Quantum Integrability of $AdS_4 - CFT_3$ to All Loops

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joint works with

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$\left(\begin{array}{l} 0904.4677 \\ 0907.nnnn \end{array} \right.$

made in Korea

Main Result

- magnon spectrum predicted by integrability and $psu(2|2)$ fits with CFT_3 perturbative computations up to 6 loops and beyond!

- symmetry considerations allow "pseudo-momentum" in general, BUT perturbative computations give rise to "lattice momentum" only.

$AdS_4 - CFT_3$

- Well-known

Type IIB on $AdS_5 \times S^5$

$\updownarrow$

4d $\mathcal{N}=4$ super Yang-Mills

- M-th. or Type IIA counterpart

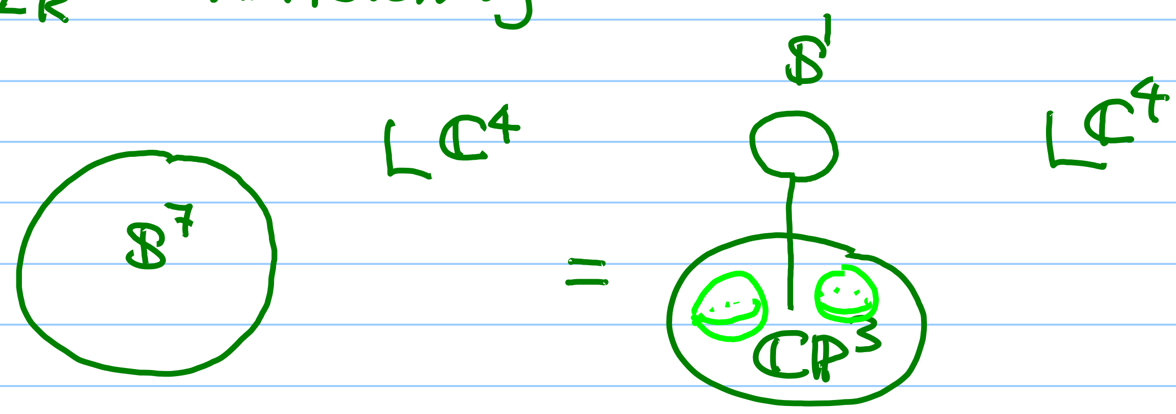
$\left(\begin{array}{l} \text{M on } AdS_4 \times (S^7/\mathbb{Z}_k) \\ \text{Type IIA on } AdS_4 \times \mathbb{CP}^3 \end{array} \right.$

$\updownarrow$

3d $\mathcal{N}=6$ super Chern-Simons

aka ABJ(M) theory

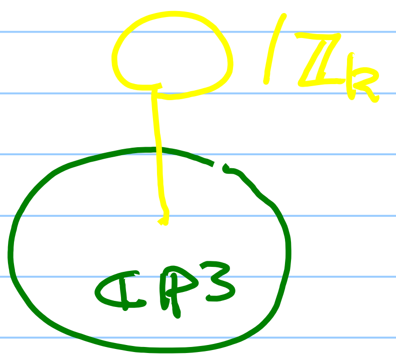
• $\mathbb{Z}_k$ orbifolding



$\mathbb{Z}_k : \; Y^I \longrightarrow \omega \cdot Y^I \qquad (\omega^k = 1)$

orbifolding along M-theory circle

$\big\downarrow$ IIA reduction



$\rightsquigarrow$ completely NEW feature of IIA side compared to IIB (and $\mathcal{N}=4$ SYM)

Coupling parameters

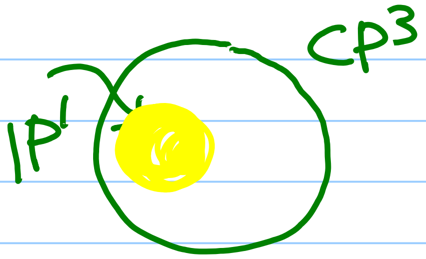
- ABJ(M) theory

* $\frac{1}{k}$ $\qquad\qquad$ $g_{YM}^{(1)}, \; g_{YM}^{(2)}$

* $M, \; N$ $\qquad\qquad$ N

$$\left[\; \lambda(M) = \frac{M}{k} \; ; \; \lambda(N) = \frac{N}{k} \qquad\qquad \lambda = g_{YM}^2 N \;\right]$$

- IIA on $AdS_4 \times \mathbb{CP}^3$



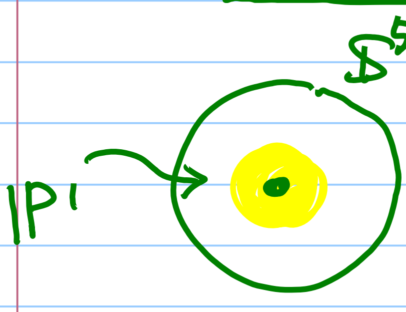
$$* \; G_4 \simeq N \cdot Vol(\mathbb{CP}^3)$$

$$G_2 \simeq k \cdot Vol(\mathbb{P}^1)$$

$$\left(B_2 - \frac{1}{2}\right) \simeq \frac{1}{k}(M-N) \cdot Vol(\mathbb{P}^1)$$

$$g_{st} \simeq \frac{1}{N}$$

cf. $\underline{AdS_5 \times S^5}$



$$G_5 \simeq N \cdot Vol(S^5)$$

$$\left(B_2 - \frac{1}{2}\right) \simeq \left(\lambda^{(1)} - \lambda^{(2)}\right) Vol(\mathbb{P}^1)$$

$$g_{st} \simeq \frac{1}{N}$$

ABJ(M) Theory

- superconf. group

$OSp(6|4,\mathbb{R}) \supset SU(4) \quad \leftrightarrow \quad SU(2,2|4) \supset SU(4)$

- gauge group

$U(M) \times U(N) \quad \leftrightarrow \quad U(N)$

- Chern-Simons level

$(k, \; -k) \quad \leftrightarrow \quad g_{YM}$

- parity (discrete torsion)

$(B - \frac{1}{2}) \equiv \left| \frac{M-N}{k} \right| \quad \leftrightarrow \quad g_{YM}^{(1)} \neq g_{YM}^{(2)}$

- fields

$A_m, \; \tilde{A}_m$ — non-dynamical

$Y^I \qquad (M, \bar{N}, \;)$

$Y_I^\dagger \qquad (N, \bar{M}, \;)$

$\psi_I \qquad (N, \bar{M}, \;)$

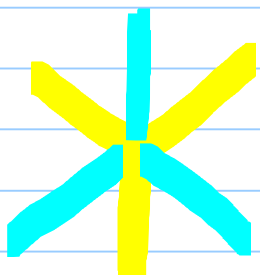
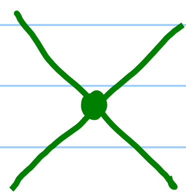
$\psi^{\dagger I} \qquad (M, \bar{N}, \;)$

- Lagrangian

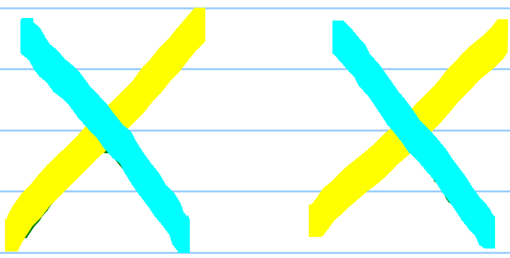
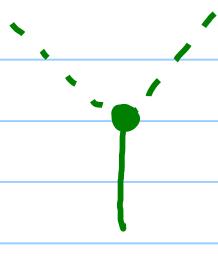
$$
\begin{aligned}
k \cdot \mathrm{tr}\Big[& \frac{1}{4\pi}\left(A\,dA + \frac{2}{3}A^3\right) \\
& - \frac{1}{4\pi}\left(\bar{A}\,d\bar{A} + \frac{2}{3}\bar{A}^3\right) \\
& + |DY^I|^2 + \psi^\dagger i \not{D} \psi \\
& - \{ Y Y^\dagger \psi \psi^\dagger + Y \psi^\dagger Y \psi^\dagger + h.c. \} \\
& - \{ Y Y^\dagger Y Y^\dagger Y Y^\dagger \} \Big]
\end{aligned}
$$

$\leftrightsquigarrow$ $\mathcal{N}=4$ SYM

So, interactions are

$\leftrightsquigarrow$

- In this talk, focus on ABJM theory.

ABJ theory modification is straightforward:

$$\boxed{\begin{array}{l} H = H(\lambda, B^2) \\ \cdots \quad \text{parity } \underline{\text{even}} \end{array}}$$

(cf. forthcoming paper

— parity-odd observables

in integrable system!)

- xral primary operators

$\mathrm{tr}[Y^1 Y_4^\dagger Y^1 Y_4^\dagger \cdots\cdots Y^1 Y_4^\dagger]$

$4 \otimes \bar{4} \otimes 4 \otimes \bar{4} \cdots\cdots$

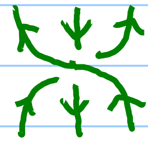
$\leftrightsquigarrow \mathrm{tr}[Z Z \cdots\cdots Z]$

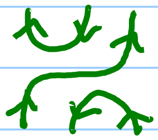
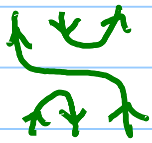
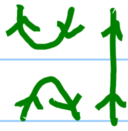
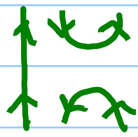
$6 \otimes 6 \otimes \cdots\cdots$

$\frac{1}{3}$ - BPS $\leftrightsquigarrow$ $\frac{1}{2}$ ~ BPS

$\rightsquigarrow$ reduced BPS property implies lesser protection against quantum correction

- 2-loop Hamiltonian

$H_2 = \frac{\lambda^2}{2} \Big(\mathbb{1} - 2$ 

$+$  $+$  $- \frac{1}{2}$  $- \frac{1}{2}$  $\Big)$

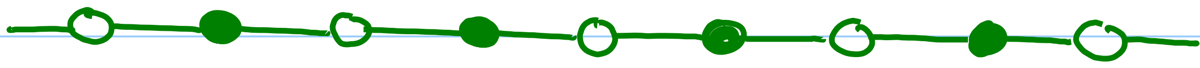
... integrable alternating SU(4) spin chain

- transfer matrix
- YBE
- BAE
- elementary excitations

Minahan + Zarembo : Bak + Rey

[cf. H. Saleur's talk]

ABJ(M) Magnons



$Y^1 \quad Y_4^\dagger$

$$Y^1 \longrightarrow (Y^2, Y^3 \,|\, \psi_1, \psi_2)$$

$$Y_4^\dagger \longrightarrow (Y_3^\dagger, Y_2^\dagger \,|\, \psi_2^\dagger, \psi_1^\dagger)$$

governed by

$$psu(2|2) \oplus psu(2|2)$$

$\mathbb{Z}_2 :\quad \bullet \longleftrightarrow \circ$

(charge conjugation
(spin chain parity

Questions

- Is ABJ(M) integrable?
- Is ABJ(M) similar to $\mathcal{N}=4$ SYM in organizing excitation structure?
  - 12 versus 16 SUSYs
    different interactions
    different quantum no.s
    ...

yet

$\boxed{\text{excitations are governed by } psu(2|2) \text{ for both } \mathcal{N}=4 \text{ SYM and ABJ(M)}}$

PSU(2|2) & Dynamic Spin Chain

- [Beisert]

centrally extended psu(2|2)

= off-shell excitation symmetry

$(R^a{}_b,\ L^\alpha{}_\beta,\ \mathcal{Q}^\alpha{}_a,\ S^a{}_\alpha)$

$$\{\mathcal{Q}^\alpha_a, S^b_\beta\} = \delta^b_a L^\alpha_\beta + \delta^\alpha_\beta R^a_b + \delta^b_a \delta^\alpha_\beta C$$

$$\{\mathcal{Q}^\alpha_a, \mathcal{Q}^\beta_b\} = \varepsilon^{\alpha\beta}\varepsilon_{ab} K$$

$$\{S^a_\alpha, S^b_\beta\} = \varepsilon^{ab}\varepsilon_{\alpha\beta} K^*$$

on $\Phi \equiv (\phi^1, \phi^2 \,|\, \psi^1, \psi^2) = (2|2)$.

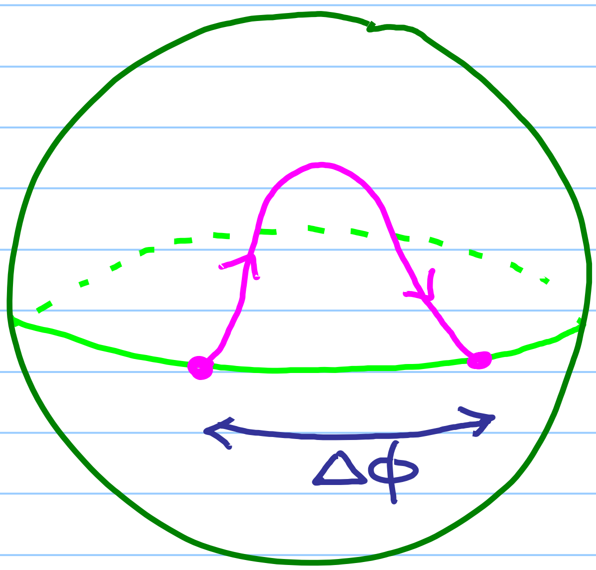
$$\mathcal{Q}^\alpha_a |\phi^b\rangle = a \, \delta^b_a |\psi^\alpha\rangle$$

$$\mathcal{Q}^\alpha_a |\psi^\beta\rangle = b \cdot \varepsilon^{\alpha\beta}\varepsilon_{ab} |\phi^b \mathcal{Z}(Y^+)\rangle$$

$$S^a_\alpha |\phi^b\rangle = c \cdot \varepsilon_{\alpha\beta}\varepsilon^{ab} |\psi^\beta \mathcal{Z}(Y^-)\rangle$$

$$S^a_\alpha |\psi^\beta\rangle = d \, \delta^\beta_\alpha |\phi^a\rangle$$

- Giant Magnon Revisited



$$\Delta\phi = P_{\text{worldsheet}}$$

<u>why</u> is this identification correct ?

More generally,
(by coordinate transform)

$$\Delta\phi = \mathcal{P}(P_{ws})$$

- closure of algebra

$\rightsquigarrow$ $ad - bc = 1$

'shortening condition'

- central changes

$$C|\Phi\rangle = \frac{1}{2}(ad + bc)|\Phi\rangle$$

$$K|\Phi\rangle = ab\,|\Phi\, G(\gamma^{+})\rangle$$

$$K^{*}|\Phi\rangle = cd\,|\Phi\, G(\gamma^{-})\rangle$$

- focus on asymptotic excitations

cf. finite-size effects yet to be explored

- On lattice momentum state

$$|p\rangle = \sum_{n=0}^{L-1} e^{inp} |\cdots \underset{(2n+1)\text{th site}}{\Phi} \cdots\rangle ,$$

$$G(Y^{\pm})\, |p\rangle = Y^{\pm} |p\rangle$$

where

$$|Y^{\pm} \Phi\rangle = e^{\mp ip} |\Phi\, Y^{\pm}\rangle .$$

- QM'cally, expect radiative corrections to $G(Y^{\pm})$

no "direct" derivations yet;
OPE + central charge relns ?

- Introduce "pseudo-momentum" (cf. Z. Bajnok's talk)

$$\mathbb{P} \equiv \mathbb{P}(p)$$

$$G(Y^{\pm})\,\Phi \equiv e^{\mp i \mathbb{P}(p)}\,\Phi\, G(Y^{\pm})$$

Physically

$$\begin{cases} \mathbb{P}(-p) = -\mathbb{P}(p) \\ \mathbb{P}(p+2\pi) = \mathbb{P}(p) \end{cases}$$

So,

$$\boxed{\begin{aligned} e^{i\mathbb{P}(p)} &= G(e^{ip}) \\ &= \exp\left(ip + i\sum_{n=1}^{\infty} b_{2n}(\lambda)\cdot \sin np\right) \end{aligned}}$$

where

perturbation theory dictates

$$\boxed{b_{2n}(\lambda) = \sum_{\ell=n}^{\infty} b_{2\ell,2n}\,\lambda^{2\ell}}$$

- on-shell condition

$$\prod_{\{\text{excitations}\}} e^{ip_k} = 1.$$

... in terms of lattice m/m, a dynamic twisted b.c.

so, defined only in asymptotic sense.

- $e^{ip(p)} = \frac{x^+}{x^-}$ $\qquad (x^+)^* = x^-$

$$\rightsquigarrow x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{h(\lambda)}$$

$$x^\pm = \frac{e^{\pm ip/2}}{4h(\lambda)\cdot \sin\frac{p}{2}}\left(1+\sqrt{1+16h^2(\lambda)\sin^2\frac{p}{2}}\right)$$

$$E = \frac{1}{2}\sqrt{1+16h^2(\lambda)\sin^2\frac{p}{2}}$$

- S-matrices, unitarity, YBE, asymptotic BAE

- In $\mathcal{N}=4$ SYM,

$$P(p) = p$$
$$X^{\pm} = x^{\pm}$$
$$h(\lambda) = \lambda$$

It poses questions :

* why ?

* in ABJ(M), with lesser SUSY + different interactions, does 'pseudo-momentum' state arise as simplifying basis ?

Spin Chain Hamiltonian

- magnon spectrum from psu(2|2) + integrability

$$E = \sqrt{\frac{1}{4} + 4h^2(\lambda)\sin^2\frac{p}{2}}$$

interpolating fcn [Sieg's talk]

$$h^2(\lambda) = \lambda^2 \sum_{l=0}^{\infty} h_{2l} \cdot \lambda^{2l} \qquad (h_0 = 1)$$

* lattice momentum

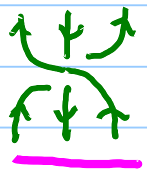
$$E = \frac{1}{2} + \left(4\sin^2\frac{p}{2}\right)\lambda^2 + \left(4h_2\sin^2\frac{p}{2} - 16\sin^4\frac{p}{2}\right)\lambda^4 + \dots$$

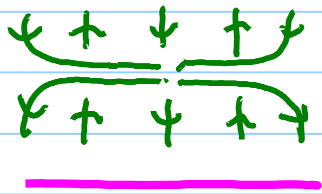
* pseudo-momentum

$$E = \frac{1}{2} + \left(e_{2,2}\sin^2\frac{p}{2}\right)\lambda^2$$
$$+ \left(e_{4,2}\sin^2\frac{p}{2} + e_{4,4}\sin^4\frac{p}{2}\right)\lambda^4$$
$$+ \left(e_{6,2}\sin^2\frac{p}{2} + e_{6,4}\sin^4\frac{p}{2} + e_{6,6}\sin^6\frac{p}{2}\right)\lambda^6$$
$$+ \dots$$

• corresponding spin chain Hamiltonian

$$H_0 = \frac{1}{2}\sum_{n=0}^{L-1} I$$

$$H_2 = e_{2,2}\sum_{n=0}^{L-1} - \frac{1}{4}$$  $+ \ldots$

$$H_4 = e_{4,4}\sum_{n=0}^{L-1} \frac{1}{16}$$  $+ \ldots$

$$H_6 = e_{6,6}\sum_{n=0}^{L-1}\left(\mathcal{O}_{6,6} + \tilde{\mathcal{O}}_{6,6}\right) + \ldots$$

where

$$\mathcal{O}_{6,6} = \frac{1}{64}\left[\left(\underline{P_{1,7}P_{3,5}} + P_{1,5}P_{3,7}\right) + P_{1,5} - 14P_{1,3} + 10\right]$$

$$\tilde{\mathcal{O}}_{6,6} = \frac{1}{64}\left[\left(1 - \underline{P_{1,7}}\right) - \left(1 - P_{1,5}\right) + 15\left(1 - P_{1,3}\right)\right]$$

cf. Serban, Staudacher; Inozemtsev

(maximal shuffling terms)

Quantum dilatation operator

- $$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_\varepsilon = \frac{C^{2L}(\varepsilon)}{(x^2)^{(1-\varepsilon)/2\cdot 2L + \gamma(\varepsilon)}}$$

$$\equiv I_\varepsilon^{2L} \cdot \exp\left[\ln(1 + A_2\lambda^2 + A_4\lambda^4 + A_6\lambda^6 + \dots)\right]$$

$$I_\varepsilon = \langle \phi(x)\phi(0)\rangle_\varepsilon$$

then

$$H_2 = -\lim_{\varepsilon\to 0} \varepsilon \cdot A_2$$

$$H_4 = -\lim_{\varepsilon\to 0} 2\varepsilon \cdot \left(A_4 - \tfrac{1}{2}A_2^2\right)$$

$$H_6 = -\lim_{\varepsilon\to 0} 3\varepsilon \cdot \left(A_6 - \tfrac{1}{2}(A_2A_4 + A_4A_2) + \tfrac{1}{3}A_2^3\right)$$

- Internal consistency check

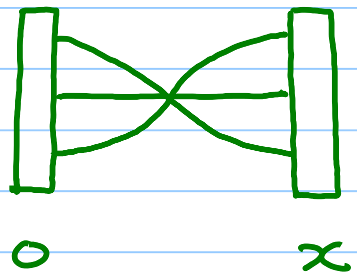
$\mathcal{O}(\frac{1}{\varepsilon})$ in H_4

$\mathcal{O}(\frac{1}{\varepsilon^2})$, $\mathcal{O}(\frac{1}{\varepsilon})$ in H_6 $= 0$

...

by renormalizability

H_2



$$= I_\varepsilon^3 (x^2\pi)^{3-2\omega} (-P_{1,3} + \cdots)$$

$$\otimes \text{(integral)}$$

(integral)

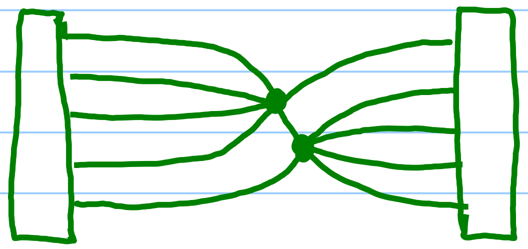
$$= -\frac{1}{4\varepsilon}\left[1 + \varepsilon(1 - \psi(\tfrac{1}{2})) + O(\varepsilon^2) \right]$$

$\therefore$ H_2 of Minahan + Zarembo
Bak + Rey

reproduced

H_4

- maximal shuffling



- operator structure obtained by iterating 2-loop operators

$\rightsquigarrow \quad \underbrace{P_{1,5}}_{\text{maximal shuffling}} + \cdots$

- $\mathcal{O}(\frac{1}{\varepsilon})$ terms cancel out

$\rightsquigarrow$ passed internal consistency

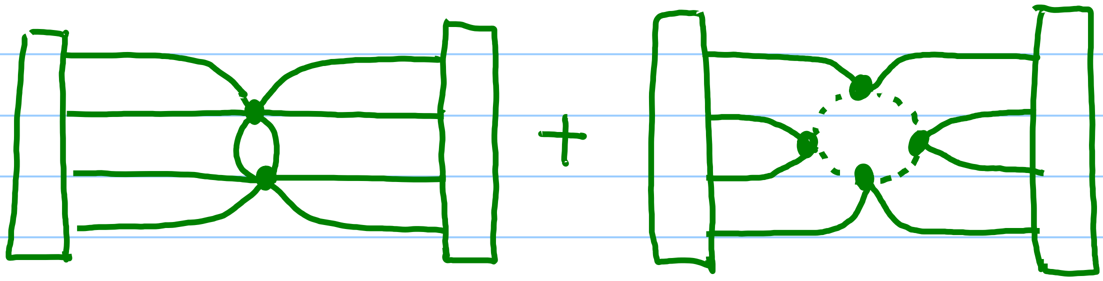
- $\mathcal{O}(\varepsilon^0)$ term computed numerically

MB
AMBRE

$$\rightsquigarrow \boxed{H_4 = \sim \sum P_{1,5} + \cdots \text{ with } c_{4,4} = -16}$$

This agrees with $PSU(2|2)$ + integrability + lattice m/m prediction.

- next-to-maximal shuffling



boson loop fermion loop

cancel each other

⤳ <u>no</u> $P_{1,4}$ terms

(decoupling between

●- sites and o-sites)

Pseudo vs. Lattice

- pseudo-momentum

$$\mathcal{P}(p) = p + \sum_{n=1}^{\infty} \sum_{l=n}^{\infty} b_{2l,2n} \lambda^{2l} \sin np$$

- interpolating ftn

$$h^2(\lambda) = \lambda^2 \sum_{l=0}^{\infty} h_{2l} \cdot \lambda^{2l} \qquad h_0 = 1$$

then

$$E = \sqrt{\frac{1}{4} + 4h^2(\lambda) \sin^2 \frac{\mathcal{P}(p)}{2}}$$

$$= \frac{1}{2}$$

$$+ (e_{2,2} \sin^2 \frac{p}{2}) \lambda^2 \qquad e_{2,2} = 4$$

$$+ (e_{4,2} \sin^2 \frac{p}{2} + e_{4,4} \sin^4 \frac{p}{2}) \lambda^4$$

$$e_{4,2} = 4h_2 + 8b_{2,2}$$

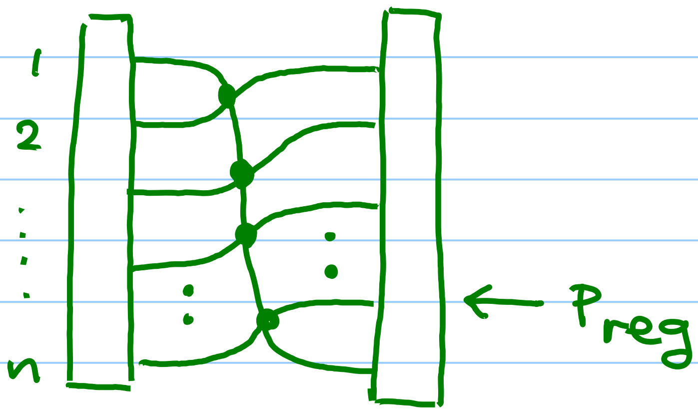
$$e_{4,4} = -16 - 8b_{2,2}$$

$$+ \cdots$$

$\therefore$ E determines $\mathcal{G}(Y^{\pm})$ and $h^2(\lambda)$ uniquely.

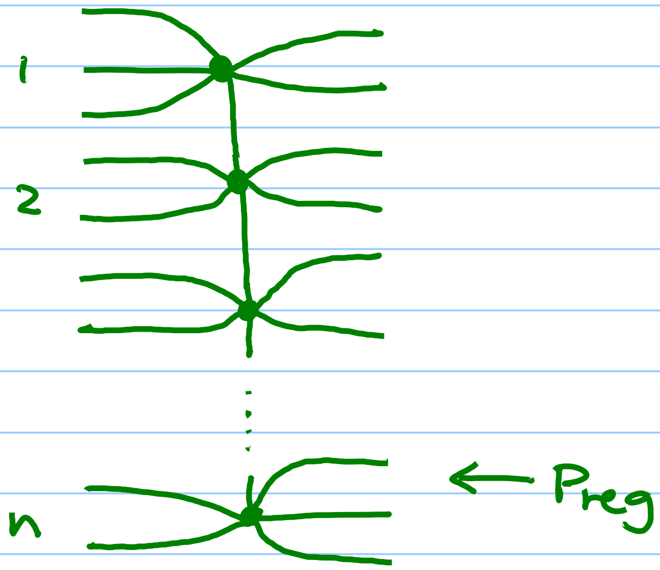
Gross, Mikhailov, Roiban

- In $\mathcal{N}=4$ SYM,
  GMR computes maximal shuffling terms to <u>all</u> orders.



- Since only ϕ^4 vertex present, <u>all</u> maximal shuffling diagrams are self-similar
- They can be summed by recursion relation to <u>all</u> orders
- reproduce $E(p)$ exactly

- For ABJ(M),

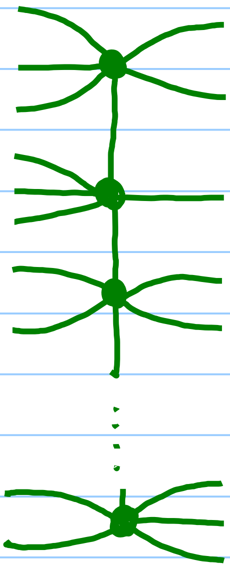


∴ these diagrams in momentum space are recursive

⇝ reproduce $E(p)$ to all orders

<u>Warning</u>: x-space evaluation is illegitimate, though looks to yield the same answer

we also have non-recursive contributions



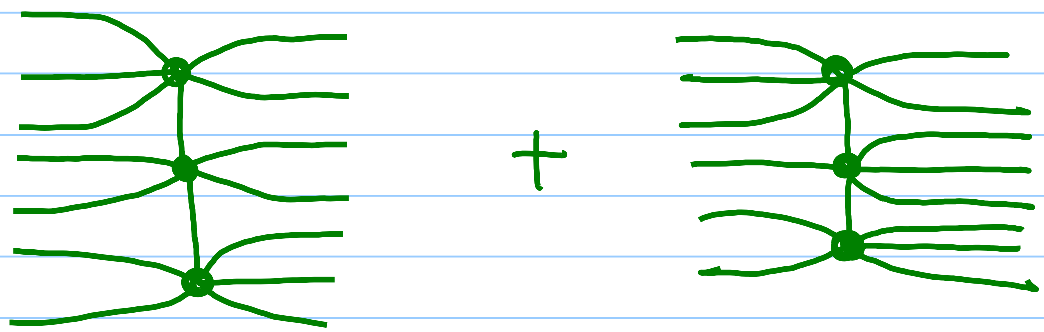
$\Leftarrow$ "impurity parts"

$\leftarrow$ $P_{reg.}$

For these diagrams, momentum regularization a lá GMR does not work!

$\rightsquigarrow$ we resort to our method and work out H_6 explicitly then generalize the argument

H_6



- From repeated application of lower order operator structure, we find

  $O_{6.6}$ and $(\tilde{O}_{6.6} - O_{6.6})$

  from each diagrams

- $(\tilde{O}_{6.6} - O_{6.6})$ gives vanishing contribution to H_6

- Feynman diagram is computed numerically [Smirnov PSLQ algorithm]

$$\boxed{\begin{array}{l} \rightsquigarrow H_6 = 128\, O_{6.6} \\ \text{with } \quad e_{6.6} = 128 \end{array}}$$

perfect agreement !

Conclusion

- maximal shuffling term in $E(p)$ agrees with $psu(2|2)$ + integrability
- <u>no</u> $\tilde{O}_{6,6}$ term contribution ... same situation as SYM
- explicitly to 6-loops
  $\rightsquigarrow$ <u>all</u> orders argument available

Under Consideration

* lattice momentum instead of pseudo-momentum.

  why?

* finite-size effect

* ABJ(M) versus IIA string elementary excitations

* Constraint on global analytic structure of $h(\lambda)$ [as opposed to perturbative computations]