Null polygonal Wilson loops and scattering amplitudes via minimal surfaces in AdS

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Based on work by F. Alday and J. M.

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Integrability 2009, Potsdam

Amplitudes and Wilson loops

- Lots of recent progress in computing the spectrum of the theory.

What is next ?

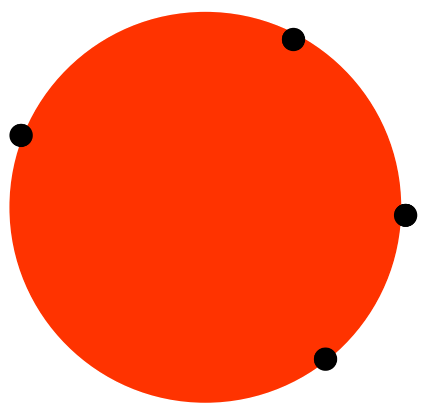
- Correlation functions
- Wilson loops
- Scattering amplitudes

Integrability

- We know that the theory is integrable.
- For each problem we need to develop some method that will enable us to solve it
- In developing these methods it is often useful to study classical solutions.
- We will study classical solutions of the sigma model that are related to scattering amplitudes and Wilson loops.

Amplitudes

- 4 dimensional scattering amplitudes are an interesting (quasi) observable of the four dimensional theory.
- Disk diagram in string theory



Amplitudes at strong coupling

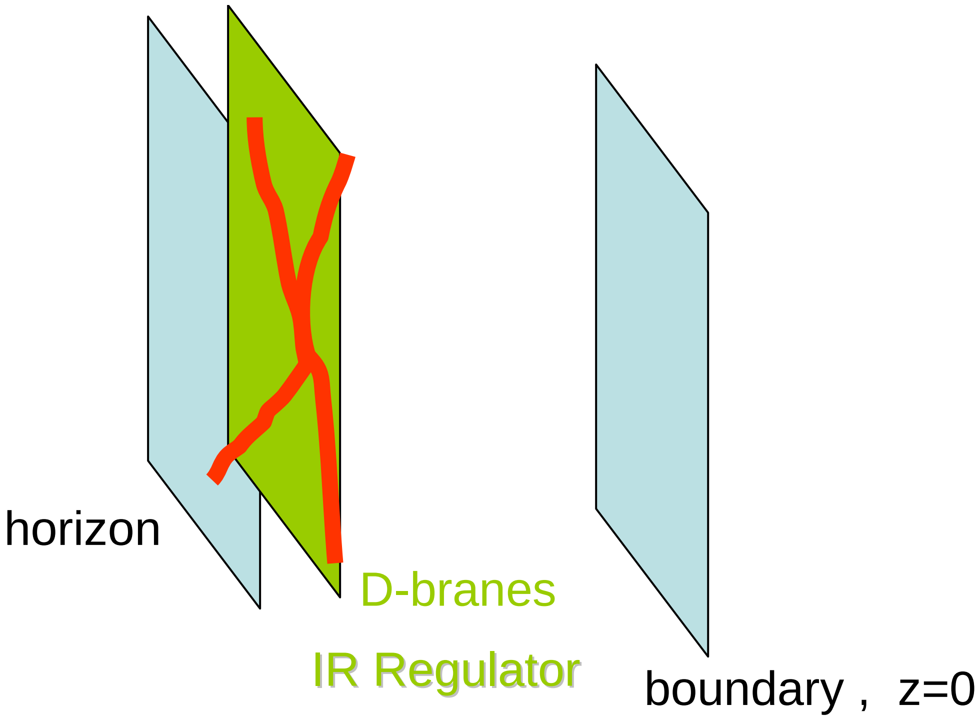
- Strong coupling has a description in terms of a simple string theory in $AdS_5 \times S^5$
- The knowledge of both weak and strong coupling helps in finding them for all coupling

- The extra symmetries associated to integrability are easier to see at strong coupling

Note: - N=4 SYM is really different than QCD at strong coupling.
- Large IR divergencies at strong coupling → hard to set up the experiment to see them (very suppressed).

Amplitudes at Strong coupling

Alday & JM

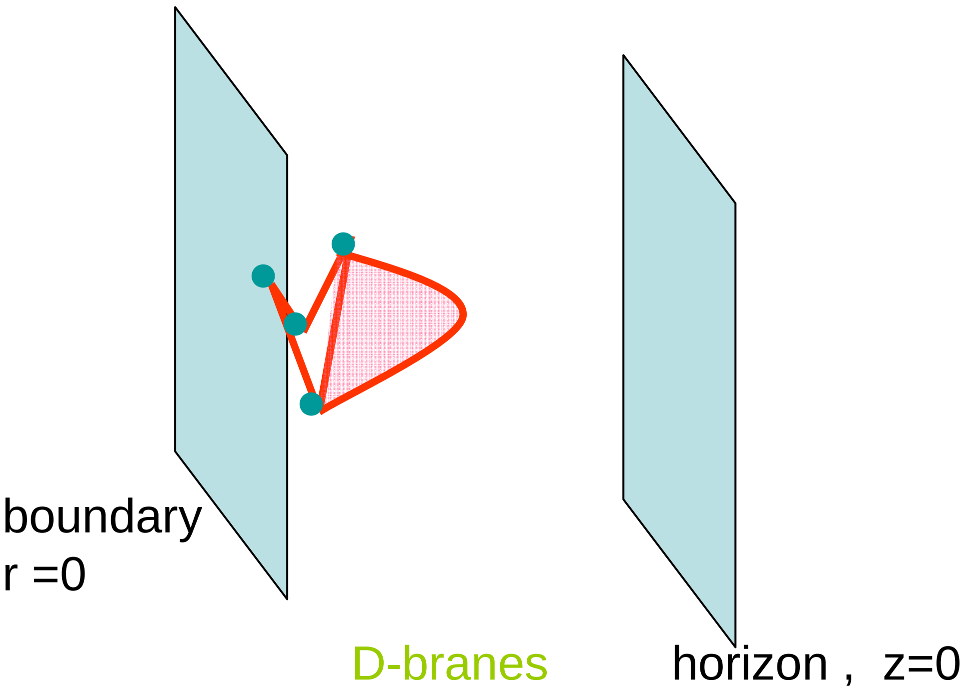


Original

$$ds^2 = \frac{dx^2 + dz^2}{z^2}$$

T – duality:

$$dy = *_2 \frac{dx}{z^2}, \qquad r = \frac{1}{z}$$



T-dual

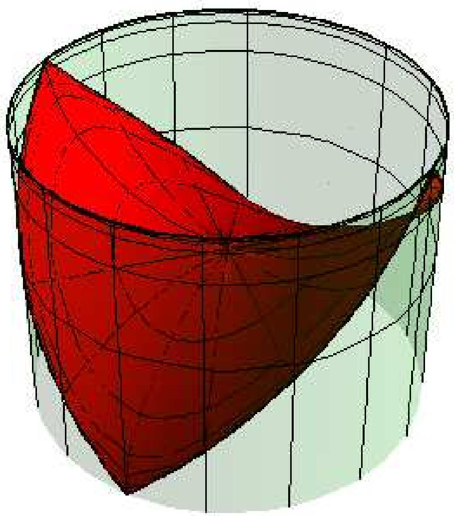
$$ds^2 = \frac{dy^2 + dr^2}{r^2}$$

Wilson loops

Area of a minimal surface in AdS that ends on the polygon

$$\mathcal{A} \sim \langle W \rangle \sim e^{-\frac{R^2}{2\pi\alpha'}(\mathrm{Area})} = e^{-\frac{\sqrt{\lambda}}{2\pi}(\mathrm{Area})}$$

Depends on
the kinematics



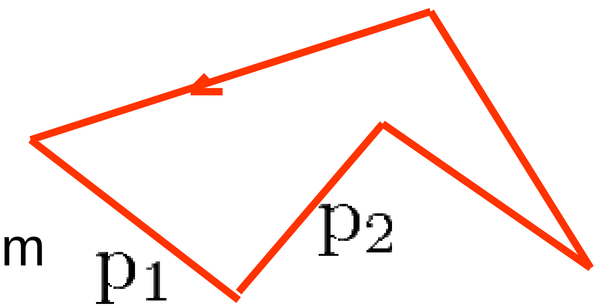
Original AdS ⟷ Dual $\widetilde{\text{AdS}}$

SO(2,4) ⟷ Dual $\widetilde{\text{SO(2,4)}}$

New Symmetry → More constraints on the amplitude

Amplitudes = Wilson loop, null polygon

- Polygon
- null sides
- each side= momentum



The new symmety is related to integrability

Beisert, Ricci, Tseytlin
Wolf
Berkovits & JM

Bosonic + fermionic T-duality → We have dual conformal symmetry to all orders in α' perturbation theory.

Dual symmetries = higher charges of the sigma model + the ordinary ones → Infinite number of conserved charges.

We have argued that the two conformal symmetries are present in the quantum theory → integrability in the quantum theory.

Present at weak coupling

Drummond, Henn, Plefka,
Bargheer, Beisert, Galleas,
Loebbert, McLoughlin

Constraints of (dual) conformal symmetry

Bern Dixon
Smirnov

One loop result

(up to single
log divergencies)

$$\langle W \rangle = \exp\left[\Gamma_{\rm cusp}({\rm Div} + {\rm BDS}) + R\right]$$

IR divergent piece

cusp anomalous dimension

Finite remainder
Function of cross
ratios

Drummond, Henn,
Korchemsky, Sokatchev

Ward identities for broken dual conformal symmetry can be proven in the Wilson loop side

Cross ratios

Define y coordinates: $$p_i^\mu = y_i^\mu - y_{i+1}^\mu$$

Cross ratios $$\chi = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}$$ Any four points

N=4, 5 no cross ratios → 4 and 5 point amplitudes computed for all values of the coupling.

Large N is harder…

N=6 3 cross ratios → 2 loops: Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich / Drummond, Henn, Korchemsky, Sokatchev. Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travaglini

N=7 6 cross ratios

N=8 9 cross ratios ← Analyze this at strong coupling

(N = number of gluons)

Problem

- Give some points on the boundary that produce a polygonal contour with null sides.
- Compute the area as a function of the position of the points.
- Compute the function R in terms of the conformal cross ratios.

Exploring the remainder function R

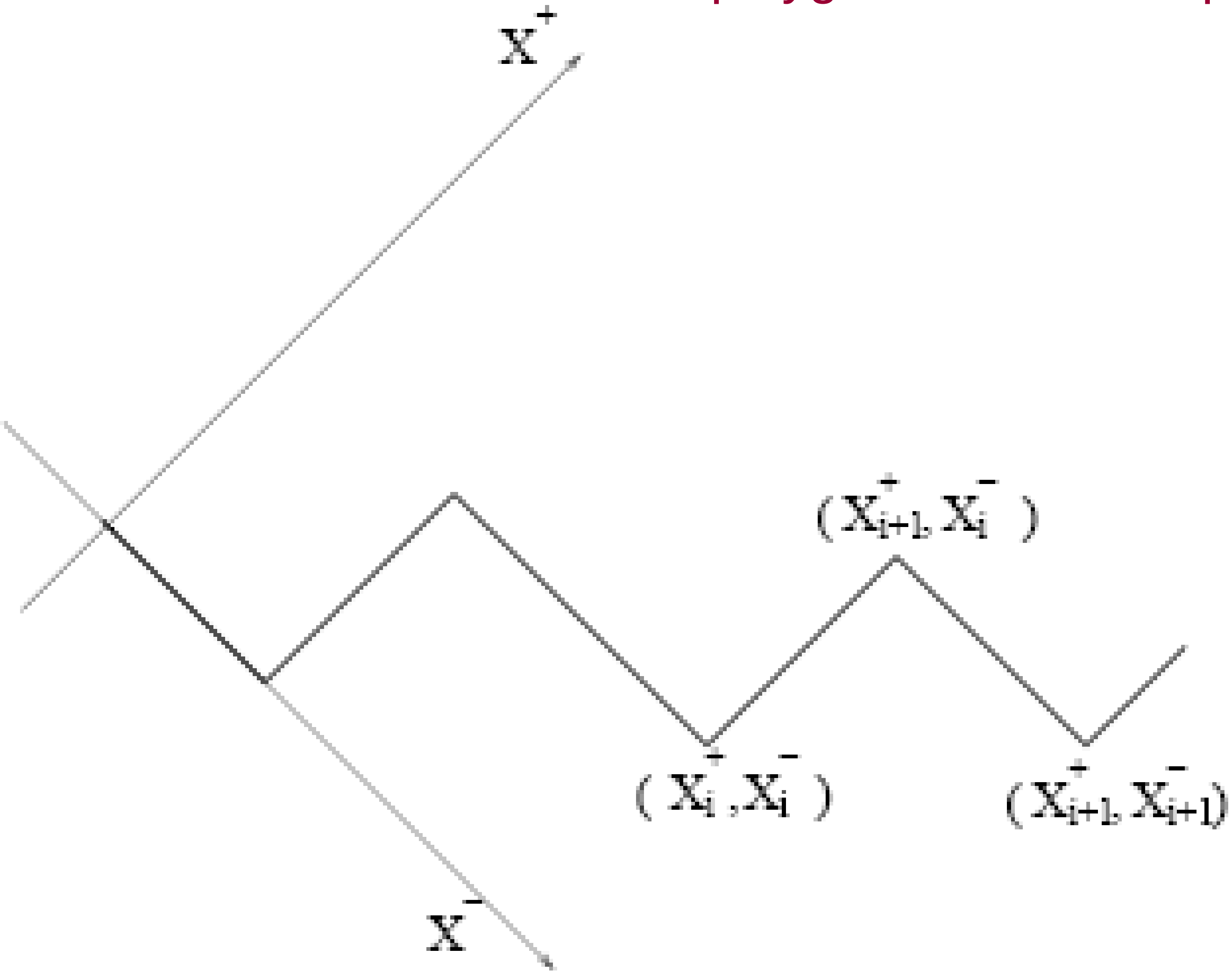
- Look at the dependence on only some of the variables.

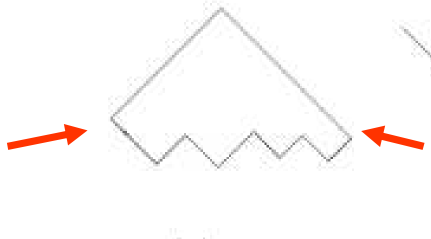
Special kinematics

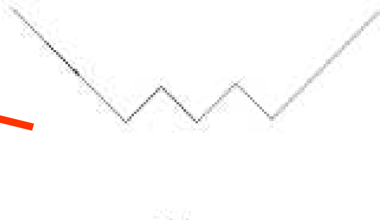
- Particles with momentum in 1 +1 dimensions. (Full 3+1 dimensional theory in the loops.)
- At strong coupling → the string surface lives in AdS_3

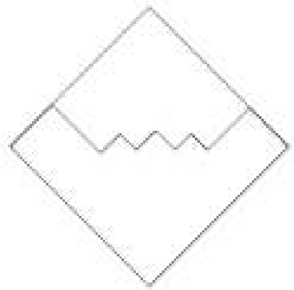
$$p_i^{\pm} \qquad i = 1, \cdots n \qquad\qquad N = 2\, n$$

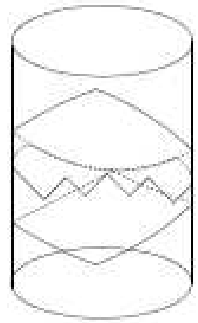
Null polygonal Wilson loop in $R^{1,1}$

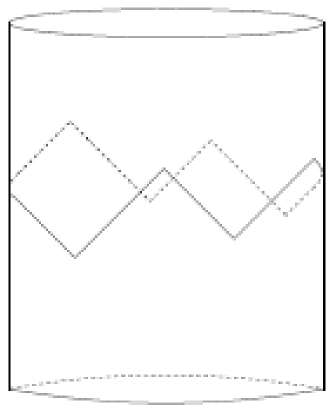


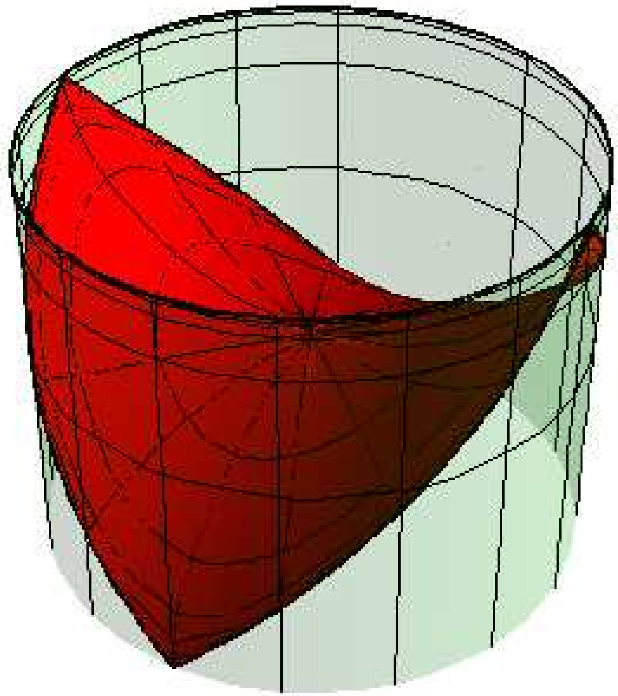












Wilson loops in $R^{1,1}$

- Number of gluons   N = 2 n
- n-3   plus cross ratios from n x_i^+
- n-3   minus cross ratios from n x_i^-

Conformal group SO(2,2) = SL(2) x SL(2) = 3 + 3 generators

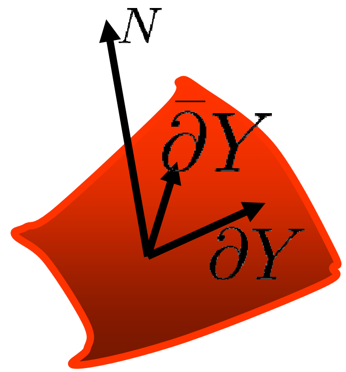
6 gluons, 6 sides , no cross ratio

8 gluons, 8 sides (n=4) , one cross ratio of each kind, 2 cross ratios.

In 4 dimensions N=8 had 9 cross ratios , here we are exploring a 2 dimensional subspace

Strings in AdS_3

Pohlmeyer,
Jevicki, Jin, Kalousios, Volovich



$$Y^M \ , \quad -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$$

$$Y^M, \quad \partial Y^M, \quad \bar\partial Y^M \ , \quad N^M = \text{normal}$$

$$e^{\alpha} = \partial Y . \bar\partial Y \ , \qquad p(z) = \partial^2 Y . N \ , \qquad \bar p(\bar z) = \bar\partial^2 Y . N$$

equations of motion + Virasoro constraints $\quad T_{zz} = T_{\bar z \bar z} = 0$

$$\bar\partial\partial\alpha - e^{\alpha} + e^{-\alpha}|p|^2 = 0$$

Generalized Sinh Gordon equation

This equation is worldsheet conformal invariant

$$z \to w(z) \ , \qquad p(z)dz^2 \to \tilde p(w) dw^2$$

p : Holomorphic function
α single degree of freedom

$$\mathbf{Area} = \int d^2 z e^{\alpha}$$

Explicitly SO(2,2) invariant in target space

Recovering the spacetime coordinates

Linear problem

$$\mathrm{d}\,\psi^L + B^L \psi^L = 0 \qquad d\psi^R + B^R \psi^R = 0$$

B^R_z, $B^R_{\bar{z}}$, B^L_z $B^L_{\bar{z}}$ are 2 x 2 matrices that depend on α and p

e.g. $\begin{pmatrix} \partial\alpha & e^{\alpha} \\ pe^{-\alpha} & -\partial\alpha \end{pmatrix}$

we have two solutions for each equation.

ψ^L_a , $\psi^R_{\dot{a}}$

$$\mathrm{Y}_{a\dot{a}} = (\psi^L_a)^t \psi^R_{\dot{a}}$$

We recover the spacetime coordinates from the solutions

$a, \dot{a}$ = 1, 2 are spacetime indices

Compare to $g = g_L(z) g_R(\bar{z})$ for WZW models

Boundary conditions

- Worldsheet: whole complex plane
- p = polynomial

For N= 2 n gluons, or 2 n sides:

$$p = z^{n-2} + m_{n-4} z^{n-4} + \cdots m_0$$

Number of non-trivial coefficients of p is n-3 → 2 (n-3) real parameters =
= number of independent cross ratios

Example: 4 Sides

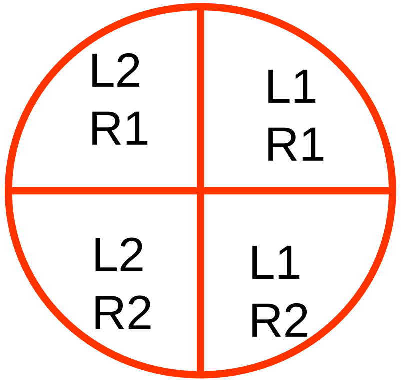
- n=2 or four sides

$$p = 1 \;, \qquad \alpha = 0,$$

$B^L, \quad B^R$ are constant and can be diagonalized

$$\psi_1^L = \begin{pmatrix} e^{z+\bar z} \\ 0 \end{pmatrix}, \qquad \psi_2^L = \begin{pmatrix} 0 \\ e^{-(z+\bar z)} \end{pmatrix}$$

$$\psi_1^R = \begin{pmatrix} e^{\frac{z-\bar z}{i}} \\ 0 \end{pmatrix}, \qquad \psi_2^R = \begin{pmatrix} 0 \\ e^{-\frac{(z-z)}{i}} \end{pmatrix}$$



$$Y_{a\dot a} = (\psi_a^L)^t \psi_{\dot a}^R$$

Area is infinite → regulate it → Usual BDS answer.

ψ grows → Y grows → goes to the boundary as z → ∞

The w "plane"

Set p → 1 via a change of coordinates

$$dw = \sqrt{p(z)}dz$$

Denote by α_w the new alpha variable: $e^{\alpha} = e^{\alpha_w} |p(z)|$

Boundary conditions on α :

- α is regular
- $\alpha_w \to 1 \quad \text{as} \quad |w| \to \infty$

- In the asymptotic regions of the w plane the solution will look like the one we had before → we reproduce the cusps asymptotically.
- We have to go around the asymptotic region of the w plane n/2 times to describe the full polygon.
- We have branch cuts on the w-plane, starting from the zeros of p.

Brute force recipe

Choose p

$$p = z^{n-2} + \cdots$$

↓

Solve Sinh-Gordon equation

$$\bar{\partial}\partial\alpha - e^{\alpha} + e^{-\alpha}|p|^2 = 0$$

↓

Solve the linear problems

$$d\,\psi^L + B^L\psi^L = 0 \qquad d\psi^R + B^R\psi^R = 0$$

↓

Determine the spacetime solution

$$Y_{a\dot{a}} = (\psi^L_a)^t \psi^R_{\dot{a}}$$

↓

Read off the spacetime cross ratios

↓

Compute the area

$$\text{Area} = \int d^2z\, e^{\alpha}$$

Better strategy

- Ask your neighbor

- The mathematics of this problem turns out to be the same as that of a problem studied by

D. Gaiotto, G. Moore & A. Neitzke - arXiv:0807.4723
- to appear

They studied BPS states of 4d N=2 theories and wall crossing

- 4d field theories on a circle ~ 3d field theory
- Hyperkahler metric on the Coulomb branch
- This metric contains the information they wanted
- It also contains the information we want.
- Only interested in a ``real'' section of the metric

Moduli space of vacua of three dimensional field theories

- D4 brane (4+1 dimensional Yang Mills theory) on a Riemman surface → 2+1 dimensional field theory

Cherkis Kapustin
Gaiotto, Moore, Neitzke

$$D_{\bar{z}}\Phi_z = 0 \ , \qquad F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0$$

$$p = \mathrm{Tr}[\ \Phi_z^2]$$

Hitchin equations,
SU(2) group

Vacuua → solutions of the equations.

Moduli → coefficients in p(z).

Metric in moduli space → hyperkahler.

The same mathematical problem, we can use those results !!

- We will describe later how they propose to find the metric (see Gaiotto’s talk)
- For now we will start with a case where the metric was already known.

Metric is known for the simplest case → $\mathrm{p} = \mathrm{z}^2 - m$ or eight sides

U(1) theory + 1 hyper

$$g_{m\bar{m}} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{|m|^2 + (n+1/2)^2}}$$

Ooguri-Vafa
Seiberg-Shenker

The area can be computed:

$$\text{Area} \sim m\partial_m K \ , \longleftarrow \partial_m \partial_{\bar{m}} K = g_{m\bar{m}}$$

In general: The moduli space has a U(1) symmetry under rotations in the plane.
The Area is the D term, or moment map, for the U(1) symmetry.

Regularizing the area

- Physical cutoff

$$r > \mu_{IR} \ , \qquad ds^2 = \frac{dx^+ dx^- + dr^2}{r^2}$$

- Need to know the solution.
- The same solution determines the position of the cusps
- → Write the answer in terms of the position of the cusps.

R = Remainder function

Usual one loop

$$A = A_{div} + A_{BDS} + A_{extra} + A^{reg}_{Kahler}$$

Usual divergent term

Previous formula with the infinity subtracted

Extra term which arises due to the regularization and depends on a certain ``magnetic'' cross ratio

Gaiotto, Moore and Neitzke have computed it

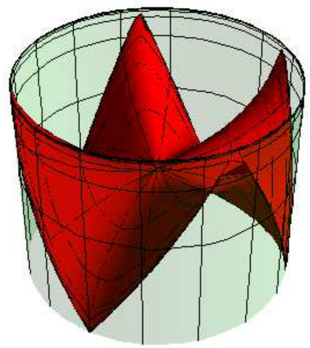
- Parameters of the polynomial in terms of the cross ratios
- Can be determined simply in this case

$$e^{\mathrm{Re}(m)} = \frac{x^+_{43} x^+_{21}}{x^+_{41} x^+_{32}} \qquad e^{\mathrm{Im}(m)} = \frac{x^-_{43} x^-_{21}}{x^-_{41} x^-_{32}}$$

Final answer for the octagon

$$R = \log\cosh \mathrm{Re}(m) \log\cosh \mathrm{Im}(m) + \frac{7\pi}{6} + \int dt \frac{(\bar{m}e^{t} - m e^{-t})}{\tanh 2t} \log\left(1 - e^{-me^{t} + me^{-t}}\right)$$

$$e^{\mathrm{Re}(m)} = \frac{x^{+}_{43} x^{+}_{21}}{x^{+}_{41} x^{+}_{32}} \qquad\qquad e^{\mathrm{Im}(m)} = \frac{x^{-}_{43} x^{-}_{21}}{x^{-}_{41} x^{-}_{32}}$$

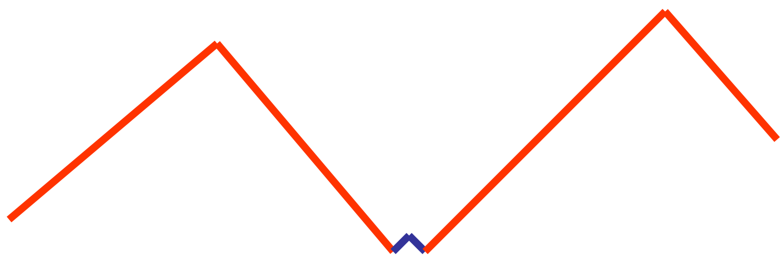


We did not need to find the explicit worldsheet solution !

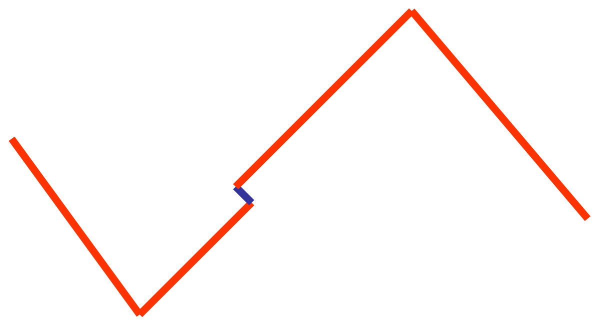
- R goes to a constant as m → Infinity

This constant is related to the solution for the hexagon.

- When m→ Infinity the cross ratios take extreme values and this corresponds to a a double soft or a soft-collinear limit



m→ infinity in a generic direction



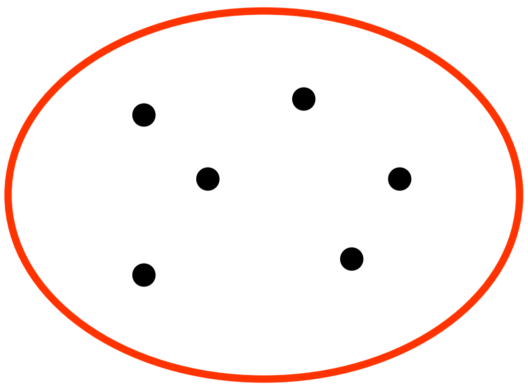
m → infinity along a Stokes line.

Integrability and the general case

- Gaoitto Moore and Neitzke write down a system of equations that should determine the answer.
- They introduce a spectral parameter and consider the cross ratios as a function of the coefficients of the polynomial and the spectral parameter.
- The problem then displays a Stokes phenomenon in the spectral parameter, as it gets small or large.

More precise relation to GMN

- The general story in GMN involves the Hitchin equations on a Riemann surface with some poles for the Higgs field and connection.



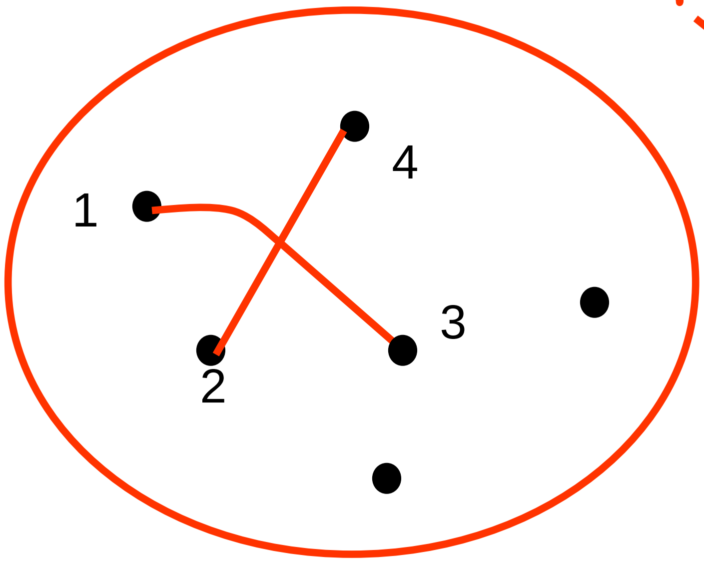
Spectral parameter

$$\mathcal{A}(\zeta) = A + \Phi/\zeta + \zeta\bar{\Phi}$$

Near each point: A small solution, **s**, and a large solution

Non-trivial information of the connection is contained in “cross ratios”

$$\chi(\zeta) = \frac{s_1 \wedge s_2 \; s_3 \wedge s_4}{s_1 \wedge s_3 \; s_2 \wedge s_4}$$



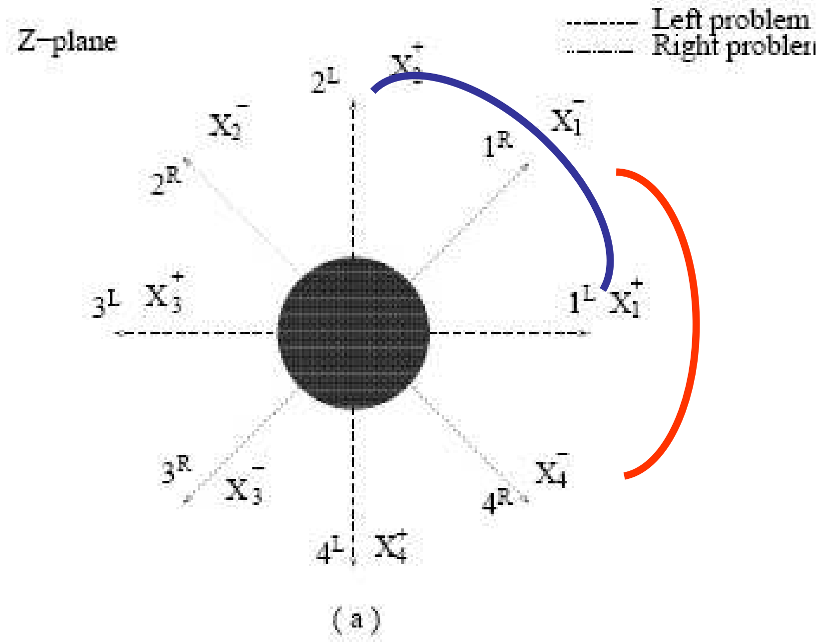
SL(2) invariant inner product

- Study analytic properties of the cross ratios as function of ς
- Formula a Riemann Hilbert problem whose solution determines the cross ratios.

Back to our problem

- All the poles are at infinity → essential singularity
- Stokes sectors
- Cross ratios formed by approaching infinity along various Stokes sectors.
- Cross ratios for $\varsigma = 1$ are the x^+ cross ratios
- Cross ratios for $\varsigma = i$ are the x^- cross ratios

Stokes Sectors



Solutions diverge in
different ways in different sectors

When we change sectors only left problem or only the right problem changes → Cusps are lightlike separated on the boundary.

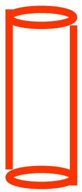
- Once the cross ratios are known as a function of the spectral parameter

$$\langle \frac{d\chi}{\chi} \frac{d\chi}{\chi} \rangle = \zeta[\ ] + \zeta^{-1}[\ ] + g_{m_i \bar{m}_i} dm_i d\bar{m}_i$$

certain inner products of cross ratios.

Extract the metric

States

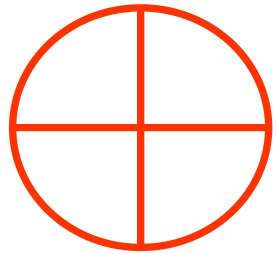


cylinder

Holonomy of the connection around the cylinder

Our case

Plane with stokes sectors



Cross ratios

New recipe

Choose p

$$p = z^{n-2} + m_{n-4} z^{n-4} + \cdots m_0$$

Introduce spectral parameter and think about cross ratios a functions of spectral parameter

$$\chi_i(\zeta; m_i, \bar{m}_i)$$

Write consistency conditions for the discontinuities of these cross ratios as functions of ς

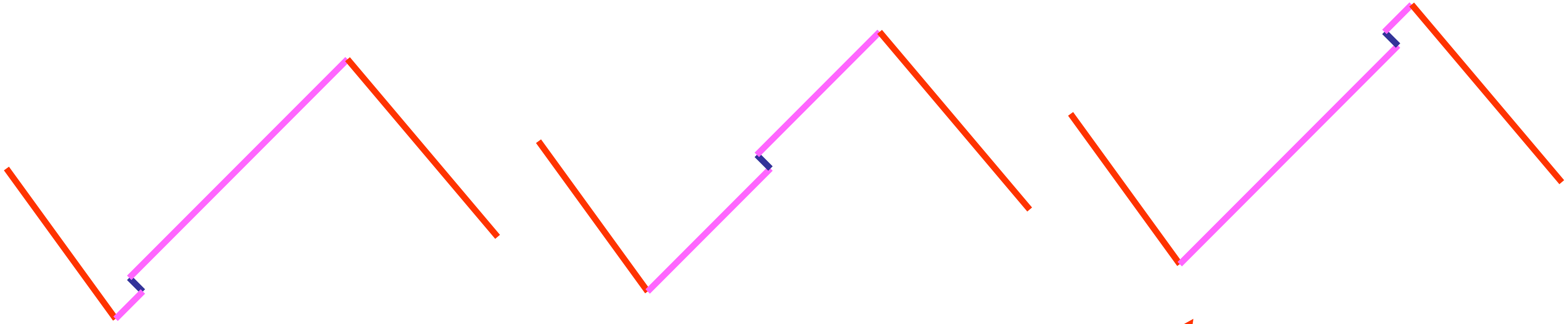
$$\chi_i \to \chi_i(1 - \chi_j)$$

Metric in moduli space

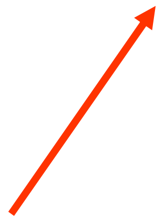
$$g_{m_i \bar{m}_j}$$

Compute the area

Wall Crossing



$$\log \epsilon^- \log \frac{\chi^+}{(1+\chi^+)}$$



$$\mathbf{A} \sim \log \epsilon^- \log \chi^+ \; .$$

$$A \sim -\log \epsilon^- \log \chi^+ + \log \chi^+ \log(1+\epsilon^-)$$

$$\chi^+ \to 0$$

$$\chi^+ \to \infty$$

$$|\log \epsilon^-| \gg |\log \chi^+|$$

Change in the coefficients of the subleading terms in the soft expansion. This change is the same at weak and strong coupling. The full coefficients are different at weak and strong coupling.

Conclusions

- We discussed amplitudes at strong coupling in N=4 SYM
- Relation to Wilson loops
- These symmetries fix the 4 and 5 gluon amplitudes
- For more gluons they leave an undetermined ``remainder'' function.

- This undetermined remainder function was computed for weak coupling at 2 loops and n=6

2 loops: Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich / Drummond, Henn, Korchemsky, Sokatchev. Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travaglini

- Here we computed it at strong coupling and for 8 gluons.

- Connection to moduli spaces of three dimensional theories (and to the wall crossing phenomenon).

- Wall crossing → changes in the subleading terms in the collinear expansion.

Future

- Do the same for minimal surfaces in AdS_5
- Use these classical equations to understand the problem of operators and correlation functions. AdS_3 , AdS_5
- Generalize to all values of the coupling.