \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\left(\theta(-u-2)+\theta(u-2)\right)\\ &+ L\check{\mathcal{E}} - \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\left(1-\frac{e^{ih_1}}{Y_+^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_+^{(2)}}\right)\star\check{K}\\ &- \log\left(1+\frac{1}{Y_{M|vw}^{(1)}}\right)\left(1+\frac{1}{Y_{M|vw}^{(2)}}\right)\star\check{K}_M + 2\log(1+Y_Q)\star\check{K}_Q^{\Sigma}\,,\end{aligned}$$

in the both small h and large L cases we get

$$\Delta = L\check{\mathcal{E}} = L\log\frac{x(u+i0)}{x(u-i0)} \neq 0 \qquad \text{for}\ \ u\in(-\infty,-2)\cup(2,\infty)\,.$$

Since Δ does not vanish, the TBA equations do NOT lead to an *analytic* Y-system.

Analyticity of Y-system

Y-system on an infinite genus surface? Y-equations valid on its particular sheet?

The Y-equation is obtained from the TBA one by applying the operator s^{-1}

$$e^{\Delta(u)}Y_1(u+\frac{i}{g}-i0)Y_1(u-\frac{i}{g}+i0)=\frac{\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)}{1+\frac{1}{Y_2}}\,.$$

From the explicit ground state solution we find that the jump discontinuity of $\log Y_1(u\pm\frac{i}{g})$ across the real u-line is given by $\pm\Delta(u)$, and, therefore,

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- Y_1-equation might hold on the u-plane with the cuts from $\pm2\pm\frac{i}{g}$ to $\pm\infty$ if the shifts upward and downward have the infinitesimal parts of the same sign.
- The equations for Y_Q $(Q\geq 2)$ would then induce infinitely many cuts on the u-plane with the branch points located at $\pm2\pm\frac{i}{g}Q$.
- Y-system equations can have the canonical form only on a particular sheet of the infinite genus Riemann surface, and would take different forms on other sheets.
- Understand the corresponding transformation properties of the Y-system.
- This is in contrast to relativistic models.
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Conclusion

- Find TBA eqs to account for excited state energies P. Dorey, Tateo '96
- Reproduce known string and field theory results by using the TBA eqs
  - Classical and one-loop spinning string energies Frolov, Tseytlin '03
  - Finite-gap integral equations Kazakov, Marshakov, Minahan, Zarembo '04
  - Finite-size giant magnon energy Arutynov, Frolov, Zamaklar '06
  - Bethe-Yang equations Beisert, Staudacher '05
  - 5-loop Konishi and twist two Bajnok, Hegedus, Janik, Lukowski '09
- Compute analytically anomalous dimension of Konishi and twist two operators up to 12 (any?) loops
- Compute numerically Konishi for any λ Gromov, Kazakov, Vieira '09(b)
- Prove $PSU(2,2|4)$ invariance of the string spectrum
- Prove the gauge independence of the string spectrum
- Understand the Y-system on the infinite genus Riemann surface


we meet again,
professor... and
this time, your
string theory
won't protect you