BACK TO THE q-DEFORMATIONS

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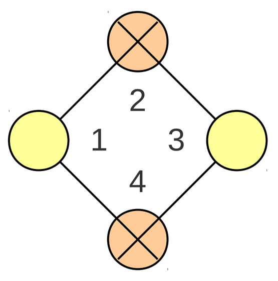
Quantum deformations

Algebra $\widehat{\mathcal{Q}}$ [1] is an affine extension of the q-deformed $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$ [2], and is a symmetry of the deformed Hubbard chain and of the q-deformed worldsheet scattering in AdS/CFT.



$$DA=\begin{pmatrix}+2&-1&0&-1\\-1&0&+1&0\\0&+1&-2&+1\\-1&0&+1&0\end{pmatrix}$$

$$D=\mathrm{diag}(1,-1,-1,-1)$$

It is parametrized by a coupling constant g and a deformation parameter $U=e^{ip}$, where p is momentum. The intertwining equation

$$\Delta^{op}(J)\,S-S\,\Delta(J)=0,\quad \forall J\in\widehat{\mathcal{Q}},\tag{1}$$

defines the bound state S-matrix uniquely up to an overall dressing phase, and it satisfies the Yang-Baxter equation.

Our goal is to find quantum affine symmetries of the q-deformed boundaries in AdS/CFT [3].

Hopf Algebra

Commutation relations ($i,j=1\dots4$):

$$K_iE_j=q^{+DA_{ij}}E_jK_i,\quad K_iF_j=q^{-DA_{ij}}F_jK_i,$$
$$[E_j,F_j\}=D_{jj}\frac{K_j-K_j^{-1}}{q-q^{-1}},\quad [E_i,F_j\}_{\substack{i\neq j\\ i+j\neq 6}}=0,$$
$$\{E_2,F_4\}=-\tilde{g}\tilde{\alpha}^{-1}(K_4-U^2K_2^{-1}),$$
$$\{E_4,F_2\}=+\tilde{g}\tilde{\alpha}(K_2-U^{-2}K_4^{-1}).\tag{2}$$

Coproducts (here $[[1]]=[[3]]=0,\ [[2]]=-[[4]]=1$):

$$\Delta(E_j)=E_j\otimes1+K_j^{-1}U^{[[j]]}\otimes E_j,$$
$$\Delta(F_j)=F_j\otimes K_j+U^{-[[j]]}\otimes F_j,\tag{3}$$

and $\Delta(X)=X\otimes X$ for U,V and K_i.

This algebra has three central elements:

$$C_1=K_1K_2^2K_3,\quad C_2=\{[E_2,E_1]_{1/q},[E_2,E_3]_q\},$$
$$C_3=\{[F_2,F_1]_{1/q},[F_2,F_3]_q\},\tag{4}$$

satisfying $\Delta(C_i)=\Delta^{op}(C_i)$, and $C_1=V^{-2}$.

Representation

A q-oscillator (supersymmetric-short) representation:

$$\begin{aligned}
E_1&=\mathrm{a}_2^\dagger\mathrm{a}_1, & F_1&=\mathrm{a}_1^\dagger\mathrm{a}_2,\\
E_2&=a\,\mathrm{a}_4^\dagger\mathrm{a}_2+b\,\mathrm{a}_1^\dagger\mathrm{a}_3 & F_2&=c\,\mathrm{a}_3^\dagger\mathrm{a}_1+d\,\mathrm{a}_2^\dagger\mathrm{a}_4,\\
E_3&=\mathrm{a}_3^\dagger\mathrm{a}_4, & F_3&=\mathrm{a}_4^\dagger\mathrm{a}_3,\\
E_2&=\tilde{a}\,\mathrm{a}_4^\dagger\mathrm{a}_2+\tilde{b}\,\mathrm{a}_1^\dagger\mathrm{a}_3 & F_2&=\tilde{c}\,\mathrm{a}_3^\dagger\mathrm{a}_1+\tilde{d}\,\mathrm{a}_2^\dagger\mathrm{a}_4.
\end{aligned}\tag{5}$$

The representation labels are:

$$a=g_M\gamma,\qquad b=\frac{g_M\alpha}{\gamma}\frac{x^--x^+}{x^-},$$
$$c=\frac{g_M\gamma}{\alpha V}\frac{iq^{\frac{M}{2}}\tilde{g}}{g(x^++\xi)},\quad d=\frac{g_M\tilde{g}q^{\frac{M}{2}}}{ig\gamma V^{-1}}\frac{x^+-x^-}{\xi x^++1},\tag{6}$$

and satisfy $ad-bc=1$; here $g_M:=\sqrt{\frac{g}{[M]_q}}$, and

$$V^2=\frac{1}{q^M}\frac{\xi x^++1}{\xi x^-+1}=q^M\frac{x^+}{x^-}\frac{x^-+\xi}{x^++\xi},\quad U^2=\frac{x^+}{x^-}V^{-2}.$$

The affine labels $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ are acquired by the rule:

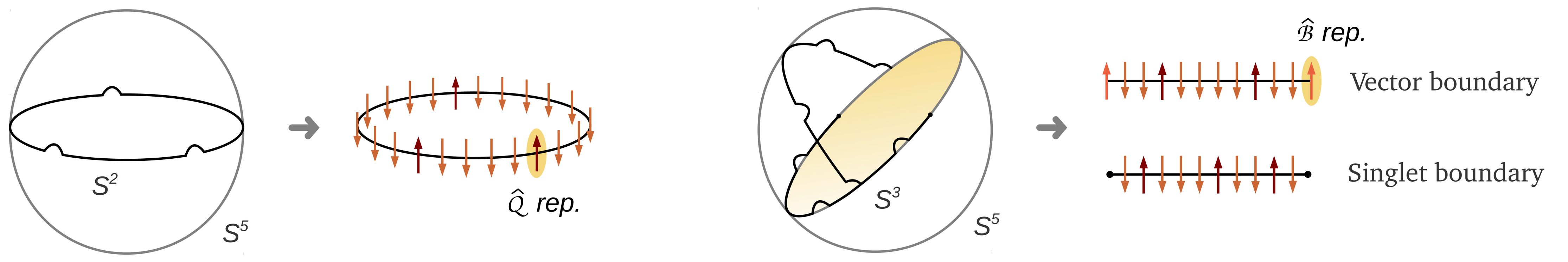
$$\gamma\to\frac{i\tilde{\alpha}\gamma}{x^+},\quad \alpha\to\alpha\,\tilde{\alpha}^2,\quad x^\pm\to\frac{1}{x^\pm}.\tag{7}$$

References

[1] N. Beisert, W. Galleas and T. M., *A Quantum Affine Algebra for the Deformed Hubbard Chain*, preprint, (2011).
[2] N. Beisert and P. Koroteev, *Quantum Deformations of the One-Dimensional Hubbard Model*, J. Phys. A 41 (2008).
[3] R. Murgan and R. Nepomechie, *q-deformed su(2|2) boundary S-matrices via the ZF algebra*, JHEP 0806 (2008).

Spin chains and boundary scattering in AdS/CFT

Certain AdS/CFT superstrings with infinite light-cone momentum are dual to quantum spin-chains. Quantum affine algebras give an elegant and uniform approach to bulk and boundary scattering for such systems.

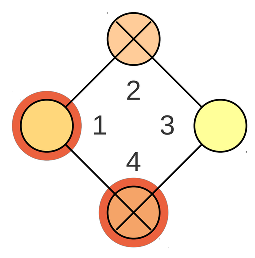


Boundary symmetries are encoded in coideal subalgebras $\widehat{\mathcal{B}}\subset\widehat{\mathcal{Q}}$ satisfying $\Delta\widehat{\mathcal{B}}\in\widehat{\mathcal{Q}}\otimes\widehat{\mathcal{B}}$. The boundary intertwining equation,

$$\Delta^{ref}(J)\,K-K\,\Delta(J)=0,\quad \forall J\in\widehat{\mathcal{B}},\tag{8}$$

defines the bound state K-matrix uniquely up to an overall dressing phase, and it satisfies the boundary Yang-Baxter equation. Each boundary in AdS/CFT has a unique boundary algebra which we call $\widehat{\mathcal{B}}_Y$, $\widehat{\mathcal{B}}_Z$ and $\widehat{\mathcal{B}}_X$.

q-deformed Y=0 giant graviton



Algebra. The $Y=0$ giant graviton wraps a maximal S^3 of $AdS_5\times S^5$ given by $X^2+Z^2=R^2$. It does not respect Dynkin nodes 1 and 4 of $\widehat{\mathcal{Q}}$, and is a singlet w.r.t. the boundary algebra. The boundary Lie algebra is $\mathcal{M}_Y=\{E_2,F_2,E_3,F_3\}$. This setup induces a root-space involution

$$\Theta_Y(\alpha_2)=\alpha_2,\quad \Theta_Y(\alpha_3)=\alpha_3,\quad \Theta_Y(\alpha_1)=-\alpha_2-\alpha_3-\alpha_4,\quad \Theta_Y(\alpha_4)=-\alpha_1-\alpha_2-\alpha_3,$$

defining the quantum affine coideal subalgebra $\widehat{\mathcal{B}}_Y$ generated by $\mathcal{M}_Y$, Cartan subalgebra $\mathcal{T}$, and the twisted affine generators:

$$\begin{aligned}
\widetilde{E}_{321}&=F_4-d_y\,\tilde{\theta}(F_4), & \widetilde{E}_{21}&=(\mathrm{ad}_rF_3)\widetilde{E}_{321}, & \widetilde{E}_1&=(\mathrm{ad}_rF_2F_3)\widetilde{E}_{321}, & \widetilde{C}_2&=(\mathrm{ad}_rE_2)\widetilde{E}_{321},\\
\widetilde{F}_{321}&=E_4'-d_x\,\tilde{\theta}(E_4'), & \widetilde{F}_{21}&=(\mathrm{ad}_rE_3)\widetilde{F}_{321}, & \widetilde{F}_1&=(\mathrm{ad}_rE_2E_3)\widetilde{F}_{321} & \widetilde{C}_3&=(\mathrm{ad}_rF_2)\widetilde{F}_{321},
\end{aligned}\tag{9}$$

where $\tilde{\theta}(F_4)=(\mathrm{ad}_rE_3E_2)E_1'$, $\tilde{\theta}(E_4')=(\mathrm{ad}_rF_3F_2)F_1$, and $E_i'=E_iK_i$. The right adjoint action is given by

$$(\mathrm{ad}_rE_i)A=(-1)^{[A][E_i]}K_iAE_i-K_iE_iA,\quad (\mathrm{ad}_rF_i)A=(-1)^{[A][F_i]}AF_i-F_iK_i^{-1}AK_i.\tag{10}$$

Reflection. There exists a reflection map $\kappa:\widehat{\mathcal{Q}}\to\widehat{\mathcal{Q}}^{ref}$ defined by

$$\kappa:(E_j,F_j,K_j,U,V)\mapsto(\underline{E}_j,\underline{F}_j,\underline{K}_j,\underline{U},\underline{V}),\quad\text{such that}\quad \underline{U}=U^{-1},\quad \underline{V}=V,\quad \underline{K}_i=K_i.\tag{11}$$

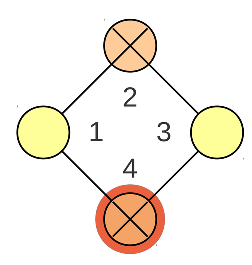
Set $\Delta^{ref}:=(\kappa\otimes id)\circ\Delta$. Then:

$$\Delta^{ref}(E_j)=\underline{E}_j\otimes1+K_j^{-1}U^{-[[j]]}\otimes E_j,\quad \Delta^{ref}(F_j)=\underline{F}_j\otimes K_j+U^{+[[j]]}\otimes F_j.\tag{12}$$

The reflected labels $\underline{a},\underline{b},\underline{c},\underline{d}$ are obtained from (6) by the map $x^\pm\mapsto-\frac{x^\mp+\xi}{\xi x^\mp+1}$.

Finally, coreflectivity $\Delta^{ref}(\widetilde{C}_i)=\Delta(\widetilde{C}_i)$ of (9) constrains $d_y=-\frac{\tilde{g}}{g\alpha\tilde{\alpha}}$ and $d_x=\alpha\tilde{\alpha}\frac{\tilde{g}}{g}$.

q-deformed Z=0 giant graviton



Algebra. The $Z=0$ giant graviton is defined by $X^2+Y^2=R^2$ and respects all but affine Dynkin node of $\widehat{\mathcal{Q}}$, and is a vector w.r.t. to the boundary algebra. Thus $\mathcal{M}_Z=\{E_1,F_1,E_2,F_2,E_3,F_3\}$, giving:

$$\Theta_Z(\alpha_i)=\alpha_i\quad\text{for}\quad i=1,2,3,\qquad\text{and}\qquad \Theta_Z(\alpha_4)=-\alpha_4-2\alpha_3-2\alpha_2-2\alpha_1.$$

The affine part of the boundary algebra $\widehat{\mathcal{B}}_Z$ is generated by the twisted affine generators:

$$\begin{aligned}
\widetilde{E}_{312}&=F_4-d_y\,\tilde{\theta}(F_4), & \widetilde{E}_{12}&=(\mathrm{ad}_rF_3)\widetilde{E}_{312}, & \widetilde{E}_{32}&=(\mathrm{ad}_rF_1)\widetilde{E}_{312}, & \widetilde{C}_2&=(\mathrm{ad}_rE_2)\widetilde{E}_{312},\\
\widetilde{F}_{312}&=E_4'-d_x\,\tilde{\theta}(E_4'), & \widetilde{F}_{12}&=(\mathrm{ad}_rE_3)\widetilde{F}_{312}, & \widetilde{F}_{32}&=(\mathrm{ad}_rE_1)\widetilde{F}_{312}, & \widetilde{C}_3&=(\mathrm{ad}_rF_2)\widetilde{F}_{312},
\end{aligned}\tag{13}$$

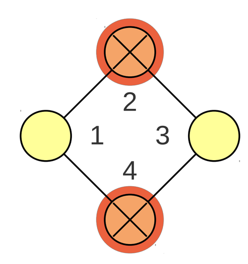
where $\tilde{\theta}(F_4)=(\mathrm{ad}_rE_1E_3E_2E_3E_2E_1)E_4'$ and $\tilde{\theta}(E_4')=(\mathrm{ad}_rF_1F_3F_2F_3F_2F_1)F_4$.

Boundary representation. Commutation relations (2) and coreflectivity of C_i in (4) and $\widetilde{C}_i$ in (13) constrains boundary labels to be:

$$a_B=g_M\gamma_B,\quad b_B=\frac{g_M\alpha}{\gamma_B},\quad c_B=\frac{g_M\gamma_B}{\alpha}\frac{i\tilde{g}}{g}\frac{q^{M/2}}{V_B(x_B+\xi)},\quad d_B=\frac{g_M\tilde{g}}{ig\gamma_B}\frac{V_Bq^{M/2}(x_B+\xi)}{\xi x_B+1},$$
$$\tilde{a}_B=\frac{ig_M\gamma_B\tilde{\alpha}}{x_B},\quad \tilde{b}_B=\frac{g_M\alpha\tilde{\alpha}}{i\gamma_B}(x_B+2\xi),\quad \tilde{c}_B=-\frac{g_M\tilde{g}\,q^{\frac{M}{2}}\gamma_B}{g\alpha\tilde{\alpha}(1+\xi x_B)\widetilde{V}_B},\quad \tilde{d}_B=\frac{g_M\tilde{g}\,q^{-\frac{M}{2}}}{g\tilde{\alpha}\gamma_B\widetilde{V}_B}\frac{1-\xi(x_B+2\xi)}{\xi^2-1},\tag{14}$$

where $V_B^2=q^M\frac{x_B}{x_B+\xi}=q^{-M}\frac{1+\xi x_B}{1-\xi^2}$, $V_B^2\widetilde{V}_B^2=1+\frac{\xi^2}{\xi^2-1}$, and gives $d_y=(\alpha\tilde{\alpha})^{-2}$, $d_x=-(\alpha\tilde{\alpha})^2$.

q-deformed "left" D7-brane



Algebra. The left factor of the $Z=0$ D7-brane is non-supersymmetric and is a singlet. The boundary Lie algebra is $\mathcal{M}_X=\{E_1,F_1,E_3,F_3\}$, giving:

$$\Theta_X(\alpha_1)=\alpha_1,\quad \Theta_X(\alpha_3)=\alpha_3,\quad \Theta_X(\alpha_2)=-\alpha_4-\alpha_3-\alpha_1,\quad \Theta_X(\alpha_4)=-\alpha_2-\alpha_3-\alpha_1.$$

The affine part of the boundary algebra $\widehat{\mathcal{B}}_X$ is generated by:

$$\begin{aligned}
\widetilde{E}_{312}&=F_4-d_y\,\tilde{\theta}(F_4), & \widetilde{E}_{12}&=(\mathrm{ad}_rF_3)\widetilde{E}_{312}, & \widetilde{F}_{32}&=(\mathrm{ad}_rE_1)\widetilde{F}_{312}, & \widetilde{\widetilde{C}}_2&=\{\widetilde{E}_{12},\widetilde{E}_{32}\},\\
\widetilde{F}_{312}&=E_4'-d_x\,\tilde{\theta}(E_4'), & \widetilde{E}_{32}&=(\mathrm{ad}_rF_1)\widetilde{E}_{312}, & \widetilde{F}_{12}&=(\mathrm{ad}_rE_3)\widetilde{F}_{312}, & \widetilde{\widetilde{C}}_3&=\{\widetilde{F}_{12},\widetilde{F}_{32}\},
\end{aligned}\tag{15}$$

where $\tilde{\theta}(F_4)=(\mathrm{ad}_rE_3E_1)E_2'$ and $\tilde{\theta}(E_4')=(\mathrm{ad}_rF_3F_1)F_2$. Coreflectivity of $\widetilde{\widetilde{C}}_i$ gives

$$d_y=\frac{\tilde{g}}{g\,\alpha\tilde{\alpha}}V_B'\quad\text{and}\quad d_x=-\frac{g\,\alpha\tilde{\alpha}}{\tilde{g}}V_B'(1-\xi^2),\quad\text{where}\quad V_B'=q\frac{1-\xi x_B'}{1-\xi^2}=q^{-1}\frac{x_B'}{x_B'-\xi}.\tag{16}$$