Surprises in the AdS algebraic curve constructions: Wilson loops and correlation functions

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Introduction

The algebraic curve shows up in the context of AdS/CFT correspondence in one of the steps of the spectral problem investigation — namely, the classification of the classical string solutions [KMMZ04]. Due to this notable role in the bigger picture, one of the questions that one could ask was whether and how a classification like this could be applied to a wider class of configurations. These include:

- Wilson loops, observables defined by contours in (boundary) space, corresponding via the AdS/CFT dictionary to minimal surfaces in the AdS bulk spanning these contours,
- correlation functions of local operators with other objects, be it another operators, Wilson loops or the like.

The surprises we have discovered are as follows:

- The algebraic curves arise in the case of Wilson loops, even though due to the lack of non-contractible loops the monodromy matrix and pseudomomenta are necessarily trivial, what renders the traditional algebraic curve analysis futile. An (almost) unambiguous construction of a Lax matrix that shares all the necessary properties of monodromy allows to circumvent this issue.
- Different solutions (and different algebraic curves) may arise for the correlators even if pseudomomenta are the same. This is also surprising from the point of view of the monodromy construction, and somewhat challenges the usual point of view that the space of solutions with given pseudomomentum is finite dimensional.
- The general properties of functions defined on the algebraic curves [BBT03] can be constraining enough to allow a complete reconstruction of a corresponding world-sheet configuration with minimal additional input. This procedure is shown to succeed in a handful of examples.

We note that our reasoning is local (starting from a PDE solution), while traditionally a knowledge of the whole world-sheet is required.

AdS_3 σ-model integrability

The world-sheet coordinates are encoded in a group element

$$g = \frac{1}{z}\begin{pmatrix} ix_1 + x_2 & 1 \\ -x_1^2 - x_2^2 - z^2 & ix_1 - x_2 \end{pmatrix}$$

given here for Euclidean AdS_3 in Poincaré patch. It is used to define currents

$$j = g^{-1}\partial g \qquad \bar{j} = g^{-1}\bar{\partial} g$$

that enter into the σ-model action (with world-sheet variables $w, \bar{w}$)

$$S = \frac{\sqrt{\lambda}}{4\pi}\int \operatorname{tr} j\bar{j}\, d^2 w$$

which yields the following equations of motion:

$$0 = \bar{\partial} j + \partial \bar{j} \qquad 0 = \bar{\partial} j - \partial \bar{j} + [j, \bar{j}]$$

Integrability of the model implies the existence of the following flat currents, or flat connection, depending on an arbitrary complex spectral parameter x:

$$J = j/(1-x) \qquad \bar{J} = \bar{j}/(1+x)$$

It encodes the equations of motion in the flatness condition

$$\partial \bar{J} - \bar{\partial} J + [J, \bar{J}] = 0 \qquad \forall x$$

For any loop C on the world-sheet a monodromy matrix is defined as

$$\Omega(w_0, \bar{w}_0; x) = P \exp \int_C J\, dw + \bar{J}\, d\bar{w}$$

The notation already takes into account that the monodromy for a given reference point $w_0, \bar{w}_0$ is in fact path-independent, as a consequence of the flatness of J.

A change in the reference point induces the following transformation:

$$\Omega(w_1, \bar{w}_1; x) = U\Omega(w_0, \bar{w}_0; x)U^{-1}$$

for some matrix U. The eigenvalues of Ω (in our 2×2 case usually expressed as $e^{\pm ip(x)}$) thus do not depend on $w, \bar{w}$. As they are functions of x, an infinite set of conserved charges can be seen here, ie. in the Taylor coefficients of pseudomomentum $p(x)$.

The algebraic curve traditionally is defined by the characteristic polynomial of the following matrix

$$L(w, \bar{w}; x) = -i\frac{\partial}{\partial x}\log \Omega(w, \bar{w}; x)$$

However, in the case of Wilson loops, where the minimal surfaces have no features and all contours C are contractible, $\Omega = UU^{-1} = I$, pseudomomenta are trivial and $L = 0$. The algebraic curve needs to be introduced in a completely different way.

It is important to note that both Ω and L solve the following system

$$\partial L + [J, L] = \bar{\partial} L + [\bar{J}, L] = 0$$

called Lax equation. Next we will learn how to construct different matrices that solve it as well, also in the case of trivial Ω.

Auxiliary linear problem

To this end, we introduce the auxiliary linear problem, a system of PDEs

$$\partial\Psi + J\Psi = 0 \qquad \bar{\partial}\Psi + \bar{J}\Psi = 0$$

for which the flatness of J is the compatibility condition. If its two independent solutions are arranged as columns of $\hat{\Psi}$, we can define

$$L = \hat{\Psi}(w, \bar{w}; x) \cdot A(x) \cdot \hat{\Psi}(w, \bar{w}; x)^{-1}$$

with A arbitrary and constant in $w, \bar{w}$. L solves the Lax equation.

It can be considered as a source for algebraic curve for this type of solutions. However, we need to resolve ambiguity still present in A. If we set $A(x) = f(x) \cdot \operatorname{diag}(1, -1)$, then the polynomial is just $y^2 = f(x)^2$. Lax matrices are usually taken to be rational in x; we will refine this condition and choose $f(x)$ so that L is polynomial in x.

Example: the null cusp

This contour is related to the real-world gluon scattering process. The minimal world-sheet coordinates are as follows [RoTs07]:

$$t = e^{-\sqrt{2}\sigma}\cosh\sqrt{2}\tau \qquad x = -e^{-\sqrt{2}\sigma}\sinh\sqrt{2}\tau \qquad z = \sqrt{2}e^{-\sqrt{2}\sigma}$$

and the emerging auxiliary problem leads to the following solutions:

$$\Psi_\pm(w, \bar{w}; x) = e^{\mp\frac{1-i}{2\sqrt{2}}\left(iw\sqrt{\frac{1+x}{1-x}} + \bar{w}\sqrt{\frac{1-x}{1+x}}\right)}\begin{pmatrix} \alpha \\ \alpha^{-1}(-ix \pm \sqrt{1-x^2}) \end{pmatrix}$$

with $\alpha = \exp\frac{1+i}{2\sqrt{2}}(-iw + \bar{w})$. Now, if we set $\hat{\Psi} = (\Psi_+; \Psi_-)$, a choice of $A(x) = \sqrt{1-x^2}\operatorname{diag}(1, -1)$ yields the following polynomial matrix

$$L = \hat{\Psi}A\hat{\Psi}^{-1} = \begin{pmatrix} ix & e^{\frac{1+i}{\sqrt{2}}(-iw+\bar{w})} \\ e^{-\frac{1+i}{\sqrt{2}}(-iw+\bar{w})} & -ix \end{pmatrix}$$

As advertised above, the algebraic curve emerging here is defined by

$$y^2 = 1 - x^2$$

Reconstruction

It turns out that the properties of analytic functions on algebraic curves are so rigid, that a program can be sketched to reconstruct the solutions Ψ without really solving the problem. Ordinarily it starts from $p(x)$ and Ω; the Lax equation for the latter implies that it has an eigenbasis common with $\partial + J, \bar{\partial} + \bar{J}$. The solutions can be then chosen from it:

$$\Omega\Psi_n(w, \bar{w}; x) = e^{ip(x)}\Psi_n(w, \bar{w}; x) = e^{ip(x)}\binom{1}{\psi(w, \bar{w}; x)}$$

with Ψ_n called the normalised eigenvector. ψ usually has genus $+ 1$ poles (in terms of x-dependence), one of which lies at $x = \infty$ and the others (dynamical) move when $w, \bar{w}$ are altered. The full solution also has a scalar pre-factor

$$\Psi(w, \bar{w}; x) = f_{BA}(w, \bar{w}; x)\Psi_n(w, \bar{w}; x)$$

It is a Baker-Akhiezer function, which means that it needs to satisfy a number of conditions: it should vanish at dynamical poles, behave in a prescribed singular manner at the special points $x = \pm 1$ and cancel the $w, \bar{w}$-dependence of Ψ_n at $x \to \infty$.

In the case of AdS_3 σ-model, the pre-factor can be inferred without Ω, namely from the following relations between $J, \bar{J}$ and L:

$$J = [P_+(L, x)]^-_{x=1} \qquad \bar{J} = [P_-(L, x)]^-_{x=-1}$$

($[\cdot]^-$ denotes singular part at the given point) for some polynomials $P_\pm$. Their form can be restricted here to $P_\pm(y, x) = \frac{c_\pm y}{1 \mp x}$ and finally

$$f_{BA} = e^{-\frac{c_+ y w}{1-x} - \frac{c_- y \bar{w}}{1+x}} \cdot \text{regular}$$

Constants $c_\pm$ can be redefined at will by a suitable diffeomorphism.

The 'physical' quantities can finally be recovered from the solutions:

$$j = -\partial\hat{\Psi} \cdot \hat{\Psi}^{-1}|_{x=0} \qquad \bar{j} = -\bar{\partial}\hat{\Psi} \cdot \hat{\Psi}^{-1}|_{x=0} \qquad g = \sqrt{\det\hat{\Psi}} \cdot \hat{\Psi}^{-1}|_{x=0}$$

Example: the null cusp again

The relevant algebraic curve is of genus 0, so no dynamical poles are expected. It admits the following uniformisation:

$$y = \frac{2t}{1+t^2} \qquad x = \frac{1-t^2}{1+t^2}$$

Two points lie above $x = \infty$ ($t = \pm i$) and L is diagonal there, so the asymptotic behaviours are distinct:

$$\Psi_n|_{x=\infty^+} = \binom{1}{0} \qquad \Psi_n|_{x=\infty^-} = \binom{1}{\infty}$$

At this point we can write (mixing variables x, y and t to emphasise the origin of respective parts; $c_\pm$ chosen deliberately)

$$\exp\left(-\frac{1+i}{2\sqrt{2}}\frac{y}{1-x}w - \frac{1-i}{2\sqrt{2}}\frac{y}{1+x}\bar{w}\right) b(w, \bar{w})\begin{pmatrix} 1 \\ a(w, \bar{w})\frac{t-i}{t+i} \end{pmatrix}$$

Functions a, b are subsequently fixed so that Ψ is constant at $x = \infty$:

$$b(w, \bar{w})^{-1} = a(w, \bar{w})b(w, \bar{w}) = \exp\left(+\frac{1+i}{2\sqrt{2}}iw - \frac{1-i}{2\sqrt{2}}i\bar{w}\right)$$

The reconstruction is now complete and the result is exactly the same as the straightforward solution of the auxiliary problem.

A trickier example: the $q\bar{q}$ potential

The contour is now an infinitely elongated rectangle. The minimal surface is now more complicated [Mald98]:

$$z = z_0 \operatorname{cn}\sigma \qquad x_1 \equiv t = z_0\tau/\sqrt{2} \qquad x_2 \equiv x = z_0 F(\sigma)/\sqrt{2}$$

with $F(\sigma) = 2E(\operatorname{am}\sigma|\frac{1}{2}) - \sigma$, the special functions being elliptic integrals and Jacobi functions. The auxiliary problem leads to a (bad-looking) solution that produces a polynomial Lax matrix in the familiar way for

$$A(x) = \sqrt{x}\sqrt{1-x^2}\operatorname{diag}(1, -1)$$

A curiosity that will reappear in the reconstruction is that in the $x \to \infty$ limit, L is non-diagonalisable (proportional to $\begin{pmatrix} 0 & 0 \\ x^2 & 0 \end{pmatrix}$).

The curve is now of genus 1, so a dynamical pole is to be expected:

$$y^2 = x(1-x^2)$$

It is conveniently uniformised using periodic products of theta functions on a lattice with quasi-periods $2(i)K(\frac{1}{2})$. f_{BA} written straightforwardly contains disallowed poles at $x = \infty$; removing these by hand destroys periodicity. It is then restored by a regular factor

$$\frac{\theta(z - \gamma(\sigma, \tau))}{\theta(z - \gamma(0, 0))} \quad \text{with} \quad \gamma(\sigma, \tau) = -i\sigma$$

where γ is the position of the dynamical pole.

A branch point at infinity allows only one asymptotic relation, therefore it is taken at the next-to-leading order as well. This allows to fix everything except a constant factor in the lower component; aside from this, the result matches the solution, albeit numerically (to hundreds of decimal places) — an analytical identity between expressions in terms of Jacobi and theta functions has not been investigated.

The trickiest example: correlation functions

In the cases of $\langle \operatorname{tr}\bar{Z}^J \operatorname{tr} Z^J\rangle$, $\langle W(C) \operatorname{tr} Z^J\rangle$ correlators (C being a circular contour), both being described by the pseudomomentum

$$p(x) = \frac{2\pi j x}{x^2 - 1} \quad \text{with} \quad j = \frac{J}{\sqrt{\lambda}}$$

it is noticeable that they correspond to different respective curves:

$$y^2 = 1 \quad \text{and} \quad y^2 = (1 + 2jx - x^2)^2$$

The former consists of two distinct sheets; the latter has degeneracies instead of branch cuts, seen as the double zeroes of the r.h.s.:

$$x = j \pm \sqrt{1 + j^2}$$

The reconstruction now involves a number of tricks and is possibly not unique. To avoid essential singularities in the usual form of f_{BA}, y is replaced by $y(1)$ and $y(-1)$ in the first and second term, respectively. Also, an additional relation comes from the requirement that the two solutions coincide at the points of degeneracy. Aside from that, the process follows the usual route and again the result is as expected.

Summary and outlook

We have managed to associate algebraic curves to AdS/CFT configurations for which this has not been done before, and to infer physical solutions from purely mathematical data. Some ideas that stem from here are as follows:

- The reconstruction procedure is now a barely trodden path, with a few caveats present; these could be investigated by applying the program to further cases of more complicated (ie. higher-genus) algebraic curves.
- The correlators are not only the trickiest, but also the most intriguing. Aside from resolving the concerns hinted at above, it is presently a mystery how do the three-point functions relate to the pseudomomenta and algebraic curves, three of which should be probably considered at once.
- One could try to extend the notion of Wilson loop monodromy (retaining its relation to algebraic curves) to contain some new quantity (eg. reflection matrices) for paths touching the boundary.

Selected references

A complete list of references is available in our paper (see header).

[BBT03] O. Babelon, D. Bernard, M. Talon
Introduction to Classical Integrable Systems
Cambridge University Press, 2003

[KMMZ04] V. Kazakov, A. Marshakov, J. Minahan, K. Zarembo
Classical/quantum integrability in AdS/CFT
JHEP **0405** (2004) 024, `arXiv:hep-th/0402207`

[Mald98] J. Maldacena
Wilson loops in large N field theories
Phys. Rev. Lett. **80** (1998) 4859, `arXiv:hep-th/9803002`

[RoTs07] R. Roiban, A. Tseytlin
Strong-coupling expansion of cusp anomaly from quantum superstring
JHEP **0711** (2007) 016, `arXiv:0709.0681 [hep-th]`

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