# Resurgent Analysis of Large N Matrix Models


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## 1 Introduction

The hermitian one–matrix model with polynomial potential $V(z)$ is, generically, very hard to solve exactly. Instead one often uses the large $N$ limit: $N \to +\infty$ while $t = g_s N$ fixed ['t Hooft]. In this case the free energy $F = \log Z$ has a perturbative genus expansion,

$$F \simeq \sum_{g=0}^{+\infty} F_g(t)\, g_s^{2g-2}.$$

Large–order $F_g \sim (2g)!$ renders the topological genus expansion as an asymptotic approximation [Shenker].

How can one recover the exact solution from the asymptotic expansion? One needs to consider all distinct eigenvalue partitions $\Rightarrow$ amounts to all distinct instanton sectors! The sum over all possible canonical multi–cut backgrounds yields a grand–canonical, manifestly background independent partition function [Eynard-Mariño]. This construction may be made explicit via: Resurgence and Transseries.

But, as it turns out, this construction will further go beyond "standard" instantons and beyond multi–cut configurations [IA-RS-Vonk]! The transseries construction reconstructs the "original" nonperturbative partition function behind the large $N$ expansion.

## 2 Resurgent Transseries and Asymptotics

How do we associate values to (factorially) divergent sums? Use the Borel transform of the asymptotic series, which has finite convergence radius

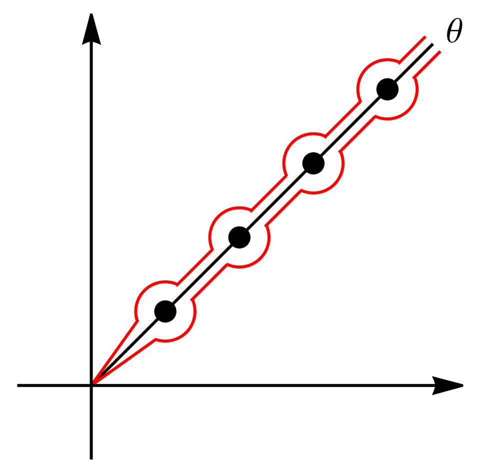
$$\mathcal{B}[F](s) = \sum_{g=0}^{+\infty} \frac{F_g}{g!}\, s^g.$$

The Borel resummation of $F(g_s)$ along $\theta$ is

$$\mathcal{S}_\theta F(g_s) = \int_0^{e^{i\theta}\infty} ds\, \mathcal{B}[F](s)\, e^{-\frac{s}{g_s}}.$$

$\mathcal{S}_\theta F(g_s)$ has, by construction, the same asymptotic expansion as $F(g_s)$ and may provide a solution to our original question. This holds *except* along *singular directions* $\theta$: directions along which there are singularities in the Borel plane (in the original complex $g_s$–plane such directions are known as Stokes lines).

One needs to introduce *lateral* Borel resummations along $\theta$, $\mathcal{S}_{\theta^\pm} F(g_s)$:




But the choice of contour introduces a nonperturbative ambiguity

$$\mathcal{S}_{\theta^+} F(g_s) - \mathcal{S}_{\theta^-} F(g_s) \propto e^{-A/g_s}.$$

Resurgent functions allow for the resummation of asymptotic series along *any* direction in the complex $s$–plane $\Rightarrow$ This first yields a family of sectorial analytic functions $\{\mathcal{S}_\theta F\}$ $\Rightarrow$ But one further needs to "connect" these sectorial solutions together [Écalle].

The connection of distinct sectorial solutions on both sides of Stokes lines entails understanding their "jump", accomplished via the Stokes' automorphism, $\underline{\mathfrak{S}}_\theta$,

$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta^-} \circ \left(\mathbf{1} - \mathrm{Disc}_{\,\theta^-}\right).$$

The action of $\underline{\mathfrak{S}}_\theta$ on resurgent functions translates into the required connection of distinct sectorial solutions, across any singular direction $\theta$.

Final solutions are written as a transseries *ansatz* for the resurgent function,

$$F(\boldsymbol{\sigma}, g_s) = \sum_{\boldsymbol{n}\in\mathbb{N}^k} \boldsymbol{\sigma}^{\boldsymbol{n}}\, e^{-\frac{\boldsymbol{n}\cdot\boldsymbol{A}}{g_s}}\, \Phi_{(\boldsymbol{n})}(g_s),$$

with $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k)$ the nonperturbative ambiguities/transseries parameters.

Matrix models/minimal/topological strings [IA-RS-Vonk]:

- "Generalized" instanton sectors are labeled by $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$.
- $\boldsymbol{n} = (0, \ldots, 0)$ sector is the usual perturbative sector.
- $\boldsymbol{n} = (n, 0, \ldots, 0)$ sectors are multi–instanton sectors.
- Generically $A_i \in \mathbb{C} \Rightarrow$ Many new sectors!
- Sectors with $n_i \neq n_j, \forall_{i,j} \Rightarrow$ Generically $\Phi_{(\boldsymbol{n})}$ has an expansion in $g_s$ (open string like).
- Sectors with $\boldsymbol{n}\cdot\boldsymbol{A} = 0 \Rightarrow$ Generically $\Phi_{(\boldsymbol{n})}$ has an expansion in $g_s^2$ (closed string like).

Exact knowledge of the above Stokes automorphism yields exact large–order formulae. Can illustrate this by writing down the first few terms in the double–series,

$$F_g^{(0)} \simeq \frac{S_1}{2\pi i} \frac{\Gamma(g-\beta)}{A^{g-\beta}} \left( F_1^{(1)} + \frac{A}{g-\beta-1} F_2^{(1)} + \cdots \right) + \frac{S_1^2}{2\pi i} \frac{\Gamma(g-2\beta)}{(2A)^{g-2\beta}} \left( F_1^{(2)} + \cdots \right) + \cdots.$$

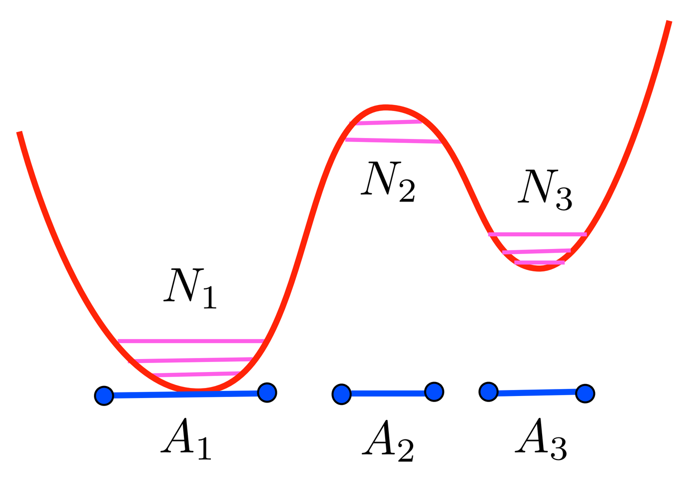
Can further obtain all multi–instanton exact large–order formulae! For example,

$$F_g^{(n)} \simeq \frac{S_1}{2\pi i} (n+1) \frac{\Gamma(g-\beta-1)}{(A)^{g-\beta-1}} F_1^{(n+1)} + \cdots + \frac{S_{-1}}{2\pi i} (n-1) \frac{\Gamma(g+\beta-1)}{(-A)^{g+\beta-1}} F_1^{(n-1)} + \cdots.$$

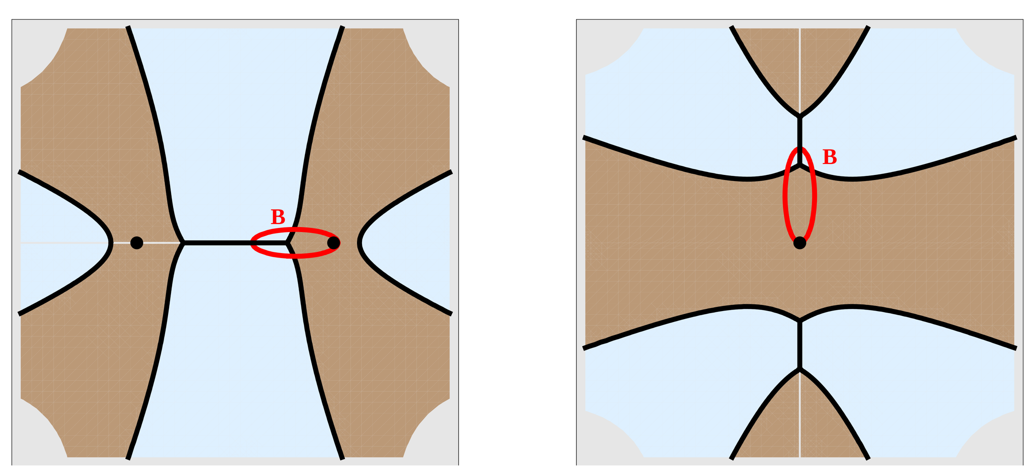
This yields the *full* large–order information in terms of a (possibly) infinite sequence of Stokes invariants $S_\ell \in \mathbb{C}$, $\ell \in \{1, -1, -2, -3, -4, \cdots\}$. It further allows for numerical checks of extremely high precision!

## 3 Resurgence of the Quartic Matrix Model

The quartic potential $V(z) = \frac{1}{2}z^2 - \frac{\lambda}{24}z^4$ generically admits a three–cuts solution. Transseries may be constructed around the one–cut and the two–cuts backgrounds.



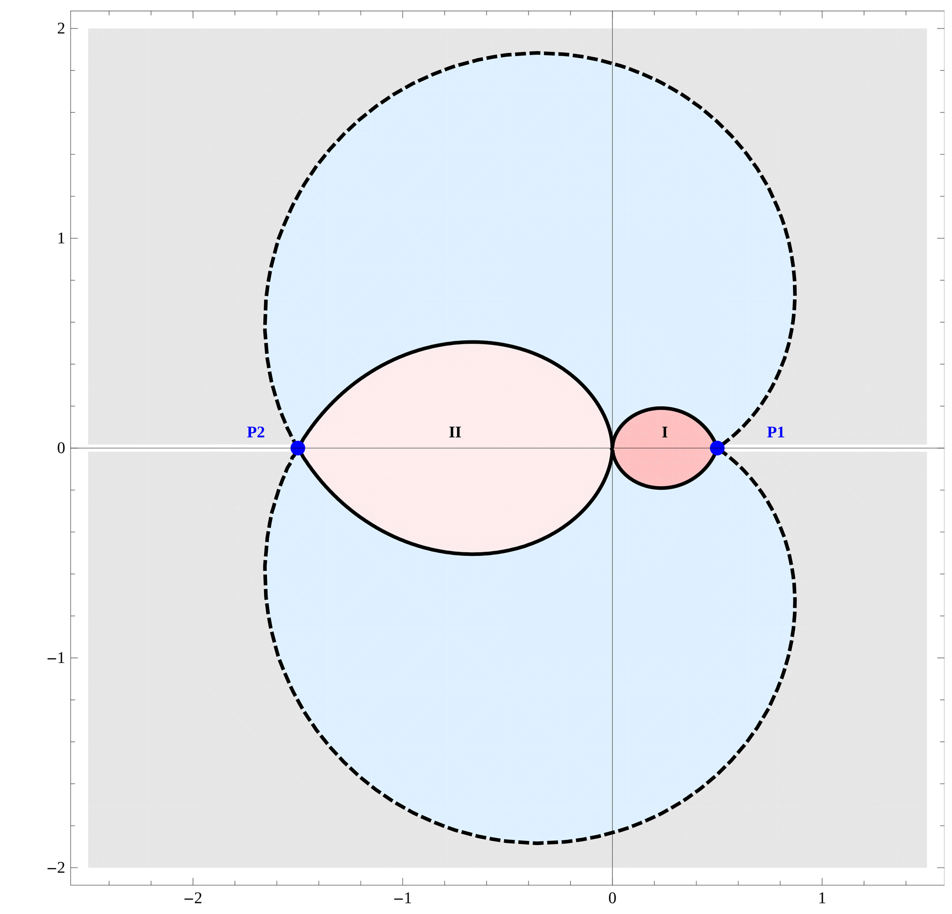

In these backgrounds, instantons arise from B–cycles [David,Seiberg-Shih,Mariño-RS-Weiss,RS-Vaz].




The instanton actions in these backgrounds yield the:

- Stokes lines ("jumps" in Borel plane): $\mathbb{I}\mathrm{m}\left(\frac{A(t)}{g_s}\right) = 0$.
- Anti–Stokes lines (phase boundaries): $\mathbb{R}\mathrm{e}\left(\frac{A(t)}{g_s}\right) = 0$.

In this way one may construct the quartic phase diagram for complex 't Hooft coupling:




One further finds a three–cuts anti–Stokes phase [Eynard-Mariño,Mariño-Pasquetti-Putrov,IA-RS-Vonk], and a "new" trivalent–tree phase [David,Bertola,IA-RS-Vonk].

The transseries solution to the quartic string equation

$$\mathcal{R}(x) \left\{ 1 - \frac{\lambda}{6} \big( \mathcal{R}(x-g_s) + \mathcal{R}(x) + \mathcal{R}(x+g_s) \big) \right\} = x$$

requires both "instanton" actions $+A$ and $-A$, leading to the transseries

$$\mathcal{R}(x) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m\, e^{-(n-m)A(x)/g_s} \sum_{g=\beta_{nm}}^{+\infty} g_s^g\, R_g^{(n|m)}(x).$$

This is a fully nonperturbative solution $\Rightarrow$ Via Stokes transitions one can move anywhere in the above phase diagram. Further:

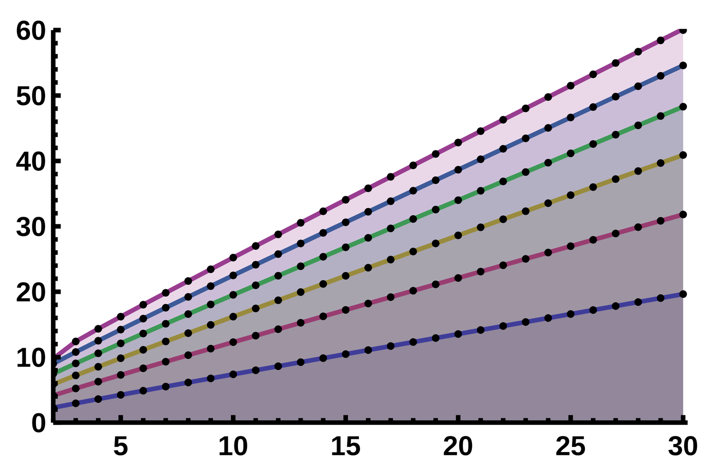
- 1–parameter resummation: yields the theta functions of the grand–canonical sum over all multi–cut configurations! [Eynard-Mariño, IA-RS-Vonk]
- 2–parameters resummation: yields the asymptotics of the trivalent–tree phase? [work in progress]

One may further study the transseries solution in the double–scaling limit yielding the Painlevé I equation $u^2(z) - \frac{1}{6}u''(z) = z$. The general two–parameteres transseries solution is ($x = z^{-5/4}$)

$$u = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m\, e^{-(n-m)\frac{A}{x}} \left( \sum_{k=0}^{\min(n,m)} \log^k(x) \cdot \Phi_{(n|m)}^{[k]}(x) \right).$$

- Checked the validity of all nonperturbative sectors via detailed large–order analysis.
- The physical interpretation of the "generalized" instanton series is still open!

Resurgence allows for extremely accurate tests: at genus $g = 30$, including six instantons corrections, our results are correct up to 60 decimal places!




In here, the Stokes constant $S_1^{(0)}$ is computed from first principles (one–loop result around the one–instanton sector) in both the matrix model and the double–scaling limit, $S_1^{(0)} = -i\frac{3^{1/4}}{2\sqrt{\pi}}$ [David]. But all other Stokes constants $S_\ell^{(k)}$, $\widetilde{S}_\ell^{(k)}$ so far have been only computed numerically $\Rightarrow$ Requires extra physical input! But there are many (as yet unexplained) relations between these constants...

## 4 Acknowledgements


We are very grateful to Marcel Vonk for a most stimulating collaboration in the results reported herein.