Bulk viscosity of the Gross-Neveu model and integrability

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GOETHE UNIVERSITÄT FRANKFURT AM MAIN
Institut für Theoretische Physik

Daniel Fernandez-Fraile

danfer@th.physik.uni-frankfurt.de

Alexander von Humboldt Stiftung/Foundation

ABSTRACT

A calculation of the bulk viscosity for the massive Gross-Neveu model at zero fermion chemical potential is presented in the large-N limit. This model resembles QCD in many important aspects: it is asymptotically free, has a dynamically generated mass gap, and for zero bare fermion mass it is scale invariant at the classical level (broken through the trace anomaly at the quantum level). For our purposes, the introduction of a bare fermion mass is necessary to break the integrability of the model, and thus to be able to study momentum transport. The main motivation is, by decreasing the bare mass, to analyze whether there is a correlation between the maximum in the trace anomaly and a possible maximum in the bulk viscosity, as recently conjectured [1]. We also analyze whether there is a contribution from bulk viscosity to the sum rule in our model.

VISCOSITIES (1)

The energy-momentum tensor of a fluid is modified by viscosities. To linear order in gradients,

$$\langle \hat{T}_{ij}\rangle = P_{\mathrm{eq}}\delta_{ij} + \boldsymbol{\eta}\left(\partial_i U_j + \partial_j U_i - \frac{2}{3}\delta_{ij}\partial_k U^k\right) - \boldsymbol{\zeta}\,\partial_k U^k\,\delta_{ij}\ .$$

shear viscosity (η) — bulk viscosity (ζ)

Linear-Response Theory: Consider a small time-dependent perturbation in the Hamiltonian: $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$

$$\delta\langle\hat{O}(t)\rangle \equiv \langle\Psi(t)|\hat{O}(t)|\Psi(t)\rangle - \langle\Psi^{(0)}(t)|\hat{O}(t)|\Psi^{(0)}(t)\rangle$$
$$= -\mathrm{i}\int_0^\infty \mathrm{d}t'\ \langle\Psi^{(0)}(0)|\theta(t-t')[\hat{O}_{\mathrm{H}}(t),\hat{V}_{\mathrm{H}}(t')]|\Psi^{(0)}(0)\rangle$$

Applied to the shear and bulk viscosities (***Kubo formulas***),

$$\eta = \frac{1}{20}\lim_{q^0\to 0^+}\lim_{|\boldsymbol{q}|\to 0^+}\frac{\partial\rho_\eta(q^0,|\boldsymbol{q}|)}{\partial q^0} \qquad \zeta = \frac{1}{2}\lim_{q^0\to 0^+}\lim_{|\boldsymbol{q}|\to 0^+}\frac{\partial\rho_\zeta(q^0,|\boldsymbol{q}|)}{\partial q^0}$$

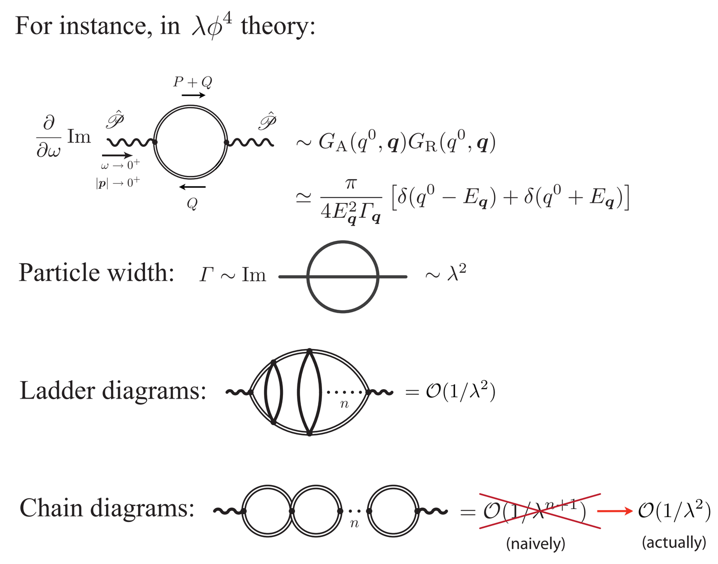
$$\rho_\eta(q^0,|\boldsymbol{q}|) = 2\,\mathrm{Im}\,\mathrm{i}\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}q\cdot x}\,\theta(t)\langle[\hat{\pi}_{ij}(x),\hat{\pi}^{ij}(0)]\rangle\ , \quad \pi_{ij}\equiv T_{ij} - g_{ij}T^k_k/3$$

$$\rho_\zeta(q^0,|\boldsymbol{q}|) = 2\,\mathrm{Im}\,\mathrm{i}\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}q\cdot x}\,\theta(t)\langle[\hat{\mathscr{P}}(x),\hat{\mathscr{P}}(0)]\rangle\ , \quad \mathscr{P}\equiv -T^k_k/3$$

CALCULATION IN QUANTUM FIELD THEORY (2)

Transport coefficients are intrinsically non-perturbative quantities ⇒ Even to leading order in the coupling, a resummation of diagrams is necessary.

- For instance, in $\lambda\phi^4$ theory:



The resummation is equivalent to an effective Kinetic Theory formulation in terms of thermal excitations [2,3].

- Result for $\lambda\phi^4$: $T\gg m\ :\ \zeta\sim\lambda T^3\ln^2\lambda\ ;\quad T\ll m\ :\ \zeta\sim \mathrm{e}^{2\beta m_{\mathrm{phys}}}/\lambda^4$

THE GROSS-NEVEU MODEL (3)

Lagrangian in 1+1 dimensions [4]:

$$\mathcal{L} = \bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{g^2}{2}(\bar{\psi}\psi)^2\ , \qquad \psi = (\psi_1,\ldots,\psi_N)$$

Similar properties to massless QCD in the large N limit ($\lambda\equiv g^2N$ kept constant):

- Renormalizable and asymptotically free.
- Classically scale invariant, but it has a dynamically generated mass gap reflected as a peak in the trace anomaly.
- Spontaneus breaking in vacuum of the discrete chiral symmetry $\psi\mapsto\gamma_5\psi$.

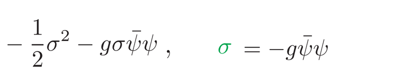
In addition:

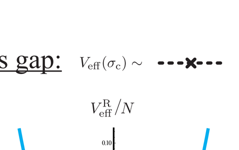
- No confinement

A Kinetic Theory treatment is possible in terms of the fundamental fields in the large-N limit ($g\sim 1/\sqrt{N}$).

In the large-N limit, it is convenient to introduce an auxiliary field:

$$\mathcal{L} = \bar{\psi}\mathrm{i}\partial\!\!\!/\psi - \frac{1}{2}\sigma^2 - g\sigma\bar{\psi}\psi\ , \qquad \sigma = -g\bar{\psi}\psi$$

Mass gap: $V_{\mathrm{eff}}(\sigma_c)\sim$ 



$$\Rightarrow\ M_0 = g\,\sigma_0\exp\left(1-\frac{\pi}{g^2N}\right)$$

Asymptotic freedom:

$$\left[\sigma_0\frac{\partial}{\partial\sigma_0} + \beta(g)\frac{\partial}{\partial g}\right]M_0(\sigma_0,g) = 0 \quad\Rightarrow\quad \beta(g) = -\frac{g^3N/2\pi}{1+g^2N/2\pi}$$

INTEGRABILITY AND TRANSPORT (4)

The Gross-Neveu model is integrable both at the classical and quamtum levels.

It has an infinite number of conserved charges [5,6]:

$$\hat{Q}_n|p_1,p_2,\ldots,p_k\rangle = [f_n(p_1)+f_n(p_2)+\ldots+f_n(p_k)]|p_1,p_2,\ldots,p_k\rangle\ , \qquad n=1,\ldots,\infty$$

$$f_n(p)\sim p^n$$

$$\hat{Q}_n|p_1,p_2,\ldots,p_k;\mathrm{in}\rangle = \hat{Q}_n|p'_1,p'_2,\ldots,p'_r;\mathrm{out}\rangle\quad\forall n\quad\Rightarrow\quad k=r$$

inelastic amplitudes vanish

no momentum transport, because binary collisions in 1+1 dimensions don't change the distribution of momenta:

and an arbitrary elastic scattering is factorized in terms of binary collisions [5]

In order to study the bulk viscosity, we consider the massive Gross-Neveu model:

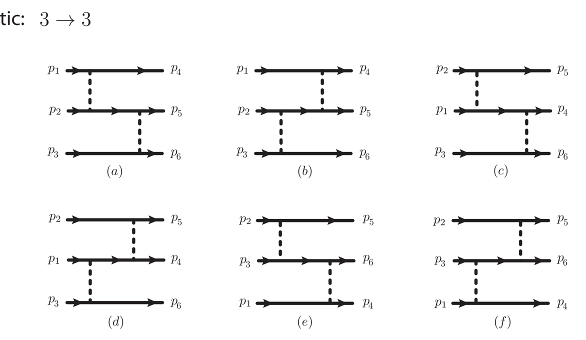
$$\mathcal{L} = \bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{g^2}{2}(\bar{\psi}\psi - Nm)^2 = \bar{\psi}\mathrm{i}\partial\!\!\!/\psi - \frac{1}{2}\sigma^2 - g\sigma\bar{\psi}\psi + Nmg\sigma$$

- Non-integrable in the large-N limit (cf. #6).
- The bare mass suppresses kink-anti-kink configurations in the thermodynamic limit, and makes the $1/N$ expansion well defined [7,8].

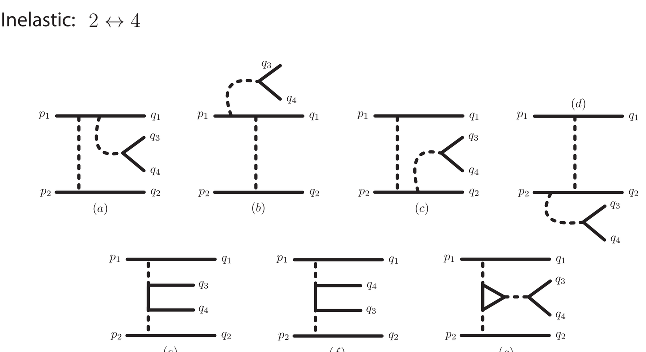
LEADING-ORDER SCATTERING AMPLITUDES (5)

The diagrams which contribute at leading order, i.e. $\mathcal{O}(1/N^2)$, to momentum transport are:

Elastic: $3\to 3$

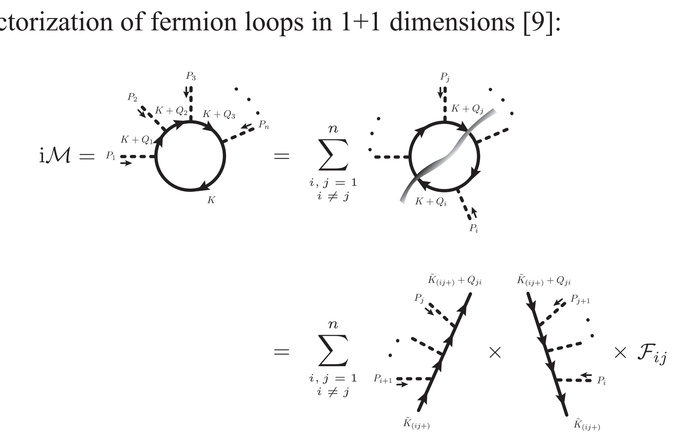


Inelastic: $2\leftrightarrow 4$



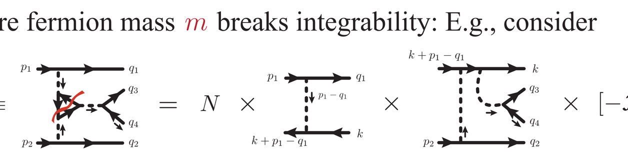
BREAKING OF INTEGRABILITY (6)

Factorization of fermion loops in 1+1 dimensions [9]:



(an analogous result can also be obtained at finite temperature [10])

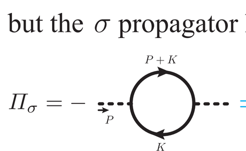
The bare fermion mass m breaks integrability: E.g., consider



with

$$\mathcal{F}_{ji}\equiv\frac{1}{4\pi Q^2_{ji,\mathrm{E}}\beta(Q_{ji,\mathrm{E}})}\ln\left(\frac{\beta(Q_{ji,\mathrm{E}})+1}{\beta(Q_{ji,\mathrm{E}})-1}\right),$$

but the σ propagator has the form

$\Pi_\sigma = -$  $\Rightarrow$

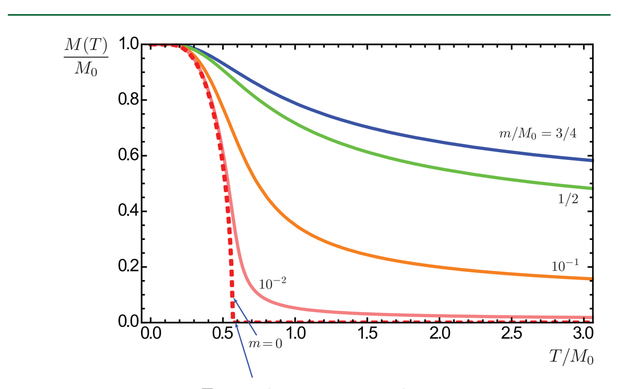
$$[\Delta^{T=0}_\sigma(P_{\mathrm{E}})]^{-1} = \frac{Nm}{M_0} + \frac{N}{2\pi}\beta(P_{\mathrm{E}})\ln\left(\frac{\beta(P_{\mathrm{E}})+1}{\beta(P_{\mathrm{E}})-1}\right),\qquad \beta(P_{\mathrm{E}})\equiv\sqrt{1+4M_0^2/P^2_{\mathrm{E}}}\ .$$

(it resummates a chain of bubbles)

$$\Rightarrow\quad |\mathcal{M}|^2\propto\left(\frac{m}{M_0}\right)^2$$

Hence by increasing or decreasing m, the bulk viscosity would in principle become arbitrarily small or large respectively. But this is not a problem to test the relationship with the trace anomaly.

THERMAL MASS GAP (7)



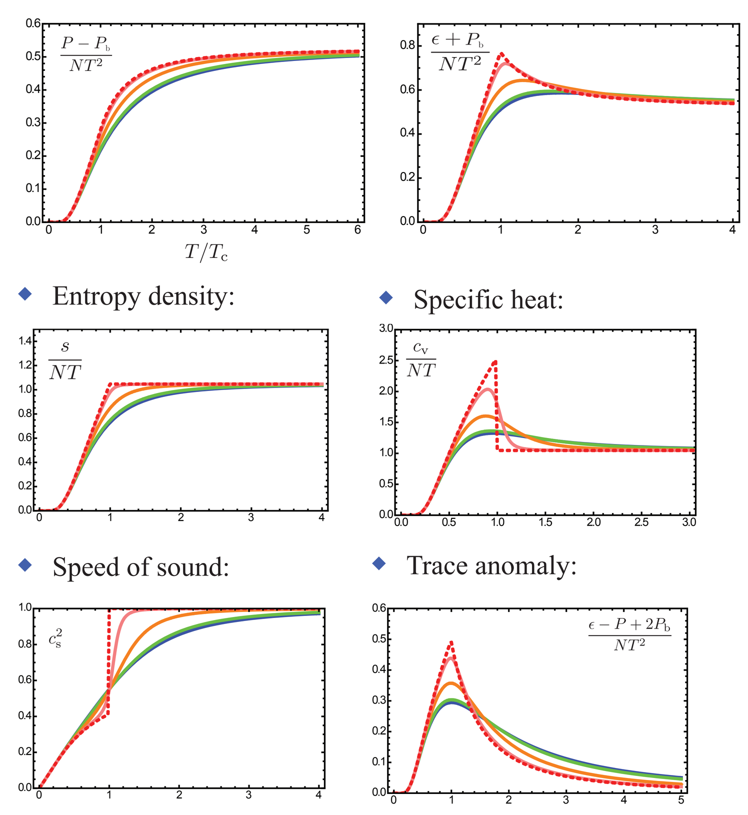
T_c ("critical temperature" corresponding to the mean-field result for $m=0$)

For $T<T_c$, $\frac{m}{M(T)}\xrightarrow[m\to 0]{}0$; For $T>T_c$, $\frac{m}{M(T)}\xrightarrow[m\to 0]{}\mathrm{const.}$

Integrability is never restored for $T>T_c$.

THERMODYNAMICS (8)

If we take first the large-N limit, and then the thermodynamic limit, we find a second-order phase transition at zero chemical potential [11]:



For $T>T_c$ and $m=0$:

$$s=\frac{\pi NT}{3}\ ,\quad c_V = s\ ,\quad c_s^2 = 1\ ,\quad \frac{\epsilon-P+2P_b}{NT^2} = \frac{2P_b}{NT^2}\ ,\ P_b = NM_0^2/(4\pi)$$

CALCULATION IN KINETIC THEORY (9)

In the large-N limit, the interaction between fermions is suppressed by powers of N:

$\Gamma\sim$  $=\mathcal{O}\left(\frac{1}{N}\right)$

⇒ a kinetic theory approach should be applicable [12].

Energy-momentum tensor: $T^{\mu\nu}(t,x) = \sum_a\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{(2\pi)E_k}\underline{k}^\mu\underline{k}^\nu f^a(t,x,\underline{k})$

The space-time evolution of f^a is determined by the Boltzmann equation:

$$\left(\frac{\partial}{\partial t}+\frac{k}{E_k}\frac{\partial}{\partial x}\right)f^a = \left.\frac{\partial f^a}{\partial t}\right|_{\mathrm{gain}} - \left.\frac{\partial f^a}{\partial t}\right|_{\mathrm{loss}} \equiv \frac{1}{E_k}C^a_k[\boldsymbol{f}]$$

To obtain the bulk viscosity we consider a small departure from equilibrium:

$$f^a(t,x,\underline{k}) = f^a_{\mathrm{eq}}(t,x,\underline{k}) + \delta f^a(t,x,\underline{k})\quad,\quad f^a_{\mathrm{eq}}(t,x,\underline{k}) = \frac{1}{\mathrm{e}^{\beta\underline{k}\cdot u}+1}$$

Linearizing the LHS and RHS of the Boltzmann equation with respect to spatial gradients:

$$|\mathcal{S}\rangle = \hat{\mathscr{C}}|B\rangle\ ,$$

$$\mathcal{S}(\underline{p})\equiv p^2 - c_s^2(E_p^2 - MT\,\mathrm{d}M/\mathrm{d}T)\ ,\ \delta f^a_k = -\beta n_{\mathrm{F}}(E_k)[1-n_{\mathrm{F}}(E_k)]B^a_k\frac{\partial u^1}{\partial x},$$

Collision matrix in the large-N limit:

$$C_{ij}\simeq N^3\beta\int_{-\infty}^{\infty}\Big[\prod_{l=1}^{6}\frac{dp_l}{(2\pi)2E_l}\Big]\Big\{(2\pi)^2\delta^{(2)}(\underline{p}_1+\underline{p}_2-\underline{p}_3-\underline{p}_4-\underline{p}_5-\underline{p}_6)n_{F,1}n_{F,2}(1-n_{F,3})(1-n_{F,4})(1-n_{F,5})$$
$$\times(1-n_{F,6})\Big[|\mathcal{M}_{12\to3456}|^2+\frac{3}{2}|\mathcal{M}_{1\bar{2}\to3\bar{4}56}|^2\Big][\phi_i(p_1)+\phi_i(p_2)-\phi_i(p_3)-\phi_i(p_4)-\phi_i(p_5)-\phi_i(p_6)]$$
$$\times[\phi_j(p_1)+\phi_j(p_2)-\phi_j(p_3)-\phi_j(p_4)-\phi_j(p_5)-\phi_j(p_6)] + (2\pi)^2\delta^{(2)}(\underline{p}_1+\underline{p}_2+\underline{p}_3-\underline{p}_4$$
$$-\underline{p}_5-\underline{p}_6)n_{F,1}n_{F,2}n_{F,3}(1-n_{F,4})(1-n_{F,5})(1-n_{F,6})\Big[\frac{1}{6}|\mathcal{M}_{123\to456}|^2+\frac{3}{2}|\mathcal{M}_{12\bar{3}\to45\bar{6}}|^2\Big][\phi_i(p_1)+\phi_i(p_2)$$
$$+\phi_i(p_3)-\phi_i(p_4)-\phi_i(p_5)-\phi_i(p_6)][\phi_j(p_1)+\phi_j(p_2)+\phi_j(p_3)-\phi_j(p_4)-\phi_j(p_5)-\phi_j(p_6)]\Big\}$$

Bulk viscosity:

$$\zeta = \langle\mathcal{S}|B\rangle = \langle\mathcal{S}|\hat{\mathscr{C}}^{-1}|\mathcal{S}\rangle\quad\Rightarrow\quad\zeta=\mathcal{O}(N^3)$$

with $\langle\chi|\psi\rangle\equiv\beta\sum_a\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{(2\pi)E_k}n_{\mathrm{F}}(E_k)[1-n_{\mathrm{F}}(E_k)]\,\chi^a(\underline{k})\psi^a(\underline{k})$.

RESULTS (10)

- Variational solution:

$$Q[\chi]\equiv\langle\chi|\mathcal{S}\rangle - \frac{1}{2}\langle\chi|\hat{\mathscr{C}}|\chi\rangle\ ,\qquad\Rightarrow\qquad \zeta = 2\,Q_{\max}\ .$$

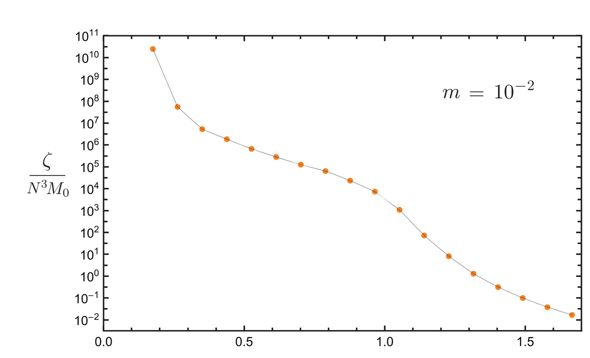
For smaller subspaces, what we obtain is a lower bound eventually converging as we increase the basis:

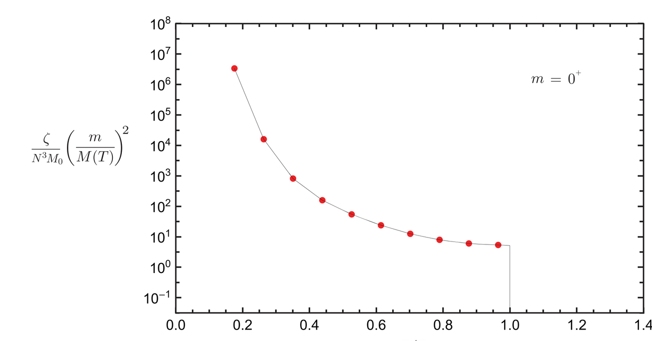
$$\zeta = \tilde{S}^{\mathrm{t}}\tilde{C}^{-1}\tilde{S}\ ,\quad S_i\equiv\langle\phi_i|\mathcal{S}\rangle\ ,\quad C_{ij}\equiv\langle\phi_i|\hat{\mathscr{C}}|\phi_j\rangle\ .$$

A convenient basis turns out to be

$$\phi_i(k) = \frac{(|k|/\langle|k|\rangle)^{i-1}}{(1+|k|/\langle|k|\rangle)^{n-3}}\ ,\quad i=1,\ldots,n\ ,\quad \begin{cases}\sim 1\ ,\ |k|\to 0\\ \sim k^2\ ,\ |k|\to\infty\end{cases}$$

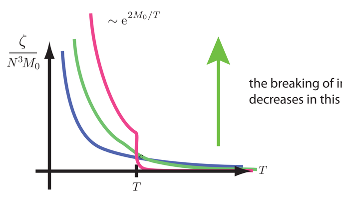
- Results ($n=3$) [10]:





For $T<T_c$, $\frac{m}{M(T)}\xrightarrow[m\to 0]{}0$

For $T>T_c$, $\frac{m}{M(T)}\xrightarrow[m\to 0]{}\mathrm{const.}(T)$



⇒ no correlation with the trace anomaly.

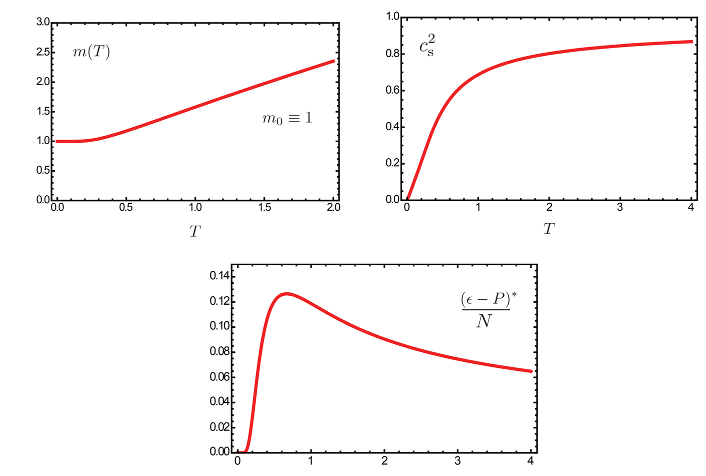
NON-LINEAR SIGMA MODEL (11)

From the analysis for the Gross-Neveu model, we can now extract similar conclusions for the Non-linear Sigma model [10]:

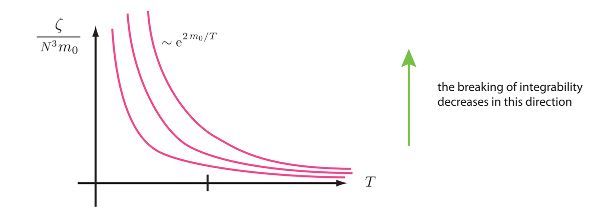
$$\mathcal{L} = \frac{1}{2g^2}\partial_\mu\phi_a\partial^\mu\phi_a\ ,\ a=1,\ldots,N\quad\text{with the condition}\quad\phi_a\phi_a = 1$$

$$\Rightarrow\quad\mathcal{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}\mathrm{i}\alpha(\phi_a\phi_a - 1/g^2)$$

- Asymptotically free, dynamical mass gap, no phase transition:



expected behavior



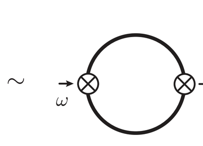
SUM RULE (12)

Following the method of [13], it is not difficult to derive:

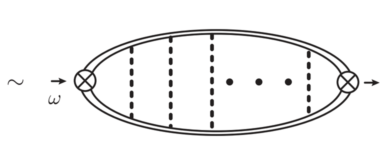
$$(\epsilon+P)(1-c_s^2) - 2(\epsilon-P) = \frac{2}{\pi}\int_0^\infty\frac{\mathrm{d}\omega}{\omega}\,\delta\rho^{\mathrm{bulk}}(\omega)\ .$$

$$\mathcal{O}(N)$$

Different regimes of frequencies:

$(i)\quad\omega\sim M_0\ :\quad\frac{\delta\rho^{\mathrm{bulk}}}{\omega}\sim$  $=\mathcal{O}(N)$

$(ii)\quad\omega\sim\gamma_{\mathrm{F}}\sim 1/N\ :$

$\frac{\delta\rho^{\mathrm{bulk}}}{\omega}\sim$  $=\mathcal{O}(N^2)$

$(iii)\quad\omega\sim 1/N^w\ ,\ w>2\ :\quad\frac{\delta\rho^{\mathrm{bulk}}}{\omega} = \mathcal{O}(N^3)$

But then the bulk viscosity does not contribute to the sum rule below the critical temperature.

CONCLUSIONS

- There is not direct correlation between the trace anomaly and the bulk viscosity.
- It is not always possible to extract information about the bulk viscosity from sum rules.
- There is no universal lower bound for the bulk viscosity, even away from conformality.
- The introduction of a bare fermion mass in the Gross-Neveu model causes the breaking of integrability and therefore allows the system to relax back to equilibrium after a perturbation in the distribution of momenta.
- Integrability is not restored above the critical temperature when we take the limit of the bare fermion mass going to zero.
- The same conclusions as for the Gross-Neveu model below the critical temperature apply to the Non-Linear Sigma Model at any temperature.

REFERENCES

[1] Kharzeev & Tuchin, JHEP 0809, 093 (2008); [2] Jeon, PRD 52, 3591 (1995); [3] Jeon & Yaffe, PRD 53, 5799 (1996); [4] Gross & Neveu, PRD 10, 3235 (1974); [5] Zamolodchikov & Zamolodchikov, AP 120, 253 (1979); [6] Witten, NPB 142, 285 (1978); [7] Dashen, Ma, & Rajaraman, PRD 11, 1499 (1975); [8] Barducci, Casalbuoni, Modugno, Pettini, & Gatto, PRD 51, 3042 (1995); [9] Berg, Nuovo Cim. A41, 58 (1977); [10] Fernandez-Fraile, PRD 83, 065001 (2011); [11] Schon & Thies, hep-th/0008175; [12] Aarts & Martinez-Resco, JHEP 02, 061 (2004); [13] Romatschke & Son, PRD 80, 065021 (2009).

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