Yangian Democracy – *On the Equivalence of Different Realizations* –

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Motivation

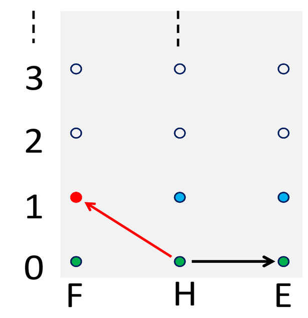
It is known that there are several realizations of Yangian. However the equivalences (or isomorphisms in precise) among them are not completely established. This is an attempt to give an equal suffrage for them to join the integrability business, especially in the AdS/CFT.

Realizations of Yangian

Minimal Realization (Min) [cf. Tolstoy(2002)]

In addition to the rank$\mathfrak{g}$-sets of Lie alg. generators (E_i, H_i, F_i), only one level-1 generator is sufficient to define the whole Yangian $Y(\mathfrak{g})$. For example, $\mathfrak{g} = \mathfrak{sl}(4)$ case,

$$\begin{pmatrix} H & E_1 & & \\ F_1 & H & E_2 & \\ & F_2 & H & E_3 \\ E_4 & & F_3 & H \end{pmatrix} \qquad \text{Fund Repr.} \quad E_4 \simeq u\big[[F_3, F_2], F_1\big]$$



The coproduct $\Delta(E_4)$ is non-local and not simple expression.

For $\mathfrak{sl}(4)$ case, the isomorphism from Min to D1 is given by

$$E_4 = [[\widehat{F}_3, F_2], F_1]$$

The Lie algebra generators are mapped to themselves.

Drinfeld's First Realization (D1) [Drinfeld(1985)]

The Yangian $Y(\mathfrak{g})$ associated with a Lie algebra $\mathfrak{g}$ is generated by the level-0 Lie algebraic generators J^A and the level-1 Yangian generators $\widehat{J}^A$ with $A = 1, \cdots, \dim\mathfrak{g}$, and satisfying the following relations,

$$[J^A, J^B] = J^C f_C{}^{AB} \qquad [J^A, \widehat{J}^B] = \widehat{J}^C f_C{}^{AB} \qquad [J^{(A}, [J^B, J^{C)}]] = 0$$

$$[\widehat{J}^A, [J^B, \widehat{J}^C]] - [J^A, [\widehat{J}^B, \widehat{J}^C]] = \frac{\hbar^2}{24} f_L{}^{AI} f_M{}^{BJ} f_N{}^{CK} f_{IJK} \{J^L, J^M, J^N\}$$

The unfriendly RHS of Serre relation (last one) is required for the compatibility of coproduct,

$$\Delta(\widehat{J}^A) = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \frac{\hbar}{2} f^A{}_{BC} J^B \otimes J^C$$

The generators $\widehat{J}^A$ in D1 is corresponding to D2 generators $(x^+_{i,1}, x^-_{i,1}, h_{i,1})$ up to quadratic Lie alg. terms. For $Y(\mathfrak{sl}(4))$ case,

$$\widehat{E}_i = x^+_{i,1} - \frac{1}{4}\Big(\{H_i, E_i\} + \sum_{\beta=1,2,3} \{F_\beta, [E_\beta, E_i]\}\Big)$$

and similarly for $\widehat{F}_i$'s and $\widehat{H}_i$'s.

Drinfeld's Second Realization (D2) [Drinfeld(1988)]

In 1988, Drinfeld proposed a *new* realization which is more suitable for the representation theory. This formulation includes all levels of simple root generators,

$$x^+_{i,r}, \quad x^-_{i,r}, \quad h_{i,r} \qquad \text{with} \quad r = 0, 1, 2, \cdots \quad \text{and} \quad i = 1, \cdots, \text{rank}\mathfrak{g}$$

The defining relations are not cubic as D1 but quadratic at most, and it looks like a natural *quantization* of the loop algebra $U(\mathfrak{g})[[u]]$ with the identification $x^\pm_{i,r} \simeq u^r x^\pm_i$.

The relation between D2 and RTT is explained by Gauss decomposition of $T(u)$ [Brundan-Kleshchev(2004)]. For the simplest $Y(\mathfrak{sl}(2))$ case,

$$T(u) = \begin{pmatrix} 1 & 0 \\ f(u) & 1 \end{pmatrix} \begin{pmatrix} d_1(u) & 0 \\ 0 & d_2(u) \end{pmatrix} \begin{pmatrix} 1 & e(u) \\ 0 & 1 \end{pmatrix}$$

and it is related to the D2 currents as

$$h(u) = d_1(u+\tfrac{1}{2})^{-1} d_2(u+\tfrac{1}{2}), \quad x^+(u) = e(u+\tfrac{1}{2}), \quad x^-(u) = f(u+\tfrac{1}{2})$$

$$\text{with} \quad h(u) = 1 + \sum_{r\geq 0} h_{1,r} u^{-r-1}, \quad x^\pm(u) = \sum_{r\geq 0} x^\pm_{1,r} u^{-r-1}$$

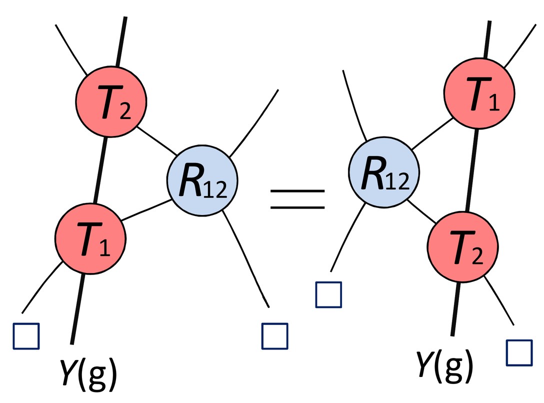
RTT Realization (RTT) [Faddeev's school (1980s)]

Yangian is also defined by the YBE associated with a rational R-matrix. In fact, plugging *Yang's R-matrix* $R_{12}(u) = 1 - P_{12}u^{-1}$ with the following RTT relations, we obtain the defining relations of $Y(\mathfrak{gl}(N))$:

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v)$$

$$T_1(u) = \sum_{i,j=1}^N t_{ij}(u) \otimes e_{ij} \otimes 1, \quad T_2(u) = \sum_{i,j=1}^N t_{ij}(u) \otimes 1 \otimes e_{ij}$$

$$t_{ij}(u) = \delta_{ij} + t^{(1)}_{ij} u^{-1} + t^{(2)}_{ij} u^{-2} + \cdots \in Y(\mathfrak{g})[[u^{-1}]]$$



where e_{ij} is a $N \times N$ matrix units. The coproducts are nicely expressed as

$$\Delta(t_{ij}(u)) = \sum_{k=1}^N t_{ik}(u) \otimes t_{kj}(u) \ .$$

Comparison

Which is the *best* formulation?

	Higher generators	Good	*Not* so good
Min	One level-1 $\widehat{E}_n$	Minimal set	
D1	All level-1 $\widehat{J}^A$		cubic Serre relations
L	Simple roots of Level-1	*Bridge* ↓	
D2	Simple roots of All levels	Repr. theory	lots of relations
RTT	Everything	nice coproducts	top down

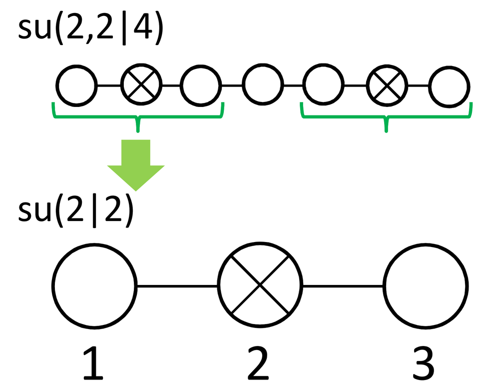
RTT is a sophisticated formulation but the relations to the others are not so obvious.
D1/Min is bottom-up construction and easy to treat, but not suitable for repr. theory.
L would fill the gap between D1/Min(bottom-up) and RTT(top-down).

Application for AdS/CFT Yangian

Centrally Extended Lie Algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ [Beisert(2005,2007)]

It was shown that the magnon scattering on the infinitely extended world-sheet in $AdS_5 \times S^5$ target space is described by two copies of $\mathfrak{su}(2|2)$ algebra.

Furthermore, one of the central charges C of the extended $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ algebra ($a, b = 1, 2$ and $\alpha, \beta = 3, 4$)



$$\left(\begin{array}{c|c} R^a{}_b & Q^\alpha{}_b \\ \hline S^a{}_\beta & L^\alpha{}_\beta \end{array}\right) \ltimes \{C, P, K\} \qquad \begin{array}{l} \{Q^\alpha{}_a, Q^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab} P \\ \{S^a{}_\alpha, S^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta} K \end{array}$$

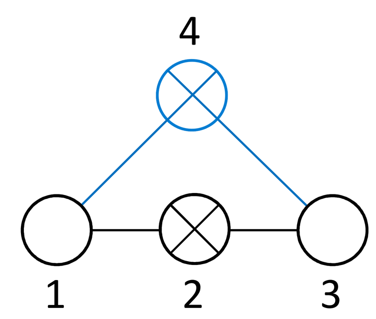
gives the all-loop magnon dispersion relation. Here R, L are the $\mathfrak{su}(2)$ generators and Q, S are the supercharges.
In addition, the S-matrix has bonus Yangian symmetries $Y(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$!

Minimal Realization of AdS/CFT Yangian [Molev-TM]

The Yangian $Y(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$ is generated by the standard Chevalley-Serre basis (E_i, H_i, F_i) with associated O-X-O diagram and one Yangian generator $E_4 = \widehat{S}^1{}_3$. The non-trivial relations are

$$\{E_4, F_2\} = \widehat{K} \cong i\alpha^{-1}(1 + U^{-2}) C$$

$$\{E_4, E_4\} = \frac{\hbar^2}{12}\big(\{F_{321}, F_3, F_{21}\} - \{F_{321}, F_{32}, F_1\} + \{F_1, F_3, K\}\big)$$



where $F_{ij} = [F_i, F_j]$, $F_{ijk} = \big[[F_i, F_j], F_k\big]$ and $\hbar = 1/ig$.

It is hard to construct D2 from Min (or D1) in direct. For this purpose, the following realization interpolate these realizations with the map,

$$\{E_4, E_{321}\} = -\big(\widehat{H}_1 + \widehat{H}_2 + \widehat{H}_3\big)$$

$$h'_{1,1} = \widehat{H}_1 - \frac{\hbar}{4}\big(2\{R^1{}_2, R^2{}_1\} + \{S^2{}_\mu, Q^\mu{}_2\} - \{S^1{}_\mu, Q^\mu{}_1\}\big).$$

Levendorskii's Realization (L) of AdS/CFT Yangian [Molev-TM]

This realization is similar for D2, but only includes level-0 and -1 as well as Min (or D1). In particular, the generator $h'_{i,1}$ plays a role of *the Boost Operator* (see $*$).

$$[h_{i,0}, h_{j,0}] = [h'_{i,1}, h_{j,0}] = [h'_{i,1}, h'_{j,1}] = 0$$

$$[h'_{i,1}, x^\pm_{j,0}] = \pm DA_{ij} x^\pm_{j,1} \qquad *$$

$$[x^+_{i,1}, x^-_{j,0}] = \delta_{ij} D_{ii} h_{i,1} \equiv \delta_{ij} D_{ii}(h'_{i,1} + \tfrac{\hbar}{2} h^2_{i,0})$$

$$[x^\pm_{i,1}, x^\pm_{j,0}] - [x^\pm_{i,0}, x^\pm_{j,1}] = \pm\frac{\hbar}{2} DA_{ij}\{x^\pm_{i,0}, x^\pm_{j,0}\}$$

$$[h'_{i,1}, [x^+_{j,1}, x^-_{j,1}]] = 0$$

$$[[x^\pm_{1,1}, x^\pm_{2,0}], [x^\pm_{3,0}, x^\pm_{2,0}]] = P^\pm_1 \qquad \text{: level-1 centers}$$

$$DA_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix} \qquad D = \text{diag}(+1, -1, -1)$$

Now we are ready to construct D2 with the use of the boost operator $h'_{i,1}$. The higher charges are inductively constructed by,

$$x^\pm_{i,r+1} \equiv \pm DA^{-1}_{ii}[h'_{i,1}, x^\pm_{i,r}] \qquad \text{for} \qquad i = 1, 3$$

$$x^\pm_{2,r+1} \equiv \pm DA^{-1}_{12}[h'_{1,1}, x^\pm_{2,r}] \qquad h_{i,r} \equiv D_{ii}[x^+_{i,r}, x^-_{i,0}].$$

D2 of AdS/CFT Yangian [Spill-Torrielli(2008), Molev-TM]

After elementary but technical induction, we would obtain the following set of D2 realization of $Y(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$.

$$[h_{i,r}, h_{j,s}] = 0 \quad [x^+_{i,r}, x^-_{j,s}] = \delta_{ij} D^{-1}_{ii} h_{i,r+s} \quad [h_{i,0}, x^\pm_{j,r}] = \pm DA_{ij} x^\pm_{j,r}$$

$$[h_{i,r+1}, x^\pm_{j,s}] - [h_{i,r}, x^\pm_{j,s+1}] = \pm\tfrac{\hbar}{2} DA_{ij}\{h_{i,r}, x^\pm_{j,s}\}$$

$$[x^\pm_{i,r+1}, x^\pm_{j,s}] - [x^\pm_{i,r}, x^\pm_{j,s+1}] = \pm\tfrac{\hbar}{2} DA_{ij}\{x^\pm_{i,r}, x^\pm_{j,s}\}$$

$$[x^\pm_{2,r}, x^\pm_{2,s}] = [x^\pm_{i,r}, x^\pm_{j,s}] = 0 \qquad \text{for} \qquad i + j = 4$$

$$[x^\pm_{j,r}, [x^\pm_{j,s}, x^\pm_{2,t}]] + [x^\pm_{j,s}, [x^\pm_{j,r}, x^\pm_{2,t}]] = 0 \qquad \text{for} \qquad j = 1, 3$$

$$[[x^\pm_{1,r}, x^\pm_{2,0}], [x^\pm_{3,s}, x^\pm_{2,0}]] = P^\pm_{r+s} \qquad \text{: infinitely many central charges!}$$

D2 realization associated with ⊗–⊗–⊗ basis was proposed by Spill-Torrielli(2008).

- The relation to RTT formulation is still open problem.
- Need to include *Secret Symmetry* ! (→ Alessandro's talk.)

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