From Q-Operators to Local Charges

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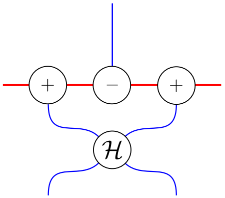

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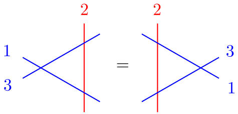
Abstract

In [1] we discuss how the Shift- and Hamiltonian operator enter the hierarchy of Baxter Q-operators in the example of $\mathfrak{gl}(n)$ homogeneous spin-chains. Building on the construction recently carried out in [2,3,4,5], we find that a reduced set of Q-operators can be used to obtain local charges. The mechanism relies on projection properties of the corresponding $\mathcal{R}$-operators on a highest/lowest weight state of the quantum space. It is intimately related to the ordering of the oscillators in the auxiliary space. Furthermore, we introduce a diagrammatic language that makes these properties manifest and the results transparent. Our approach circumvents the paradigm of constructing the transfer matrix with equal representations in quantum and auxiliary space and underlines the strength of the Q-operator construction.

Q-operator construction

Yang-Baxter Equation

- Starting point: Yang-Baxter equation $(\mathbb{C}^n \otimes V \otimes \mathbb{C}^n)$



$$R^{13}(x-y)L^{12}(x)L^{23}(y) = L^{23}(y)L^{12}(x)R^{13}(x-y)$$

- $R(z) = z + \mathbf{P}$ is an $n^2 \times n^2$ matrix
- $L(z)$ is an $n \times n$ matrix with entries in V

New solutions of the YBE

- Defines the Yangian algebra $\mathcal{Y}(\mathfrak{gl}(n))$

$$(z_1 - z_2)[L_A^B(z_1), L_C^D(z_2)] = L_A^D(z_1)L_C^B(z_2) - L_A^D(z_2)L_C^B(z_1)$$

- Realization via infinite-dimensional oscillator algebra

$$[\mathbf{a}_{\dot{b}}^{a}, \bar{\mathbf{a}}_{d}^{\dot{c}}] = \delta_d^a \delta_{\dot{b}}^{\dot{c}} \qquad a, b, c \in I, \qquad \dot{a}, \dot{b}, \dot{c} \in \bar{I} \qquad I \cup \bar{I} = \{1, \ldots, n\}$$

- New solutions of YBE (fundamental rep.)

$$\mathbf{L}_I(z) = \begin{pmatrix} (z - \frac{|\bar{I}|}{2})\delta_b^a - \bar{\mathbf{a}}_b^{\dot{a}} \mathbf{a}_{\dot{a}}^a & \bar{\mathbf{a}}_b^{\dot{a}} \\ -\mathbf{a}_{\dot{b}}^a & \delta_{\dot{b}}^{\dot{a}} \end{pmatrix} \quad \text{for} \quad I = \{1, \ldots, |I|\}$$

$\mathcal{R}$-operators for Q-operators

- Generalization for arbitrary representations

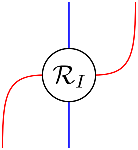
$$\mathcal{R}_I(z) = e^{\bar{\mathbf{a}}^{\dot{c}}_{c} J^{c}_{\dot{c}}} \cdot \mathcal{R}_{0,I}(z) \cdot e^{-\mathbf{a}^{c}_{\dot{c}} J^{\dot{c}}_{c}}$$

$$\mathcal{R}_{0,I}(z) = \rho_I(z) \prod_{k=1}^{|\bar{I}|} \Gamma(z - \tfrac{|\bar{I}|}{2} - \hat{\ell}^{\bar{I}}_k + 1)$$

$\hat{\ell}^{\bar{I}}_k$: operatorial shifted weights of the $\mathfrak{gl}(n)$-subalgebra $\mathfrak{gl}(\bar{I})$

ρ_I : normalization not fixed by YBE

- Diagrammatic expression

$\mathcal{R}_I(z) =$ 

Q-operators

- Q-operators are constructed as regularized traces over the oscillator space of the monodromy of $\mathcal{R}$-operators

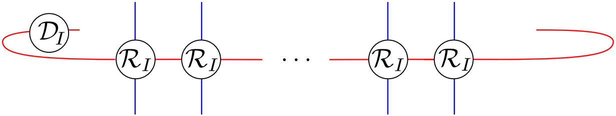
$$\mathbf{Q}_I(z) = e^{iz\phi_I}\,\widehat{\mathrm{Tr}}\Big\{\mathcal{D}_I\,\underbrace{\mathcal{R}_I(z)\otimes\ldots\otimes\mathcal{R}_I(z)}_{L}\Big\} \quad \text{with} \quad \phi_I = \textstyle\sum_{a\in I}\phi_a$$

- Regulator

$$\mathcal{D}_I = \exp\Big\{-i\textstyle\sum_{a,\dot{b}}\phi_{a\dot{b}}\,\mathbf{h}_{a\dot{b}}\Big\} \qquad \phi_{a\dot{b}} = \phi_a - \phi_{\dot{b}} \quad \mathbf{h}_{a\dot{b}} = \bar{\mathbf{a}}_a^{\dot{b}}\mathbf{a}_{\dot{b}}^a$$

- Diagrammatic form of the Q-operators

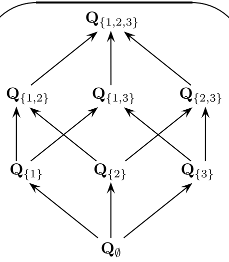
$\mathbf{Q}_I(z) =$



Properties of Q-operators

- Commuting family of operators

$$[\mathbf{Q}_I(z), \mathbf{Q}_J(z')] = 0$$



Hasse diagram $\mathfrak{gl}(3)$

- 2^n Q-operators
- Q-operators satisfy functional relations
- normalized Q-functions: $Q_I(z) = e^{iz\phi_I} \prod_{i=1}^{M} (z - z_i)$
- Bethe equations follow from a path on the Hasse diagram Tsuboi

What are we diagonalizing?

From Q-operators to the Hamiltonian

Alternative presentation of $\mathcal{R}$-operators

- Reordered $\mathcal{R}$-operators

$$\mathcal{R}_I(z) = e^{\bar{\mathbf{a}}_c^{\dot{c}} J_{\dot{c}}^c} \circ \tilde{\mathcal{R}}_{0,I}(z) \circ e^{-\mathbf{a}_{\dot{c}}^c J_c^{\dot{c}}}$$

$\circ$: opposite product in the oscillator space

- YBE: $\tilde{\mathcal{R}}_{0,I}(z)$ same defining relation as $\mathcal{R}_{0,\bar{I}}^{-1}(z+\frac{n}{2})$

$$\tilde{\mathcal{R}}_{0,I}(z) = \tilde{\rho}_I(z) \prod_{k=1}^{|I|} \frac{1}{\Gamma(z+\frac{|\bar{I}|}{2} - \hat{\ell}_k^I + 1)}$$

- Relative normalization can only be obtained directly

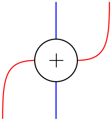
$$\mathcal{R}_{0,I}(z) \longrightarrow \sum_{k=0}^{\infty} \frac{1}{k!} (J_{\dot{c}}^c)^k \, \mathcal{R}_{0,I}(z) \, (J_c^{\dot{c}})^k$$

Diagrammatics

- Orderings of the $\mathcal{R}$-operators

$$\mathcal{R}_I(z) = e^{\bar{\mathbf{a}}^{\dot{c}}_{c} J^{c}_{\dot{c}}} \cdot \mathcal{R}_{0,I}(z) \cdot e^{-\mathbf{a}^{c}_{\dot{c}} J^{\dot{c}}_{c}} = e^{\bar{\mathbf{a}}^{\dot{c}}_{c} J^{c}_{\dot{c}}} \circ \tilde{\mathcal{R}}_{0,I}(z) \circ e^{-\mathbf{a}^{c}_{\dot{c}} J^{\dot{c}}_{c}}$$

- Toolbox of diagrams

$e^{\bar{\mathbf{a}}^{\dot{c}}_{c} J^{c}_{\dot{c}}} =$ 

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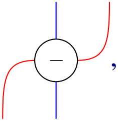
$e^{-\mathbf{a}^{c}_{\dot{c}} J^{\dot{c}}_{c}} =$ 

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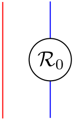
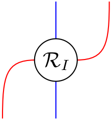
$\mathcal{R}_{0,I} =$ 

- Diagrammatic expression (bottom to top)



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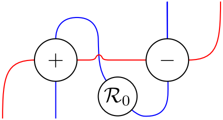

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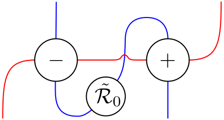


Reduction (I)

$\mathcal{R}$-operators at level $n-1$ in the fundamental representation

- Reordering causes a shift in z

$$\mathbf{L}_{n-1}(z) = \begin{pmatrix} (z-\frac{1}{2})\delta^a_b - \bar{\mathbf{a}}_b\mathbf{a}^a & \bar{\mathbf{a}}_b \\ -\mathbf{a}^a & 1 \end{pmatrix} = \begin{pmatrix} (z+\frac{1}{2})\delta^a_b - \mathbf{a}^a\bar{\mathbf{a}}_b & \bar{\mathbf{a}}_b \\ -\mathbf{a}^a & 1 \end{pmatrix}$$

- Two special points at $z = \pm\frac{1}{2}$

$$\mathbf{L}_{n-1}(+\frac{1}{2}) = \begin{pmatrix} \bar{\mathbf{a}}_b \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\mathbf{a}^a & 1 \end{pmatrix}, \qquad \mathbf{L}_{n-1}(-\frac{1}{2}) = \begin{pmatrix} \bar{\mathbf{a}}_b \\ 1 \end{pmatrix} \circ \begin{pmatrix} -\mathbf{a}^a & 1 \end{pmatrix}$$

Reduction (II)

- In general: $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ become projectors on a one dimensional subspace at $\hat{z}$ and $\check{z}$ for certain set I

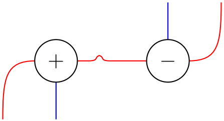
$$\mathcal{R}_I(\hat{z}) = e^{\bar{\mathbf{a}}_c^{\dot{c}} J_{\dot{c}}^c} \cdot |hws\rangle\langle hws| \cdot e^{-\mathbf{a}_{\dot{c}}^c J_c^{\dot{c}}}$$

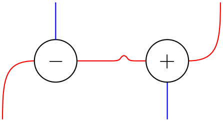
$$\mathcal{R}_I(\check{z}) = e^{\bar{\mathbf{a}}_c^{\dot{c}} J_{\dot{c}}^c} \circ |hws\rangle\langle hws| \circ e^{-\mathbf{a}_{\dot{c}}^c J_c^{\dot{c}}}$$

- Highest weight condition

$$J_a^{\dot{a}} |hws\rangle = 0, \quad J_b^a |hws\rangle = \lambda_I \delta_b^a |hws\rangle, \quad J_{\dot{b}}^{\dot{a}} |hws\rangle = \bar{\lambda}_I \delta_{\dot{b}}^{\dot{a}} |hws\rangle$$

$\mathcal{R}$-operators at the two special points

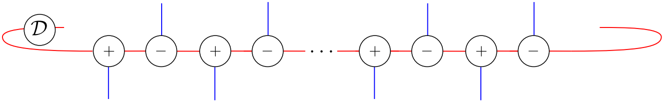
$\mathcal{R}_I(\hat{z}) =$ 

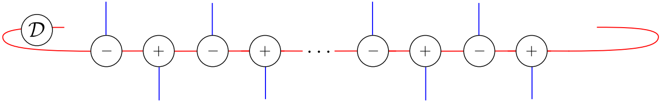
$\mathcal{R}_I(\check{z}) =$ 

- $\mathfrak{gl}(n)$ part decomposes into an outer product

Q-operators at the two special points

- Q-operators at the special points

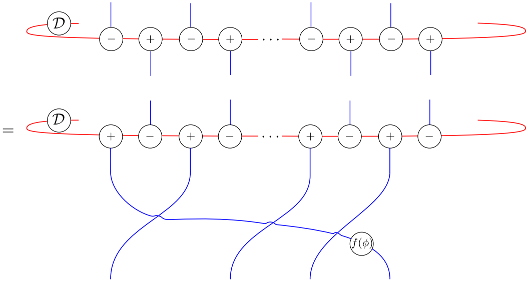
$\mathbf{Q}(\hat{z}) =$ 

$\mathbf{Q}(\check{z}) =$ 

Shift mechanism

- 'First local charge'

$$\mathbf{U} = f_1(\phi)\,\mathbf{P}_{1,2}\mathbf{P}_{2,3}\cdots\mathbf{P}_{L-1,L}$$

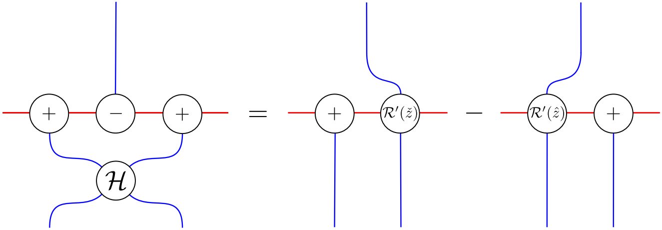


- Shift operator and eigenvalues in terms of Bethe roots

$$\mathbf{Q}(\check{z}) = \mathbf{U}\,\mathbf{Q}(\hat{z}) \quad \leftrightarrow \quad U(\{z_i\}) = e^{i(\check{z}-\hat{z})\phi_l} \prod_{i=1}^{M} \frac{\check{z}-z_i}{\hat{z}-z_i}$$

Hamiltonian

- Action of the Hamiltonian density



- $\mathbf{H} = \sum_{i=1}^{L} \mathcal{H}_{i,i+1}$ acts locally
- $\mathbf{H}$ belongs to the family of commuting operators
- Energy eigenvalues in terms of Bethe roots

$$\mathbf{HQ}(\check{z}) = \mathbf{Q}'(\check{z}) - \mathbf{U}\,\mathbf{Q}'(\hat{z}) \quad \leftrightarrow \quad E(\{z_i\}) = \sum_{i=1}^{M}\left(\frac{1}{\check{z}-z_i} - \frac{1}{\hat{z}-z_i}\right)$$

Conclusion and Outlook

- Q-ops provide an intuitive way to obtain local charges
- No reference to the *fundamental* transfer matrix

- Generalization to $\mathfrak{gl}(n|m)$
- Different representations and other integrable models
- Construct eigenvectors from Q-operators $\rightarrow$ Correlation functions

References

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