Integrability of Scattering Amplitudes in multi-Regge kinematics in $N = 4$ SUSY

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References: J. Bartels, L. N. Lipatov, A. Sabio Vera, arXiv:0802.2065; 0807.0894; L.N.Lipatov, arXiv: 0902.1444

1   Gluon reggeization
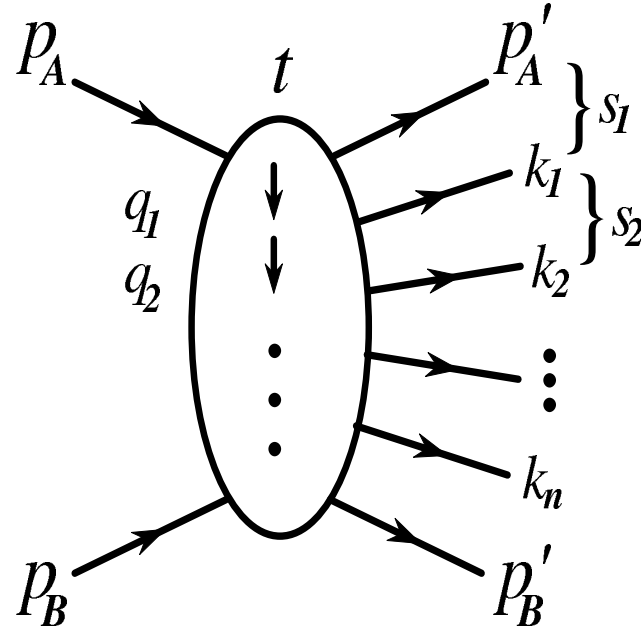
Regge kinematics of scattering amplitudes

$$s = 4E^2 \gg -t = |q|^2 \approx E^2 \, \theta^2$$

Elasic amplitude in LLA of QCD

$$M_{AB}^{A'B'}(s,t) = s \, g \, T_{A'A}^c \, \delta_{\lambda_{A'} \lambda_A} \, \frac{s^{\omega(t)} + (-s)^{\omega(t)}}{t} \, g \, T_{B'B}^c \, \delta_{\lambda_{B'} \lambda_B}$$

Gluon Regge trajectory in the leading order

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k \, \frac{|q|^2}{|k|^2 |q-k|^2}$$

$$= -\frac{\alpha_s N_c}{2\pi} \, \ln \frac{|q^2|}{\lambda^2} = -\frac{\alpha_s N_c}{2\pi} \left(\ln \frac{|q^2|}{\mu^2} - \frac{1}{\epsilon} \right)$$

Multi-Regge amplitudes in LLA (F.,K.,L. (1975))

$$M_{2 \to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2} \, g T_{c_2 c_1}^{d_1} C(q_2, q_1) \, \frac{s_2^{\omega_2}}{|q_2|^2} ... g T_{c_{n+1} c_n}^{d_n} \, C(q_{n+1}, q_n) \, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2} \, ,$$

$$C(q_2, q_1) = \frac{q_2 \, q_1^*}{q_2^* - q_1^*} \, , \ \sigma_t = \sum_n \int d\Gamma_n \, |M_{2 \to 2+n}|^2$$

2 Analyticity, unitarity and bootstrap

Steinmann relations for overlapping channels

$$\Delta_{s_r} \Delta_{s_{r+1}} M_{2 \to 2+n} = 0$$

Dispersion representation for $M_{2 \to 3}$ in the Regge ansatz

$$M_{2 \to 3} = c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)} + c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

Dispersion representation for $M_{2 \to 4}$ in the Regge ansatz

$$M_{2 \to 4} = d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2} + d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$

$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1} + d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$

$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2} \ , \ \ j_r = j(t_r)$$

Bootstrap relation in LLA

$$-i\pi\omega(t_1) M_{2 \to 2+n} = \sum_{r=(0,1,2...)} \Delta_{s_r} M_{2 \to 2+n} = \sum_t M_{2 \to 2+t} M^*_{2+t \to 2+n}$$

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3 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1, \vec{\rho}_2)\,,\ \sigma_t \sim s^{\Delta}\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^*$$

$$+\frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 - 4\psi(1)\,,\ \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,$$

$$m = \gamma + n/2\,,\ \widetilde{m} = \gamma - n/2\,,\ \gamma = 1/2 + i\nu$$

4 BKP equation (1980)

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\,,\;\; H = \sum_{k<l}\frac{\vec{T}_k\vec{T}_l}{-N_c}\,H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = h + h^*\,,\;\; h_{12} = \ln p_1 + \ln p_2 + \frac{1}{p_1}(\ln\rho_{12})\,p_1 + \frac{1}{p_2}(\ln\rho_{12})\,p_2 - 2\psi(1)$$

Holomorphic factorization of wave functions

$$\Psi(\vec{\rho}_1,\vec{\rho}_2,...,\vec{\rho}_n) = \sum_{r,s}a_{r,s}\,\Psi_r(\rho_1,...,\rho_n)\,\Psi_s(\rho_1^*,...,\rho_n^*)$$

Möbius invariance

$$M_a^2\Psi_r = m(m-1)\Psi_r\,,\;\; M_a^{*2}\Psi_s = \widetilde{m}(\widetilde{m}-1)\Psi_s$$

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5   Integrability at $N_c \to \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} , \; T(u) = A(u) + D(u) ,$$

$$L_k = \begin{pmatrix} u + \rho_k \, p_k & p_k \\ -\rho_k^2 \, p_k & u - \rho_k \, p_k \end{pmatrix} , \; [T(u), h] = 0$$

Yang-Baxter equation (L. (1993))

$$t_{r_1'}^{s_1}(u) \, t_{r_2'}^{s_2}(v) \, l_{r_1 r_2}^{r_1' r_2'}(v - u) = l_{s_1' s_2'}^{s_1 s_2}(v - u) \, t_{r_2}^{s_2'}(v) \, t_{r_1}^{s_1'}(u) , \; \hat{l} = u \, \hat{1} + i \, \hat{P}$$

Duality symmetry (L. (1999))

$$p_r \longrightarrow \rho_{r+1,r} \longrightarrow p_{r+1}$$

Heisenberg spin model (L. (1994); F., K.(1995))

6 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\,\chi(n,\gamma) + 4\,\hat{a}^2\,\Delta(n,\gamma)\,,\ \ \hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\ M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\ \beta'(z) = \frac{1}{4}\left[\Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right)\right]$$

Maximal transcendentality (2002) and integrability for γ (L (1997))

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\ -\ \frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

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7 Elastic BDS amplitude

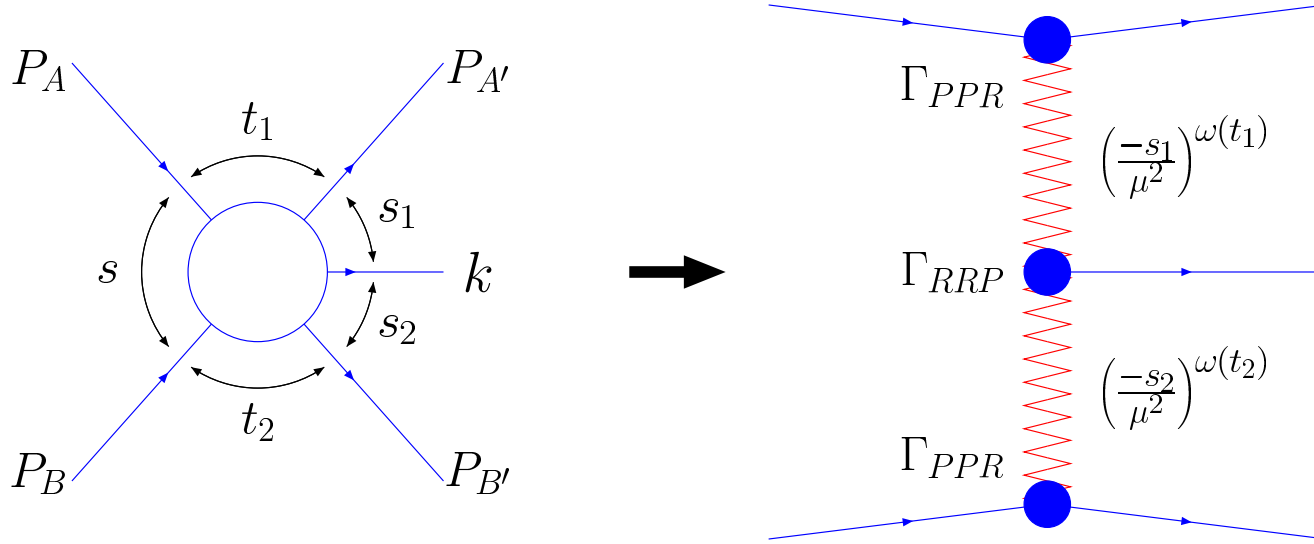$$M_{2 \to 2} = \Gamma(t) \left(\frac{-s}{\mu^2} \right)^{\omega(t)} \Gamma(t)$$

Reggeized gluon trajectory

$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a') \right)$$

Reggeon residues

$$\ln \Gamma(t) = \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2} \right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2} \zeta_2$$

$$- \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a') \right)$$
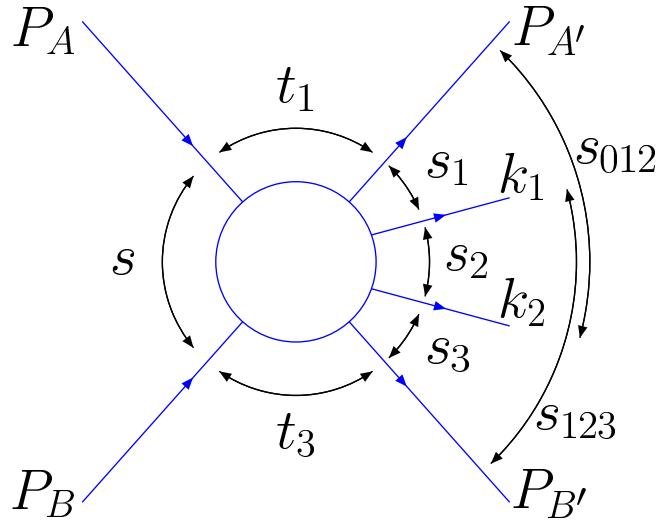
8   One particle production (BLS)

$$\ln \Gamma_{\kappa = s_1 s_2 / s} = -\frac{1}{2} \left(\omega(t_1) + \omega(t_2) - \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a') \right) \right) \ln \frac{-\kappa}{\mu^2} -$$

$$\frac{\gamma_K(a)}{16} \left(\ln^2 \frac{-\kappa}{\mu^2} - \ln^2 \frac{-t_1}{-t_2} - \zeta_2 \right) - \frac{1}{2} \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a') \right)$$

9 Regge factorization violation

$$M_{2\to4}\big|_{s,s_2>0;\,s_1,s_3<0} = \exp\left[\frac{\gamma_K(a)}{4}\,i\pi\left(\ln\frac{t_1 t_2}{(\vec{k}_1+\vec{k}_2)^2\mu^2}-\frac{1}{\epsilon}\right)\right]$$

$$\times\Gamma(t_1)\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma(t_2,t_1)\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\Gamma(t_3,t_2)\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}\Gamma(t_3)$$
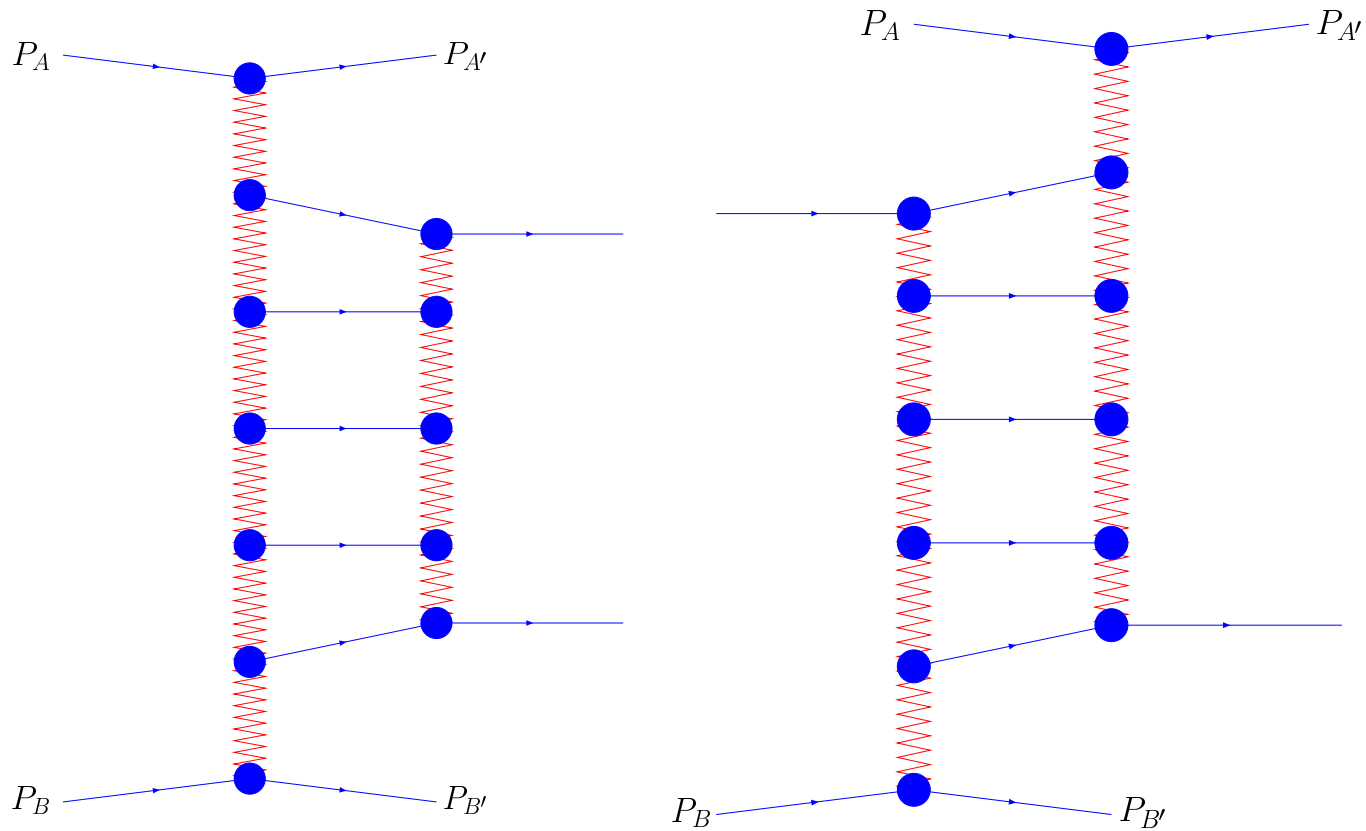
10   Mandelstam cuts in j_2-plane

Figure 1: BFKL ladders in $M_{2\to 4}$ and $M_{3\to 3}$

11   BFKL equation for octets (BLS)

Regge singularities in the t_2-channel

$$\omega(t_2) = -a\left(E + \ln\frac{-t_2}{\mu^2} - \frac{1}{\epsilon}\right)$$

BFKL hamiltonian for the partial wave f_{j_2}

$$H = \ln\frac{|p_1 p_2|^2}{|p_1 + p_2|^2} + \frac{1}{2}\,p_1 p_2^* \ln|\rho_{12}|^2 \frac{1}{p_1 p_2^*} + \frac{1}{2}\,p_1^* p_2 \ln|\rho_{12}|^2 \frac{1}{p_1^* p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left(\frac{p_1}{p_2}\right)^{i\nu+n/2}\left(\frac{p_1^*}{p_2^*}\right)^{i\nu-n/2}, \quad E_{n,\nu} = 2Re\,\psi(i\nu + \frac{|n|}{2}) - 2\psi(1)$$

Factorization of infrared divergencies in LLA

$$M^{2\to4}_{|s,s_2>0;s_1,s_3<0} = (1 + i\Delta_{2\to4})\,M^{BDS}_{2\to4},$$

12 Möbius and conformal invariances

Analytic result in LLA in the region $a \ln s_2 \sim 1$

$$\Delta_{2\to4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} (V^*)^{i\nu - \frac{n}{2}} V^{i\nu + \frac{n}{2}} \left(s_2^{\omega(\nu,n)} - 1 \right)$$

Duality transformation to the Möbius representation

$$V = \frac{q_3 k_1}{k_2 q_1} \to \frac{z_{03} z_{0'1}}{z_{0'3} z_{01}}$$

Perturbation theory expansion

$$i\Delta_{2\to4} = -2i\pi \, a^2 \ln s_2 \ln \frac{|k_1 + k_2||q_2|}{|k_2||q_1|} \ln \frac{|k_1 + k_2||q_2|}{|k_1||q_3|} + ...$$

Functions of 4-dimensional anharmonic ratios

$$i\Delta_{2\to4} = \frac{a^2}{4} Li_2(\chi) \ln \frac{\chi t_2 s_{13}}{s_3 t_1} \ln \frac{\chi t_2 s_{02}}{t_3 s_1} + ..., \ \chi = 1 - \frac{s s_2}{s_{012} s_{123}}$$

13 Multi-gluon states in octet channels

Channels with the Mandelstam cuts constructed from n gluons

$$s_1, s_2, ..., s_{n-1}, s_{n+1}, ..., s_{2n} < 0 \,, \; s, s_n > 0$$

Schrödinger equation for octet composite states

$$H\Psi = E\Psi \,, \;\; \omega(t) = a\left(-\ln\frac{-t}{\mu^2} + \frac{1}{\epsilon}\right) - \frac{a}{2}E \,, \;\; a = \frac{g^2 N_c}{8\pi^2}$$

Holomorphic separability

$$H = h + h^* \,, \;\; h = \ln\frac{p_1 p_n}{q^2} + \sum_{r=1}^{n-1} h^t_{r,r+1}$$

Helpful operator relation

$$\ln\partial = -\ln x + \frac{1}{2}\left(\psi(x\partial + 1) + \psi(-x\partial)\right)$$

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14 Möbius invariance in the p-space

$$p_k = Z_{k-1,k}\,, \;\; \rho_{k,k+1} = i\frac{\partial}{\partial Z_k} = i\partial_k$$

$$h = \ln(Z_1^2\partial_1) - 2\psi(1) + \ln\partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}\,, \;\; Z_0 = 0\,, \;\; Z_n = \infty\,,$$

$$h_{1,2} = \ln(Z_{12}^2\partial_1) + \ln(Z_{12}^2\partial_2) - 2\ln Z_{12} - 2\psi(1)$$

$$A' = Z_1 \prod_{s=1}^{n-2} Z_{s,s+1} \prod_{r=1}^{n-1} \partial_r\,, \;\; [h, A'] = 0$$

15 Integrable Heisenberg spin chain

$$[L_k(u)L_{k+1}(u), h_{k,k+1}] = -i\left(L_k(u) - L_{k+1}(u)\right)$$

Integrals of motion: $[D, h] = 0$

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k\,, \quad q'_k = -\sum_{0 < r_1 < ... < r_k < n} Z_{r_1} \prod_{s=1}^{k-1} Z_{r_s, r_{s+1}} \prod_{t=1}^{k} i\partial_{r_t}$$

Sklyanin ansatz and Baxter equation

$$\Omega = \prod_k Q(\hat{u}_k)\,\Omega_0\,, \quad \Omega_0 = \prod_{l=1}^{n-1} \frac{1}{|Z_l|^4}\,,$$

$$D(u)Q(u) = (u + i)^{n-1}Q(u + i)$$

16 Three-gluon composite state

$$\Psi = Z_2^{a_1+a_2} (Z_2^*)^{\widetilde{a}_1+\widetilde{a}_2} \int \frac{d^2 y}{|y|^2} \, y^{-a_2} \, (y^*)^{\widetilde{a}_2} \left(\frac{y-1}{y - Z_2/Z_1} \right)^{a_1} \left(\frac{y^*-1}{y^* - Z_2^*/Z_1^*} \right)^{\widetilde{a}_1}$$

$$\Psi(\vec{Z}_1, \vec{Z}_2) = \int d^2 p_1 \, d^2 p_2 \, \exp(i\vec{p}_1\vec{Z}_1) \, \exp(i\vec{p}_2\vec{Z}_2) \, \Psi(\vec{p}_1, \vec{p}_2) \,, \ \ E = E(a_1) + E(a_2)$$

$$\Psi^t(\vec{p}_1, \vec{p}_2) = P^{-a_1-a_2} (P^*)^{-\widetilde{a}_1-\widetilde{a}_2} \int d^2 u \, u \, \widetilde{u} \, Q(u, \widetilde{u}) \left(\frac{p_1}{p_2} \right)^u \left(\frac{p_1^*}{p_2^*} \right)^{u^*}$$

$$Q(u, \widetilde{u}) = \frac{\Gamma(-u)\,\Gamma(-\widetilde{u})}{\Gamma(1+u)\,\Gamma(1+\widetilde{u})} \frac{\Gamma(u-a_1)\,\Gamma(u-a_2)}{\Gamma(1-\widetilde{u}+\widetilde{a}_1)\,\Gamma(1-\widetilde{u}+\widetilde{a}_2)} \,, \ \int d^2 u = \int d\nu \sum_n$$

17 Discussion

1. Multi-Regge amplitudes and bootstrap relations.
2. Integrability of BFKL dynamics in LLA.
3. Remarkable properties of NLLA in $N = 4$ SUSY.
4. BDS amplitudes in the multi-Regge kinematics.
5. Breakdown of the Regge factorization.
6. Mandelstam cuts in the planar amplitudes M_n for $n > 5$.
7. Integrable open spin chain for scattering amplitudes.