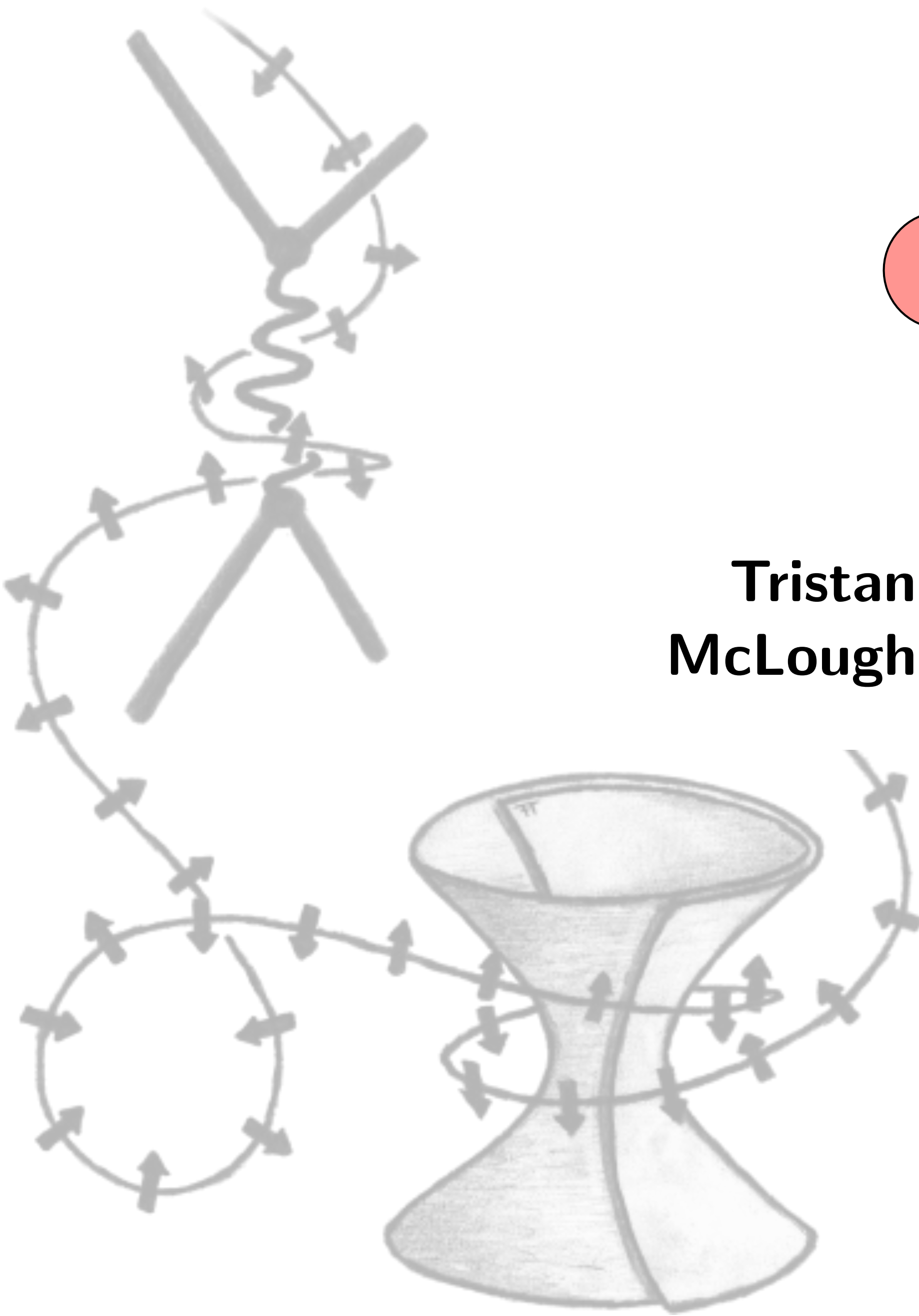
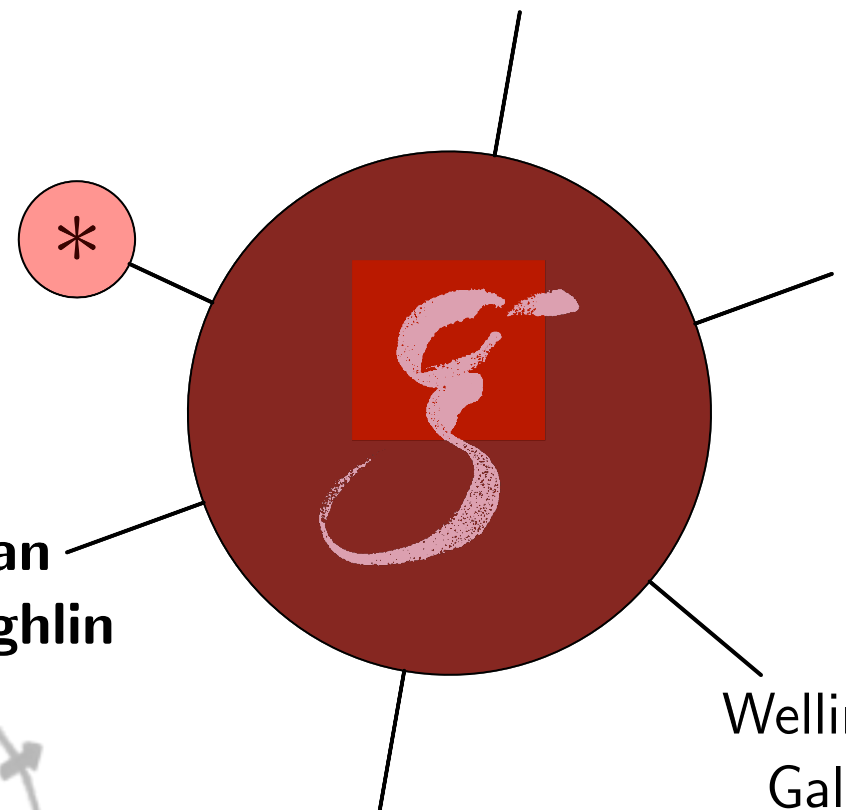
Exacting $\mathcal{N}=4$ Superconformal Symmetry*





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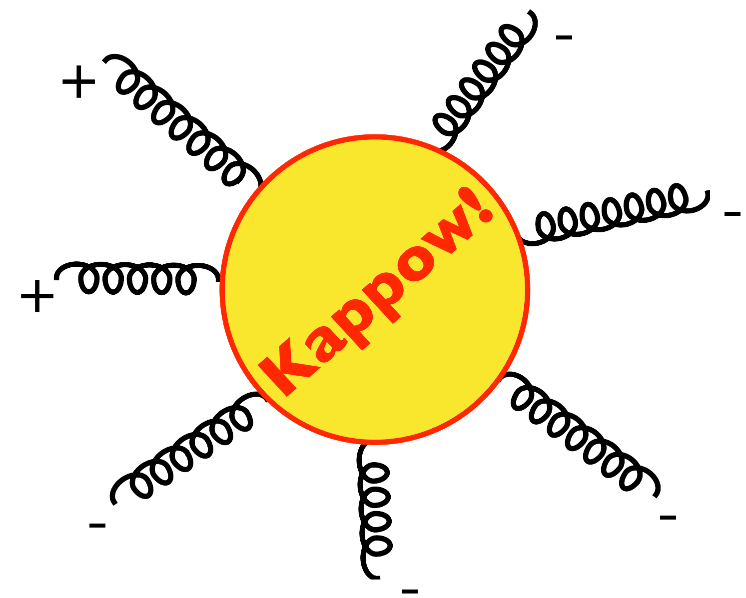
Scattering Amplitudes in $\mathcal{N} = 4$ SYM



- Scattering amplitudes in $\mathcal{N} = 4$ SU(N) SYM possess a great deal of symmetry and simplicity, particularly in the large N limit, which makes for an attractive toy model of more realistic theories. The leading colour contribution to amplitudes is given by,

$$\mathcal{A}_n(\{p_i, h_i, a_i\}) \sim \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A_n(\sigma(1), \ldots, \sigma(n))$$

- One example: all-loop MHV amplitudes: $A_n = A_n^{\mathrm{tree}} M_n$ with

[Anastasiou, Bern, Dixon, Kosower], [Bern,Dixon,Smirnov]

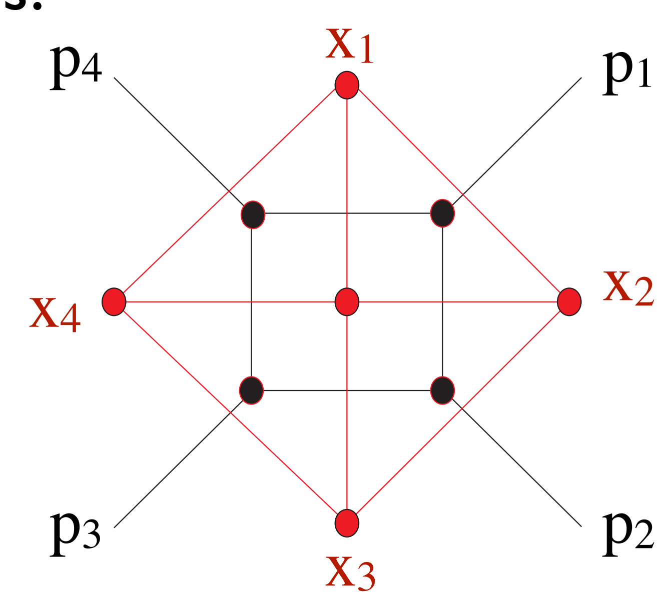
$$\ln M_n = \sum_{i=1}^{n} \frac{-1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right) + f(\lambda) F_{\mathrm{fin}}^{(1)}(\{p_i\}) + R_n(\{p_i\}, \lambda)$$

- Depends on tree level amp., one-loop amp. , cusp & collinear anomalous dimensions + R
- We can find the cusp anomalous dimension from an integrable model (also some results for collinear anom. dim). [Beisert, Eden,Staudacher] [Freyhult,Zieme]
- Are there symmetries behind the other structures of amplitudes? Can we determine R?

Dual conformal invariance

- One can introduce "dual" variables $x_i, \; i = 1, \ldots, n$ which solve the momentum constraint: $(p_i)^{a\dot{a}} = (x_i - x_{i+1})^{a\dot{a}}$ and we identify $x_{n+1} = x_1$.
- Amplitudes only depend on p's so they are trivially invariant under dual translations
- Can consider dual conformal transformations which act linearly on dual space coordinates and dual super-conformal invariance. Superamplitudes appear to be covariant under dual superconformal transformations!

[Drummond, Henn, Korchemsky, Sokatchev]



- This symmetry constrains the integrals that can appear in loop calculations. [Drummond, Henn, Smirnov, Sokatchev]
- Related to the duality between amplitudes and light-like Wilson loops seen perturbatively. [Alday, Maldacena] [Drummond, Henn, Korchemsky, Sokatchev][Brandhuber, Heslop, Travaglini]
- At strong coupling (via AdS/CFT duality) the dual conformal symmetries can be explained via a supersymmetric T-duality transformation that maps the string model to itself. [Alday, Maldacena] [Berkovits, Maldacena]
- Constrains but doesn't fix the remainder term R.

Structure of amplitudes

- The existence of the remainder function has been shown perturbatively using generalised unitarity methods [Bern, Dixon,Kosower,Roiban, Spradlin,Vergu,Volovich]
- There is further evidence from the duality with Wilson loops and high energy scattering limits [Alday, Maldacena] [Drummond,Henn Korchemsky Sokatchev] [Bartels Lipatov Sabio Vera]

- ✦ At strong coupling there is the worldsheet integrable structure based on the $\mathfrak{psu}(2,2|4)$ loop algebra seen in the calculation of string energies. Dual conformal invariance corresponds to a different embedding of $\mathfrak{psu}(2,2|4)$ into this algebra. [Bena, Polchinski, Roiban] [Beisert, Ricci, Tseytlin, Wolf][Berkovits,Maldacena]
- ✦ Yangian algebra can be defined for tree level amplitudes. [Drummond, Henn, Plefka]

- ✤ Can we use the integrable structures, which have been so useful for finding the planar spectrum, to determine the amplitudes?
- ✤ Need to carefully understand the action of symmetries for even the standard superconformal symmetries.
- ✤ Can we learn something from integrable spin chain picture for local operators?

On-shell description

- Spinor helicity formalism for massless particles gives extremely compact expressions for amplitudes.
- Given a light-like four momentum for each external leg, $p^{a\dot{a}}$, we can decompose it into two component spinors, $p^{a\dot{a}} = \lambda^a \tilde{\lambda}^{\dot{a}}$, (for 3+1 signature spinors are related by c.c. & for positive energy particles $\tilde{\lambda} = +\bar{\lambda}$).
- Decomposition is unique up to a phase: $\lambda^a \mapsto e^{i\varphi}\lambda^a, \qquad \bar{\lambda}^{\dot{a}} \mapsto e^{-i\varphi}\bar{\lambda}^{\dot{a}}$
- Standard convention: λ^a has helicity -1/2 ; $\bar{\lambda}^{\dot{a}}$ has helicity +1/2.
- Introduce spinor brackets: $\langle ij \rangle = \epsilon_{ab}\lambda_i^a \lambda_j^b$ & $[ij] = \epsilon_{\dot{a}\dot{b}}\bar{\lambda}_i^{\dot{a}}\bar{\lambda}_j^{\dot{b}}$ "Square roots" of kinematical inv.
- Onshell content of $\mathcal{N} = 4$ SYM:

State	G^+	Γ_A	S_{AB}	$\bar{\Gamma}^A$	G^-
Helicity	$+1$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

A,B=1,...,4
SU(4) R-symmetry indicies

- Introduce Graβmanian variable, η^A, and combine all states into a single super-wavefunction

$$\Phi(\lambda,\bar{\lambda},\eta) = G^+(\lambda,\bar{\lambda}) + \eta^A \Gamma_A(\lambda,\bar{\lambda}) + \tfrac{1}{2}\eta^A\eta^B S_{AB}(\lambda,\bar{\lambda})$$
$$+ \tfrac{1}{6}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\bar{\Gamma}^D(\lambda,\bar{\lambda}) + \tfrac{1}{24}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D G^-(\lambda,\bar{\lambda})$$

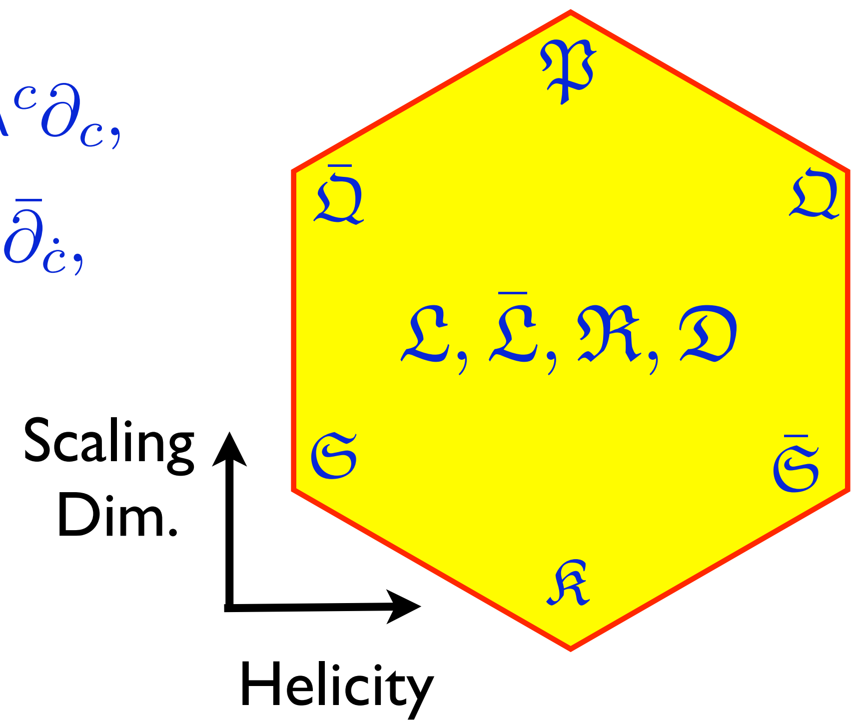
[Nair, Witten]

- Super-amplitudes combine all external states: $A_n(\lambda_1,\bar{\lambda}_1,\eta_1,\ldots,\lambda_n,\bar{\lambda}_n,\eta_n)$

$\mathfrak{psu}(2,2|4)$ symmetries

[Witten]

$$\mathfrak{L}^a{}_b = \lambda^a\partial_b - \tfrac{1}{2}\delta^a_b\lambda^c\partial_c,$$
$$\mathfrak{D} = \tfrac{1}{2}\partial_c\lambda^c + \tfrac{1}{2}\bar\lambda^{\dot c}\bar\partial_{\dot c},$$
$$\mathfrak{Q}^{aB} = \lambda^a\eta^B,$$
$$\bar{\mathfrak{Q}}^{\dot a}_B = \bar\lambda^{\dot a}\partial_B,$$
$$\mathfrak{P}^{a\dot b} = \lambda^a\bar\lambda^{\dot b},$$



$$\bar{\mathfrak{L}}^{\dot a}{}_{\dot b} = \bar\lambda^{\dot a}\bar\partial_{\dot b} - \tfrac{1}{2}\delta^{\dot a}_{\dot b}\bar\lambda^{\dot c}\bar\partial_{\dot c},$$
$$\mathfrak{R}^A{}_B = \eta^A\partial_B - \tfrac{1}{4}\delta^A_B\eta^C\partial_C,$$
$$\mathfrak{S}_{aB} = \partial_a\partial_B,$$
$$\bar{\mathfrak{S}}^B_{\dot a} = \eta^B\bar\partial_{\dot a},$$
$$\mathfrak{K}_{a\dot b} = \partial_a\bar\partial_{\dot b}.$$

- Can extend algebra to $\mathfrak{u}(2,2|4)$ by including central charge and helicity charge

$$\mathfrak{C} = \partial_a\lambda^a - \bar\lambda^{\dot c}\bar\partial_{\dot c} - \eta^C\partial_C = 2 + \lambda^a\partial_a - \bar\lambda^{\dot c}\bar\partial_{\dot c} - \eta^C\partial_C, \qquad \mathfrak{B} = \eta^C\partial_C.$$

- The action on n-point amplitudes of a single particle generator, $\mathfrak{J}$, is given by

$$\mathfrak{J} = \sum_{k=1}^{n}\mathfrak{J}_k$$

which should annihilate tree-level amplitudes, $\mathfrak{J}\cdot A_n = 0$.

- It has been shown that tree level amplitudes carry a representation of the Yangian algebra with the action of level one generators being

[Drummond, Henn, Plefka]

$$\widehat{\mathfrak{J}}_\alpha = \tfrac{1}{2}f_\alpha^{\beta\gamma}\sum_{k<\ell=1}^{n}\mathfrak{J}_{k,\beta}\mathfrak{J}_{\ell,\gamma}.$$

Consistent with cyclicity for superconformally invariant amplitudes!

Tree Level Amplitudes

- Colour ordered tree level amplitudes have a particularly simple form when written in terms of spinor brackets e.g. n-point MHV gluon amplitude

[Parke, Taylor][Berends, Giele]

$$A_n(1^-,2^-,3^+,\dots,n^+) = \frac{\langle 12\rangle^4}{\prod_{k=1}^n \langle k,k+1\rangle}\delta^{(4)}(P), \qquad P^{a\dot{a}} = \sum_{k=1}^n \lambda_k^a \bar{\lambda}_k^{\dot{a}}$$

where we have neglected overall iπ's etc but kept the momentum delta-function.

- Using the onshell superspace formulation we have a super MHV-amplitude from which we can extract component amplitudes by taking derivatives w.r.t. the η's

[Nair]

$$A_n^{\text{MHV}} = \frac{\delta^{(4)}(P)\delta^{(8)}(Q)}{\prod_{k=1}^n \langle k,k+1\rangle}, \qquad Q^{aA} = \sum_{k=1}^n \lambda_k^a \eta_k^A$$

here we have a Grassmann delta-function imposing super-momentum conservation.

- More general tree amplitudes can be written as

$$A_n = A_n^{\text{MHV}}\mathcal{P}_n \qquad \mathcal{P}_n = \mathcal{P}_n^{\text{MHV}} + \mathcal{P}_n^{\text{NMHV}} + \dots$$

where the $\mathcal{P}$'s depend on the Grassmann variables increasing in Grassmann degree by 4 at each step. For example the gluon MHV amplitude appears as

$$A_n = \eta_1^4\eta_2^4 A_n(1^-,2^-,3^+,\dots,n^+) + \dots$$

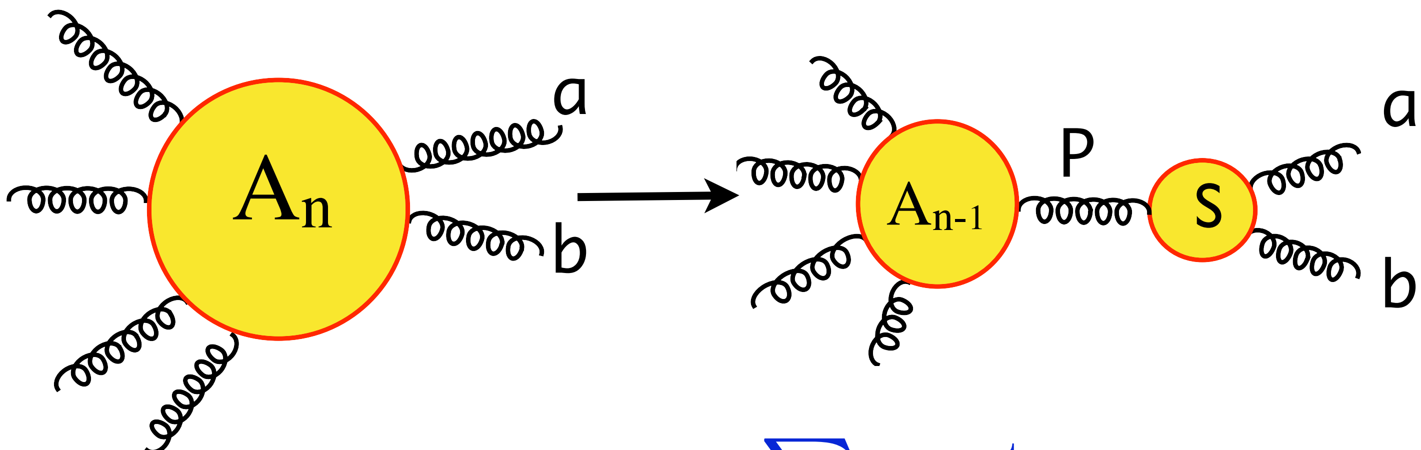
The Parke-Taylor form makes the singularity structure of the MHV amplitude transparent.

Collinear Limits

- Analytic properties provide strong physical constraints on amplitudes. In particular, tree level amplitudes have poles as various kinematic invariants vanish; colour ordered amplitudes can only have poles in sums of cyclically adjacent momenta i.e.

$$P_{i,j}^2 \to 0 \quad \text{where} \quad P_{i,j} = p_i + p_{i+1} + \cdots + p_j$$

- MHV amplitudes only have collinear (two-particle) channels and poles get reduced by angular momentum factors to "square roots" i.e. spinor brackets.



$$A_n(\ldots, a, b, \ldots) \to \sum_h \text{Split}_{-h}^{\text{tree}}(z, a, b) A_{n-1}(\ldots, P^h, \ldots)$$

Amplitudes factorize in a universal fashion with the intermediate state momenta given by $p_P = p_a + p_b$ i.e. the collinear momenta scale as $p_a \sim z\, p_P$, $p_b \sim (1-z)\, p_P$, and for e.g.

[Berends, Giele]

$$\text{Split}_{-}^{\text{tree}}(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\langle ab\rangle}$$

To consider collinear limits of superamplitudes we must also rescale the Grassmann variables $\eta_a \to \sqrt{z}\eta_P,\ \eta_b \to \sqrt{1-z}\eta_P$ and one finds

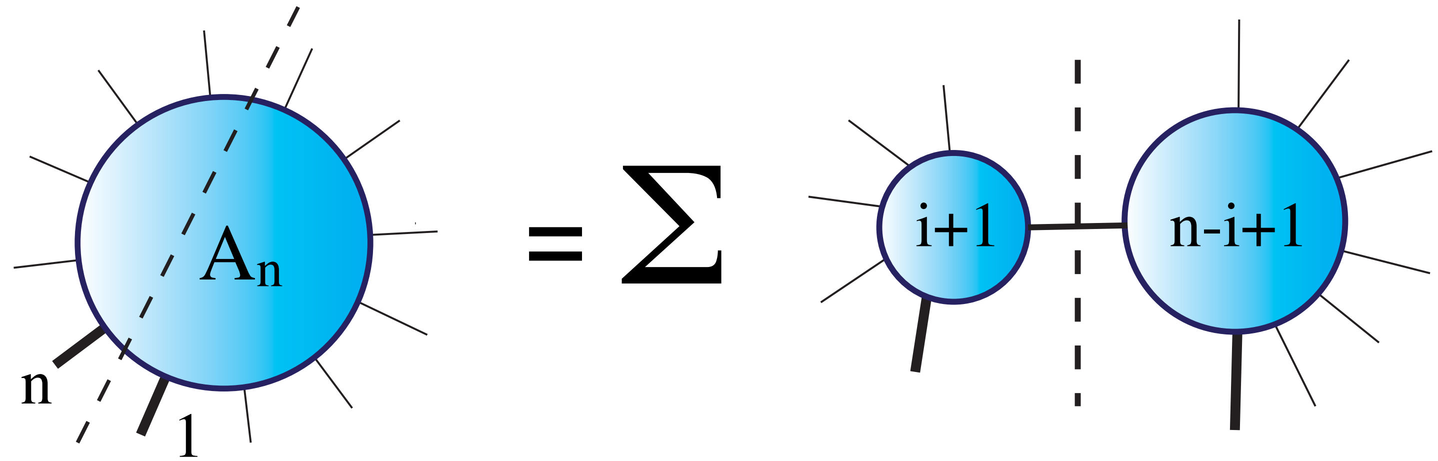
[Drummond Henn]

$$A_n(\ldots, a, b, \ldots) \to \frac{1}{\sqrt{z(1-z)}\langle ab\rangle} A_{n-1}(\ldots, \lambda_P, \bar{\lambda}_P, \eta_P, \ldots)$$

BCFW in superspace

Shifted legs

$$\hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 - z_{P_i}\tilde{\lambda}_n$$
$$\hat{\tilde{\lambda}}_n = \tilde{\lambda}_n + z_{P_i}\tilde{\lambda}_1$$



BCFW recursion relations relate the n-point scattering amplitudes to a sum over a product of amplitudes of fewer points (derived by considering the amplitudes as analytic functions of complex momenta and using Cauchy's theorem)

$$A_n = \sum_{P_i}\sum_{h} A_L^h(z_{P_i})\frac{1}{P_i^2}A_R^{-h}(z_{P_i})$$

where z_P indicates that momenta are shifted e.g. simplest case is for two adjacent legs.
The shift parameter z_P must be chosen so that the intermediate momenta

$$\hat{P}_i = -(\hat{\lambda}_1\tilde{\lambda}_1 + \sum_{j=1}^{i-1}\lambda_j\tilde{\lambda}_j)$$

is null.
The superspace version corresponds to replacing the sum over intermediate states with a superspace integral

$$A_n = \sum_{P_i}\int d^4\eta_{P_i} A_L(z_{P_i})\frac{1}{P_i^2}A_R(z_{P_i})$$

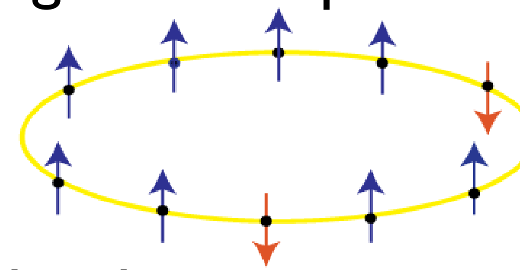
and it is also necessary to shift the the Grassmann variables $\hat{\eta}_n = \eta_n + z_{P_i}\eta_1$

The validity of these relations requires that the shifted superamplitudes vanish as $z \to \infty$

Local Operators/Spin Chains

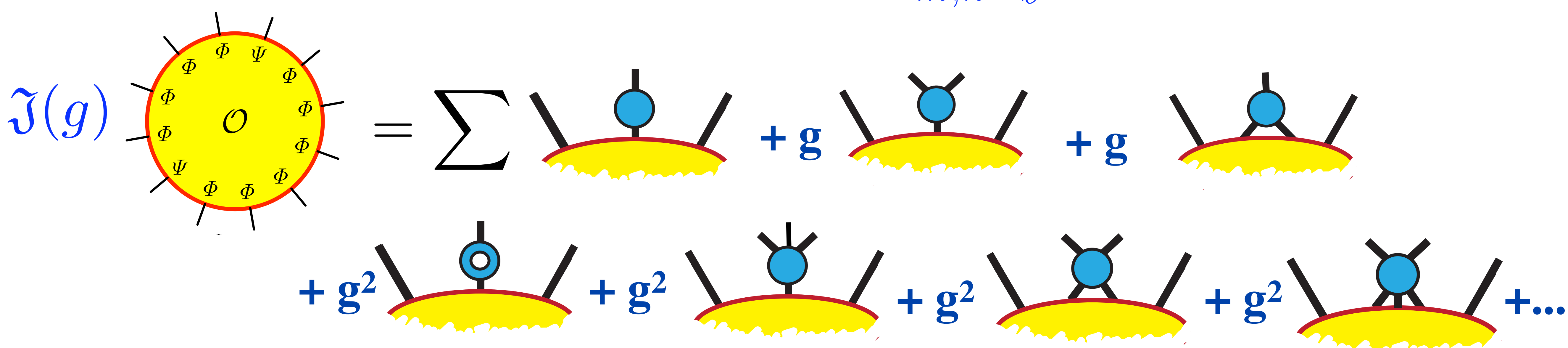
- Consider the action of symmetry generators on single trace operators (familiar spin chain picture)

$$\mathcal{O} = \mathrm{Tr}(\Phi\Phi\ldots\Phi\Psi\Phi\ldots\Phi\Psi\Phi\ldots\Phi) \quad =$$



- Free representation of superconformal algebra on local operators gets deformed at loop level as it is required to account for anomalous dimensions.
  - Tree level generators act on a single site and give back a single site however at loops we get more complicated, long range, effects. [Beisert, Kristjansen, Staudacher]
  - Action is ***dynamical:*** it can change the number of sites. Occurs even classically as susy generators exchange a fermion for a commutator of scalars. ($g \sim \sqrt{\lambda}$) [Beisert]

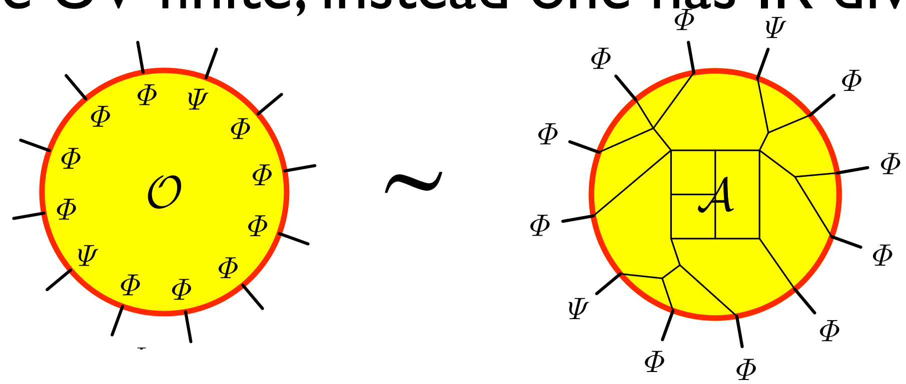
$$\mathfrak{J}(g) = \sum_{k=1}^{n} \mathfrak{J}_k(g) \quad \text{with} \quad \mathfrak{J}_k(g) = \sum_{m,n}\sum_{\ell} g^{2\ell+m+n-2}\left(\mathfrak{J}^{(\ell)}_{m,n}\right)_k$$



Can we extend this to scattering amplitudes?

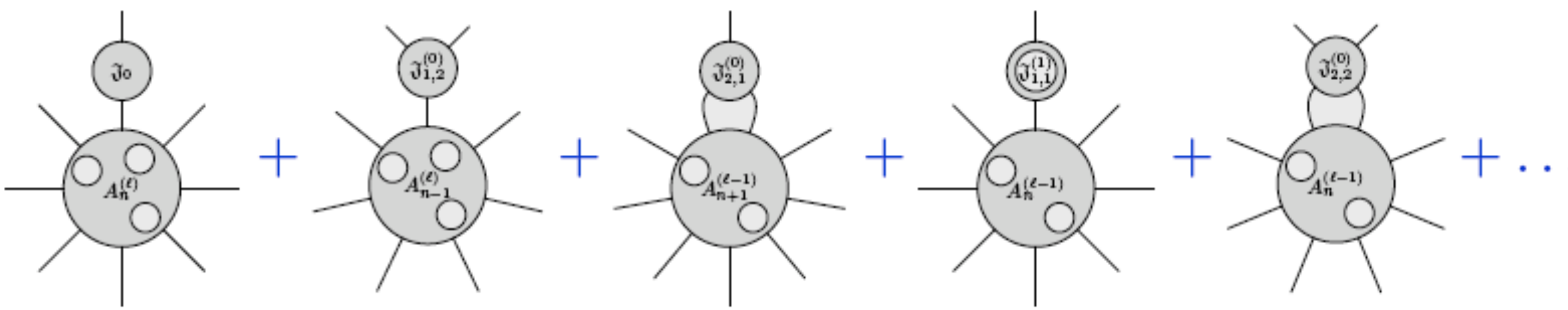
Scattering Amplitudes

- Scattering amplitudes are UV finite, instead one has IR divergences.



- Nonetheless there are similarities in their structure so one could:
  - Expect symmetry generator representation on amplitudes to be deformed but not necessarily equivalent to that carried by local operators.
  - Expect the action to be long range at higher orders and act on multiple legs.
  - Expect the action to be dynamical and change the number of legs.

Thus we have $\mathfrak{J}(g) = \mathfrak{J}_0 + g\mathfrak{J}^{(0)}_{1,2} + g\mathfrak{J}^{(0)}_{2,1} + g^2\mathfrak{J}^{(1)}_{1,1} + \dots$ acting on a linear combination of amplitudes "$\sum_{n=4}^{\infty}\sum_{\ell=0}^{\infty} g^{n-2+2\ell} A_n^{(\ell)}$" as



$$\mathfrak{J}^{(0)}_{1,1} A_n^{(\ell)} + \mathfrak{J}^{(0)}_{1,2} A_{n-1}^{(\ell)} + \mathfrak{J}^{(0)}_{2,1} A_{n+1}^{(\ell-1)} + \mathfrak{J}^{(1)}_{1,1} A_n^{(\ell-1)} + \mathfrak{J}^{(0)}_{2,2} A_n^{(\ell-1)} + \dots = 0$$

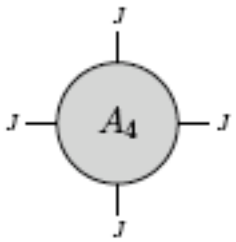
Need to combine amplitudes with different numbers of legs into a single functional.

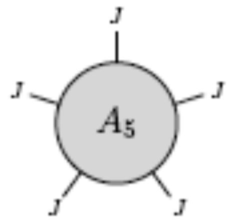
Amplitude Generating Functional

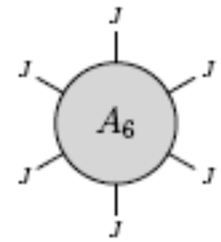
- Need to have a convenient way to combine all amplitudes: introduce a generating functional for colour ordered amplitudes (written as functions of superspace coordinates)

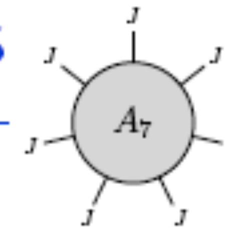
$$\mathcal{A}[J] = \sum_n \frac{g^{n-2}}{n} \int \prod d^{4|4}\Lambda \; A_n(\Lambda_1, \ldots, \Lambda_n) \; \mathrm{Tr}\, J(\Lambda_1) \ldots J(\Lambda_n)$$

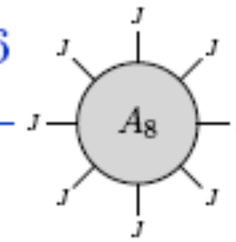
$$\Lambda = (\lambda^a, \bar{\lambda}^{\dot{a}}, \eta^A) \qquad d^{4|4}\Lambda = d^4\lambda \; d^4\eta$$

$= \frac{g^2}{4}$ 

$+ \frac{g^3}{5}$ 

$+ \frac{g^4}{6}$ 

$+ \frac{g^5}{7}$ 

$+ \frac{g^6}{8}$ 

$+ \ldots .$

- Extract amplitudes as usual by taking functional variations: $\dfrac{\delta}{\delta J(\Lambda)} = \check{J}(\Lambda)$
- Free representation of generators given by:

$$(\mathfrak{P}_0)^{a\dot{b}} = \int d^{4|4}\Lambda \; \mathrm{Tr}\, \lambda^a \bar{\lambda}^{\dot{b}} J(\Lambda)\, \check{J}(\Lambda)$$

$$(\bar{\mathfrak{Q}}_0)^{\dot{a}}_B = -\int d^{4|4}\Lambda \; \mathrm{Tr}\, \bar{\lambda}^{\dot{a}} \partial_B J(\Lambda)\, \check{J}(\Lambda) \qquad (\mathfrak{Q}_0)^{aB} = \int d^{4|4}\Lambda \; \mathrm{Tr}\, \lambda^a \eta^B J(\Lambda)\, \check{J}(\Lambda)$$

$$(\mathfrak{S}_0)_{aB} = \int d^{4|4}\Lambda \; \mathrm{Tr}\, \partial_a \partial_B J(\Lambda)\, \check{J}(\Lambda) \qquad (\bar{\mathfrak{S}}_0)^B_{\dot{a}} = -\int d^{4|4}\Lambda \; \mathrm{Tr}\, \eta^B \bar{\partial}_{\dot{a}} J(\Lambda)\, \check{J}(\Lambda)$$

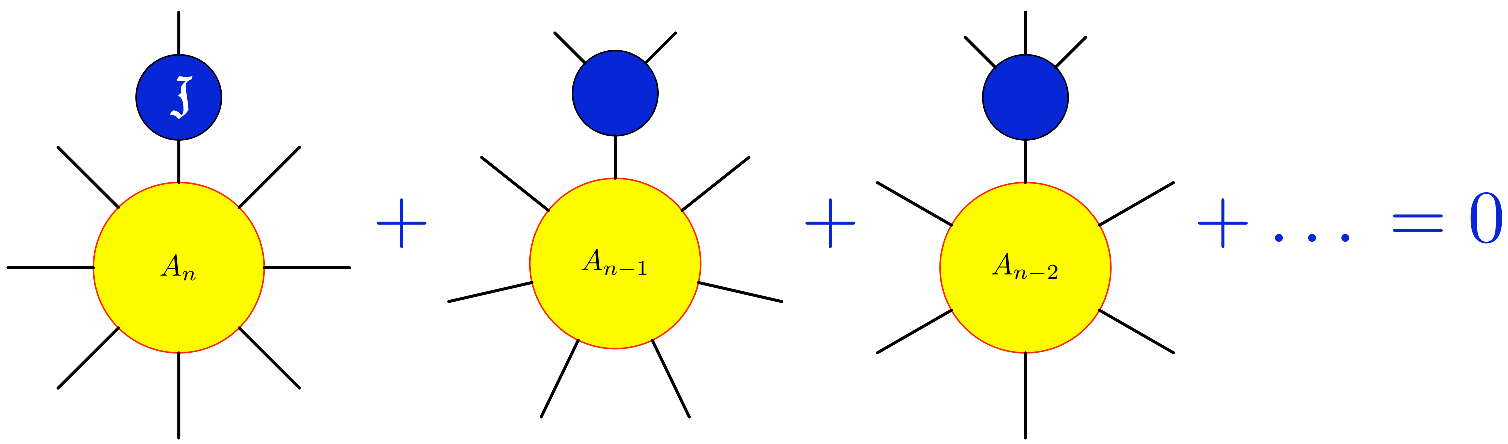
$$(\mathfrak{K}_0)_{a\dot{b}} = \int d^{4|4}\Lambda \; \mathrm{Tr}\, \partial_a \bar{\partial}_{\dot{b}} J(\Lambda)\, \check{J}(\Lambda)$$

- Expect leading deformations to be of the form:

$$\mathfrak{J}^{(0)}_{(1,2)} \sim \int \mathrm{Tr}\, J J \check{J}$$

Tree level superconformal symmetry

- We start with the simplest case, tree level, and we want to show that for all generators, $\mathcal{J}$,



$$J^{(0)}_{1,1} A_n + J^{(0)}_{1,2} A_{n-1} + J^{(0)}_{1,3} A_{n-2} + \cdots = 0$$

- We act with free generators on n-point tree level amplitudes and
  - see that amplitudes are not actually invariant
  - write in terms of generating functionals combining amplitudes with different numbers of legs
  - find a deformation of the generators such that they do annihilate the combined amplitudes
  - show that the deformed generators still satisfy the psu(2,2|4) algebra.

Start with action of $\bar{\mathfrak{S}}_0$ on MHV n-point amplitudes

$$A_n^{\mathrm{MHV}} = \frac{\delta^{(4)}(\sum \lambda_i \bar{\lambda}_i)\delta^{(8)}(\sum \lambda_i \eta_i)}{\prod \langle k, k+1 \rangle}$$

Recalling that the spinor brackets are defined as, $\langle k, k+1 \rangle = \epsilon_{ab} \lambda_k^a \lambda_{k+1}^b$, we see that the MHV amplitudes, A_n^{MHV}, are holomorphic functions of the λ's except for the argument of the momentum delta-function $P^{a\dot{a}} = \sum \lambda_i \bar{\lambda}_i$. Thus we have

$$\begin{aligned}
(\bar{\mathfrak{S}}_0)^B_{\dot{a}} A_n^{\mathrm{MHV}} &= \Big[\sum_j \eta_j^B \frac{\partial}{\partial \bar{\lambda}_j^{\dot{a}}} \delta^{(4)}(\sum \lambda_i \bar{\lambda}_i)\Big] \frac{\delta^{(8)}(\sum \lambda_i \eta_i)}{\prod \langle k, k+1 \rangle} \\
&= \Big[\sum_j \lambda_j^c \eta_j^B\Big] \frac{\partial \delta^{(4)}(P)}{\partial P^{c\dot{a}}} \frac{\delta^{(8)}(\sum \lambda_i \eta_i)}{\prod \langle k, k+1 \rangle} = 0!
\end{aligned}$$

However this is too fast; there are anomalous contributions of the form [Cachazo, Svrček, Witten]

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z} = \pi \delta^{(2)}(z)$$

which gives rise to contributions of the form (crucially depends on 3+1 signature)

$$\frac{\partial}{\partial \bar{\lambda}^{\dot{a}}} \frac{1}{\langle \lambda \mu \rangle} = \pi \delta^{(2)}(\langle \lambda \mu \rangle)\, \epsilon_{\dot{a}\dot{b}}\, \bar{\mu}^{\dot{b}}$$

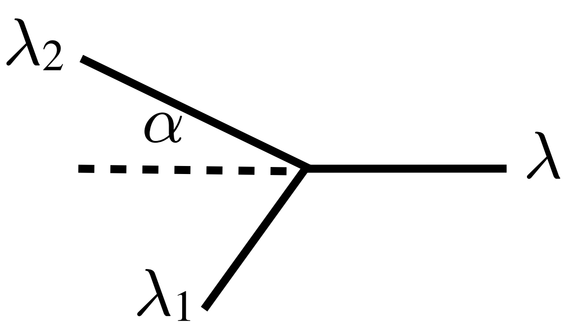
Get new terms from collinear limit: we must include this effect!

....anomalous term is:

$$(\bar{\mathfrak{S}}_0)^B_{\dot{a}} A_n^{\rm MHV} = -\pi \sum_{k=1}^{n} \varepsilon_{\dot{a}\dot{b}} (\bar{\lambda}^{\dot{b}}_{k-1} \eta^B_k - \bar{\lambda}^{\dot{b}}_k \eta^B_{k-1}) \frac{\delta^{(2)}(\langle \lambda_{k-1}, \lambda_k \rangle)\, \delta^{(4)}(P)\, \delta^{(8)}(Q)}{\langle 12 \rangle \ldots \langle k-1, k \rangle^0 \ldots \langle n1 \rangle}$$

...write this as in terms of generating functional for n-point MHV amplitudes

$$(\bar{\mathfrak{S}}_0)^B_{\dot{a}} \mathcal{A}_n^{\rm MHV}[J] = -2\pi \int d^{4|4}\Lambda \prod_{k=3}^{n} d^{4|4}\Lambda_k d^4\eta' d\alpha\, \epsilon_{\dot{a}\dot{b}} \bar{\lambda}^{\dot{b}}_1 \eta^B_2$$
$$\times A_{n-1}^{\rm MHV}(\Lambda, \Lambda_3, \ldots, \Lambda_n)\, {\rm Tr}[\hat{J}(\Lambda_1), J(\Lambda_2)] \ldots J(\Lambda_n)$$



...where we have evaluated the delta-function imposing collinearity so that

$$\lambda_1 \sim \lambda \sin\alpha \qquad \eta_1 \sim \eta \sin\alpha + \eta' \cos\alpha$$
$$\lambda_2 \sim \lambda \cos\alpha \qquad \eta_2 \sim \eta \cos\alpha - \eta' \sin\alpha$$

We see that the anomalous variation produces a (n-1)-point MHV amplitude with a modification on the first leg. Can compensate for this by including

$$\bar{\mathfrak{S}}^{(0)}_{1,2} = \bar{\mathfrak{S}}_+$$

with $$(\bar{\mathfrak{S}}_+)^{\mathbf{B}}_{\dot{\mathbf{a}}} = \mathbf{2}\pi^{\mathbf{2}} \int \mathbf{d}\Lambda^{4|4} \mathbf{d}^4 \eta' \mathbf{d}\alpha\, \epsilon_{\dot{\mathbf{a}}\dot{\mathbf{c}}} \bar{\lambda}^{\dot{\mathbf{c}}}_{\mathbf{1}} \eta^{\mathbf{B}}_{\mathbf{2}}\, {\rm Tr}[\hat{\mathbf{J}}(\Lambda_{\mathbf{1}}), \mathbf{J}(\Lambda_{\mathbf{2}})] \check{\mathbf{J}}(\Lambda)$$

Classical Representation

- We thus find a correction to the fermionic special conformal symmetry such that

$$\bar{\mathfrak{S}}_0 \mathcal{A}_n^{\rm MHV} + \bar{\mathfrak{S}}_+ \mathcal{A}_{n-1}^{\rm MHV} = 0$$

or in terms of the all-leg MHV amplitude generating functional (and to leading order)

$$\bar{\mathfrak{S}} \mathcal{A}^{\rm MHV}[J] = 0 \quad \text{where} \quad \bar{\mathfrak{S}} = \bar{\mathfrak{S}}_0 + \bar{\mathfrak{S}}_+$$

Note:

- The $+$ indicates that $\bar{\mathfrak{S}}_+$ increases, by 2, the charge $2n - 4k$, where $4k$ is the helicity charge $\mathfrak{B}$ of the amplitude.

- This gives a recursive structure which ends with $\bar{\mathfrak{S}}_+ A_4^{\rm MHV} = 0$

Corrections to $\mathfrak{S}$ can be found by considering $\overline{\rm MHV}$ amplitudes where one finds a correction $\mathfrak{S}_{1,2}^{(0)} = \mathfrak{S}_-$ which increases k by one. There are similar terms for the special conformal generator $\mathfrak{K}$

$$\mathfrak{K} = \mathfrak{K}_0 + g\, \mathfrak{K}_+ + g\, \mathfrak{K}_- + g^2\, \mathfrak{K}_{+-}$$

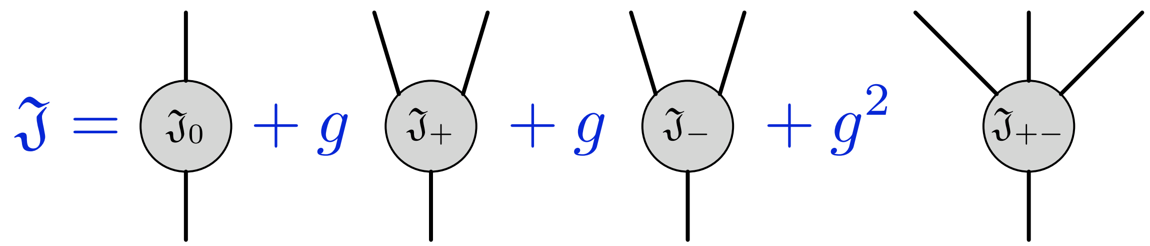
There are no deformations to $\mathfrak{D}, \mathfrak{P}, \mathfrak{Q}$ or $\bar{\mathfrak{Q}}$ at this order.

Generic amplitudes

- So far we have considered only MHV (and $\overline{\text{MHV}}$) amplitudes. We want to extend this to general amplitudes. With the notation

  $A_{n,k}$ - n-point amplitude with helicity charge 4k i.e. $A_n^{\text{MHV}} = A_{n,2},\ A_n^{\overline{MHV}} = A_{n,n-2}$

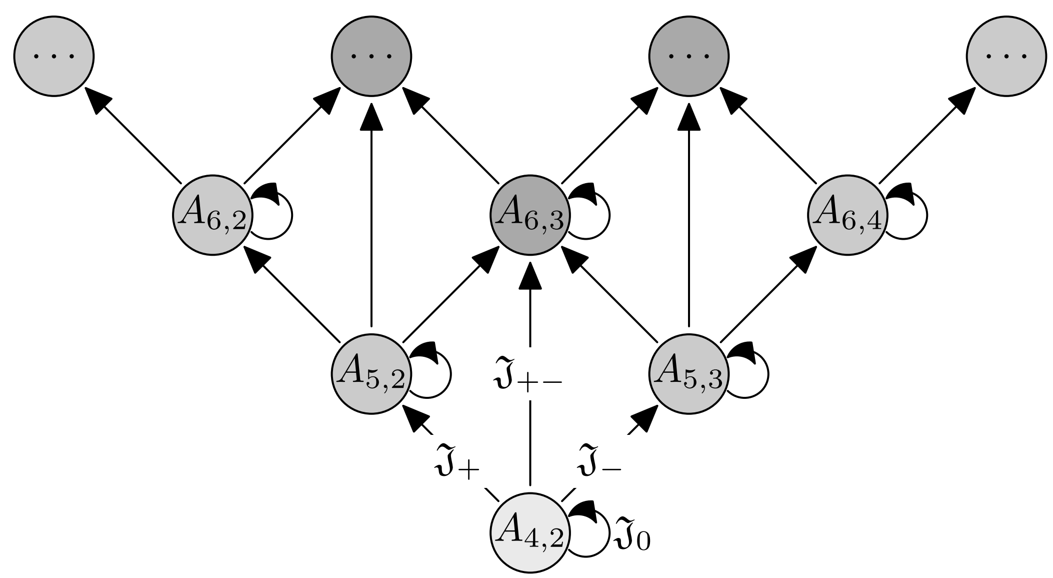
  we expect to find that the deformed generators



  will annihilate an arbitrary tree-level amplitude (i.e. they account for all collinear singularities)

$$\mathfrak{J}_0 A_{n,k} + \mathfrak{J}_+ A_{n-1,k} + \mathfrak{J}_- A_{n-1,k-1} + \mathfrak{J}_{+-} A_{n-2,k-1} = 0$$

  This gives rise to the recursive pattern of relations between all amplitudes



These relations arise from collinear singularities which are known to be universal and we expect that the action of the deformed generators extends easily to generic amplitudes. Note that the generators provide relations between all amplitudes.

- Start with a specific example -- six point NMHV amplitude.
- Explicit expression is known (in terms of dual superconformal invariants) [Drummond, Henn, Korchemsky,Sokatchev]

$$A_6^{\rm NMHV} = A_6^{\rm MHV}(\tfrac{1}{2}R_{146} + \text{cyclic})$$

- In addition to collinear singularities there are other possible sources of anomalous contributions
  - Multiparticle singularities which occur when linear combinations e.g. $p_4+p_5+p_6$ of external momenta become null. However these are of the form $\frac{1}{\bar{z}z}$ and so don't contribute.
  - Apparent singularities when e.g. p_4+p_5 is any linear combination of p_3 and p_6. These are in fact not physical and cancel when all terms are combined -- these are so-called "spurious" singularities.

- We calculate $(\mathfrak{S}_0)_{aA} A_6^{\rm NMHV}$ and find

$$(\mathfrak{S}_0)A_6^{\rm NMHV} + \mathfrak{S}_- A_5^{\rm MHV} = 0 \quad \text{i.e.} \quad (\mathfrak{S}_0)A_{6,3} + \mathfrak{S}_- A_{5,2} = 0$$

with the same deformation as found previously. Similarly we calculate $(\bar{\mathfrak{S}}_0)_{\dot{a}}^{A} A_6^{\rm NMHV}$ and find

$$(\bar{\mathfrak{S}}_0)A_6^{\rm NMHV} + \bar{\mathfrak{S}}_+ A_5^{\rm NMHV} = 0 \;\; \text{i.e.} \;\; (\bar{\mathfrak{S}}_0)A_{6,3} + \bar{\mathfrak{S}}_+ A_{5,3} = 0$$

again with the same deformation as found from MHV amplitudes.

Constraints on amplitudes

- The R_{pqr}'s that appear in the six-point NMHV amplitudes are dual superconformal invariants (i.e. their form is fixed by the symmetries). This dependence is shared by all NMHV amplitudes

$$A_n^{NMHV} = A_n^{MHV} \Big[\frac{1}{n} \sum_{p,q,r \in \mathcal{S}_n} R_{pqr} \Big]$$

  i.e. they are sums of dual conformal invariants, however the coefficients of the invariants are not fixed by the symmetries.

- Demonstrating the universality of the deformations of the symmetry generators requires that the coefficients take specific values. That is to say, it is only for these specific values that the collinear limits are correct and the spurious singularities are absent so that the deformed symmetry generators annihilate the amplitudes.

- We can turn this around and argue that demanding that the NMHV amplitudes are invariant w.r.t. the deformed generators constrains the coefficients of the dual superconformal invariants.

- Korchemsky & Sokatchev have recently shown that the tree-level amplitudes while not fixed by superconformal symmetries, dual or otherwise, are indeed uniquely determined by the collinear limits and/or the absence of spurious singularities.

- In the language of this talk that is to say that the tree level amplitudes are exactly fixed by the exact symmetries.

- To show that all amplitudes are annihilated by the deformed generators we consider the collinear behaviour as

$$\lambda_1 \to e^{i\phi}\lambda \sin\alpha, \qquad \lambda_2 \to \lambda\cos\alpha$$

and with the redefinitions

$$\eta_1 = \eta' \cos\alpha + e^{-i\phi}\eta\sin\alpha, \qquad \eta_2 = \eta\cos\alpha - e^{i\phi}\eta'\sin\alpha.$$

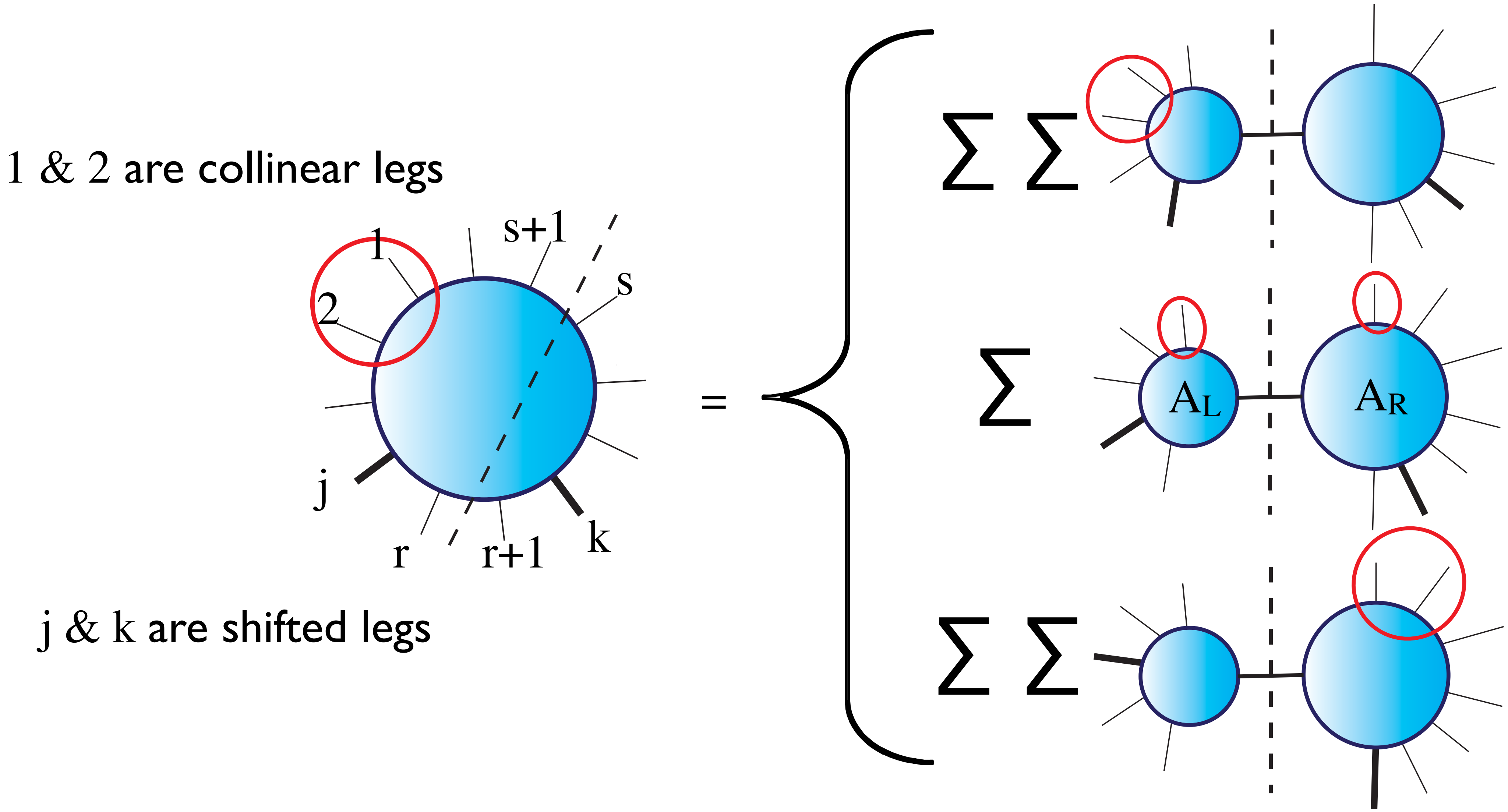
Assume that amplitude scales as

$$A_{n,k}(\Lambda_1,\ldots,\Lambda_n)|_{1||2} = \frac{e^{-i\phi}\sec\alpha\csc\alpha}{\langle 12\rangle} A_{n-1,k}(\Lambda,\Lambda_3,\ldots,\Lambda_n) + \frac{e^{i\phi}\sec\alpha\csc\alpha}{[12]}\delta^{(4)}(\eta')A_{n-1,k-1}(\Lambda,\Lambda_3,\ldots,\Lambda_n) + \text{finite terms},$$

with similar scaling in all other collinear limits and no other contributions to the anomalous action of the generators. With these assumptions it is straightforward to show that the $\mathfrak{S}_-$ and $\bar{\mathfrak{S}}_+$ cancel all the anomalous terms for all the amplitudes. One can conveniently show the above scaling inductively by making use of the BCFW recursion relations in the superspace formalism.

Essentially demonstrating the known universality of splitting functions via the BCFW relations. (See also Drummond & Henn)

Universal collinear limits from BCFW



When both collinear legs are on A_L we find the correct scaling by assumption. Similarly when both collinear legs are on A_R .When the legs lie on different partial amplitudes there is no singularity and the term is subleading in the scaling limit.

One can see the absence of spurious singularities by changing the shifted legs and noting that the singularities in individual terms change.

Closure of the algebra

- The deformed generators essentially combine the knowledge of the free symmetries with the analytic behaviour of the amplitudes in collinear limits. It is non-trivial whether this deformed representation respects the algebra i.e. whether the algebra closes.
- E.g. Consider the anti-commutator $\{\mathfrak{S}_{aA}, \mathfrak{S}_{bB}\} = 0$
- Calculate, $\{(\mathfrak{S}_0)_{(aA}, (\mathfrak{S}_-)_{bB)}\}\hat{J}(\mathit{\Lambda})$, the action of the commutator on a source (projected onto the part with a definite scaling weight under phase shifts of the arguments).
- Need to make use of vanishing of the central charge

$$\mathfrak{C}\hat{J} = 0 \text{ i.e. } (\lambda^e\partial_e - \bar{\lambda}^{\dot{e}}\bar{\partial}_{\dot{e}} - \eta^E\partial_E + 2)\hat{J} = 0$$

- Find that

$$\{(\mathfrak{S}_0)_{(aA}, (\mathfrak{S}_-)_{bB)}\}\hat{J}(\mathit{\Lambda}) = \pi^2\epsilon_{ac}[\partial_A\partial_B\hat{J}(0), \hat{J}(\mathit{\Lambda})]$$

which has the form of a field dependent gauge transformation $\hat{J}(\mathit{\Lambda}) \mapsto [X, \hat{J}(\mathit{\Lambda})]$. Thus we define the generator of gauge transformations

$$\mathfrak{G}[X] = \pi^2 \int d^{4|4}\,\mathrm{Tr}([X, J(\lambda)]\check{J}(\mathit{\Lambda}))$$

and we find that

$$\{\mathfrak{S}_{aA}, \mathfrak{S}_{bB}\} = \epsilon_{ac}\mathfrak{G}[\partial_A\partial_B J(0)]$$

i.e. the algebra closes up to gauge transformations. The other non-trivial (anti)-commutators are similar.

Holomorphic anomaly

- The holomorphic anomaly has played an interesting earlier role in the study of tree level and one-loop $\mathcal{N} = 4$ SYM scattering amplitudes. [Cachazo, Svrček, Witten]
- Amplitudes show interesting behaviour in twistor space which can be analyzed by studying the differential equations that they obey e.g collinearity, where amplitudes consist of sums of terms where all gluons lie on a union of straight lines, corresponds to being annihilated by operator

$$F_{ijk} = \langle ij \rangle \frac{\partial}{\partial \bar{\lambda}_k} + \langle jk \rangle \frac{\partial}{\partial \bar{\lambda}_i} + \langle ki \rangle \frac{\partial}{\partial \bar{\lambda}_j}$$

- CWS analyzed the unitarity cut of one-loop diagrams and showed that it is necessary to include the effect of the holomorphic anomaly to correctly interpret the differential equations; this should also be true for full amplitude because of cut constructibility. See also [Bena, Bern, Kosower & Roiban]. (Perhaps interestingly the collinearity operator appears to be related to the action of $\bar{\mathfrak{S}}$ on tree level amplitudes. [Korchemsky&Sokatchev])

- Can use the holomorphic anomaly to efficiently evaluate unitarity cuts of one-loop amplitudes of certain classes of amplitudes e.g. one can write generic one-loop amplitude as a linear combination of box integrals with unknown coefficients. By acting on the imaginary parts of these integrals as well as the cuts of amplitudes one can determine some of the coefficients of the scalar box integrals for MHV and NMHV amplitudes. [Cachazo] [Britto,Cachazo,Feng]

Conclusions & Outlook

- Tree level scattering amplitudes are *almost* invariant under superconformal transformations. Singular limits involving collinear momenta result in violations of this invariance which requires modifying the generators.
- Corrected generators are dynamical i.e they change the number of external legs. Strong similarities with spin chain pictures of local operators.
- The deformed generators still satisfy the unmodified superconformal algebra.
- Symmetries, collinear limits and absence of spurious singularities fix uniquely the tree-level scattering amplitudes.

- Extend to higher loops - start with one-loop!
- What about Yangian symmetries/**dual** superconformal symmetries?
- What about other signatures, twistor space description?
- Can we determine both the divergent and finite pieces of scattering amplitudes for all orders in coupling?
- Can we similarly use symmetries to constrain amplitudes in $\mathcal{N} = 8$ sugra? $E_{7(7)}$ generators?

Done!
Thank You!

Extra Slides

"Whoa—way too much information!"

More about dual conformal invariants

- Explicit form of the 6-point function

$$\frac{1}{2}\left(R_{146}+R_{251}+R_{362}\right)=\frac{1}{2}\Bigg[\frac{\langle 34\rangle\langle 56\rangle\langle 61\rangle\langle 45\rangle}{x_{14}^2\langle 1|x_{14}|4]\langle 3|x_{36}|6][45][56]}\delta^4\left(\eta_4[56]+\eta_5[64]+\eta_6[45]\right)$$
$$+\frac{\langle 45\rangle\langle 61\rangle\langle 12\rangle\langle 56\rangle}{x_{25}^2\langle 2|x_{25}|5]\langle 4|x_{42}|1][56][61]}\delta^4\left(\eta_5[61]+\eta_6[15]+\eta_1[56]\right)$$
$$+\frac{\langle 56\rangle\langle 12\rangle\langle 23\rangle\langle 61\rangle}{x_{36}^2\langle 3|x_{36}|6]\langle 5|x_{53}|2][61][12]}\delta^4\left(\eta_6[12]+\eta_1[26]+\eta_2[61]\right)\Bigg]$$

- Generic form of the dual conformal invariants

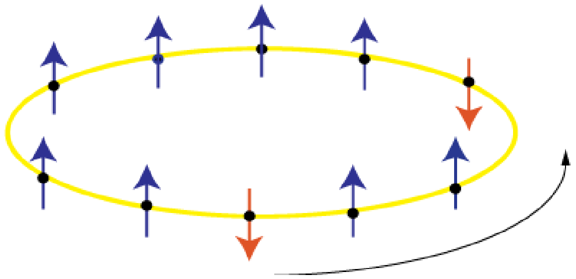
$$R_{pqr}=c_{pqr}\delta^4(\varXi_{pqr})\,,$$
$$c_{pqr}=\frac{\langle q-1,q\rangle\langle r-1,r\rangle}{x_{qr}^2\langle p|x_{pr}x_{rq-1}|q-1\rangle\langle p|x_{pr}x_{rq}|q\rangle\langle p|x_{pq}x_{qr-1}|r-1\rangle\langle p|x_{pq}x_{qr}|r\rangle}\,,$$
$$\varXi_{pqr}^A=-\langle p|\Big[x_{pq}x_{qr}\sum_{i=p}^{r-1}|i\rangle\eta_i^A+x_{pr}x_{rq}\sum_{i=p}^{q-1}|i\rangle\eta_i^A\Big].$$

Cusp from integrability

Cusp dimension known from AdS/CFT planar integrable system!

Compute cusp dimension using Bethe equations. Integral eq.: [Eden, Staudacher]

Weak coupling expansion of integral equation [Beisert, Eden, Staudacher]



$$D_{\text{cusp}}(\lambda) = \frac{1}{2}\frac{\lambda}{\pi^2} - \frac{1}{96}\frac{\lambda^2}{\pi^2} + \frac{11}{23040}\frac{\lambda^3}{\pi^2} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2} \pm \ldots .$$

Agreement with gluon scattering amplitudes. [Bern, Dixon, Smirnov] [Bern, Czakon, Dixon, Kosower, Smirnov]

Strong coupling asymptotic expansion of integral equation [Casteill, Kristjansen] [Basso, Korchemsky, Kotański]

$$E_{\text{cusp}}(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} - \frac{\beta(2)}{\pi\sqrt{\lambda}} + \ldots .$$

Agreement with semiclassical energy of spinning string. [Gubser, Klebanov, Polyakov] [Frolov, Tseytlin] [Roiban, Tirziu, Tseytlin]