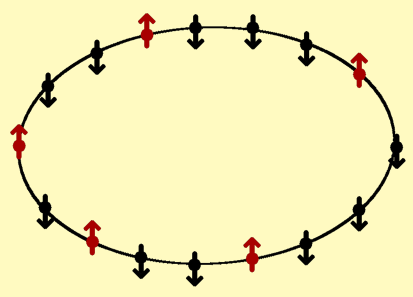
Exceptional operators in $\mathcal{N}=4$ super Yang-Mills.

Alessandro Sfondrini, with Gleb Arutyunov & Sergey Frolov. ArXiv:1205.6660.

Exceptional operators at one loop: XXX spin chain

Planar spectrum of $\mathcal{N}=4$ SYM at one loop is given by XXX spin chain:

$$\mathcal{O}=\mathrm{Tr}[ZXZZXZZZXZZZXZZX]$$



Energies can be found by **Bethe Ansatz** for magnon rapidity and **dispersion relations**:

$$1=e^{ip(u_k)\,L}\prod_{j\neq k}^{M}\frac{u_k-u_j-2i}{u_k-u_j+2i}\,,\qquad 1=\prod_j\frac{u_j+i}{u_j-i}\,,\qquad E=L+g^2\sum_{k=1}^{M}\frac{2}{1+u_k^2}\,,$$

where we also imposed cyclicity of the trace. For $M=3$ (three-magnons), $L=6,8,10,\dots$ there are **exceptional solutions** to this equation, where $p(u_k)$ and $S(u_k,u_j)$ **become singular**:

$$u_1=0\,,\quad u_2=-i\,,\quad u_3=i\,,\qquad \mathcal{O}_L=\sum_{j=1}^{L-4}(-1)^j\,\mathrm{tr}\Big(XX\,Z^jX\,Z^{L-j-3}\Big)\,,\qquad E_L=L+3g^2\,.$$

These are **truly in the spectrum** (eigenvalues of the permutation operator!). Simplest case:

$$L=6:\qquad \mathcal{O}_6=\mathrm{tr}\big(XXZXZZ-XXZZXZ\big)\,.$$

Interpretation: **infinitely tight bound states**, with momentum

$$p_1=\pi\,,\qquad p_2=-\pi/2+i\,\infty\,,\qquad p_3=-\pi/2-i\,\infty\,.$$

Several cross-checks: diagonalization, Baxter T-Q relation, and **regularization by a twist** ϕ:

$$1=e^{-i\phi}\left(\frac{u_k+i}{u_k-i}\right)^L\prod_{j\neq k}^{M}\frac{u_k-u_j-2i}{u_k-u_j+2i}\,,\qquad \prod_{j=1}^{M}e^{i\,p_j}=e^{iM\phi/L}\,,$$

which is a Leigh-Strassler deformation, and regularizes the solution

$$u_1\sim\phi\,,\quad u_2\sim -i-\phi-i\,\phi^L\,,\quad u_3\sim +i-\phi+i\,\phi^L\,,\qquad E_L=L+3g^2+O(\phi)\,.$$

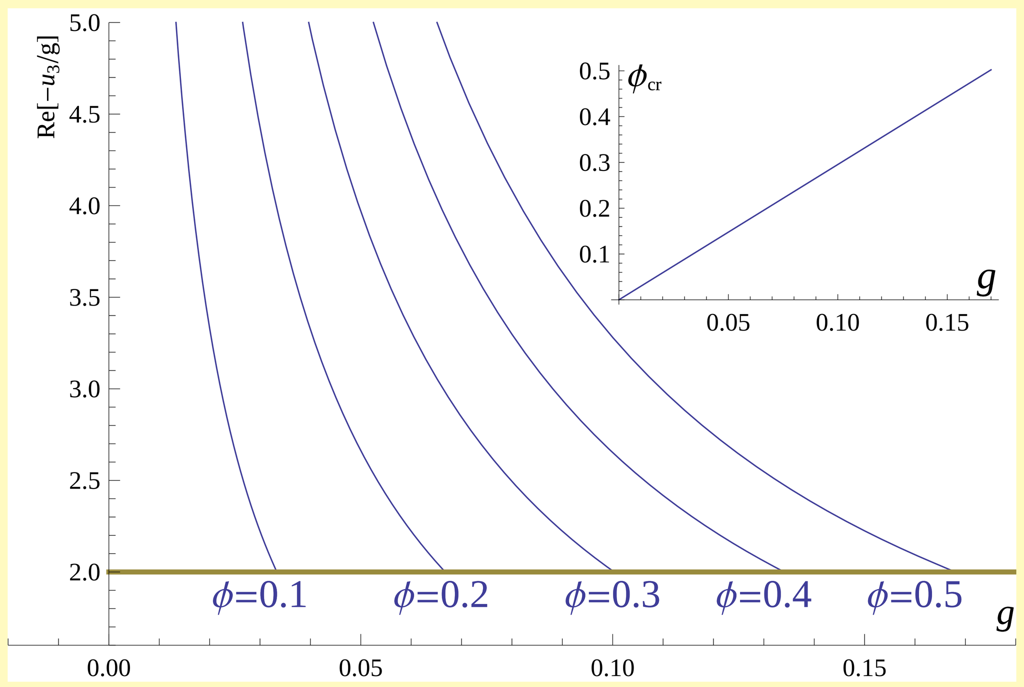
The twisted theory is physical: $E(\phi)$ should always be regular, in particular when $\phi\to0$.

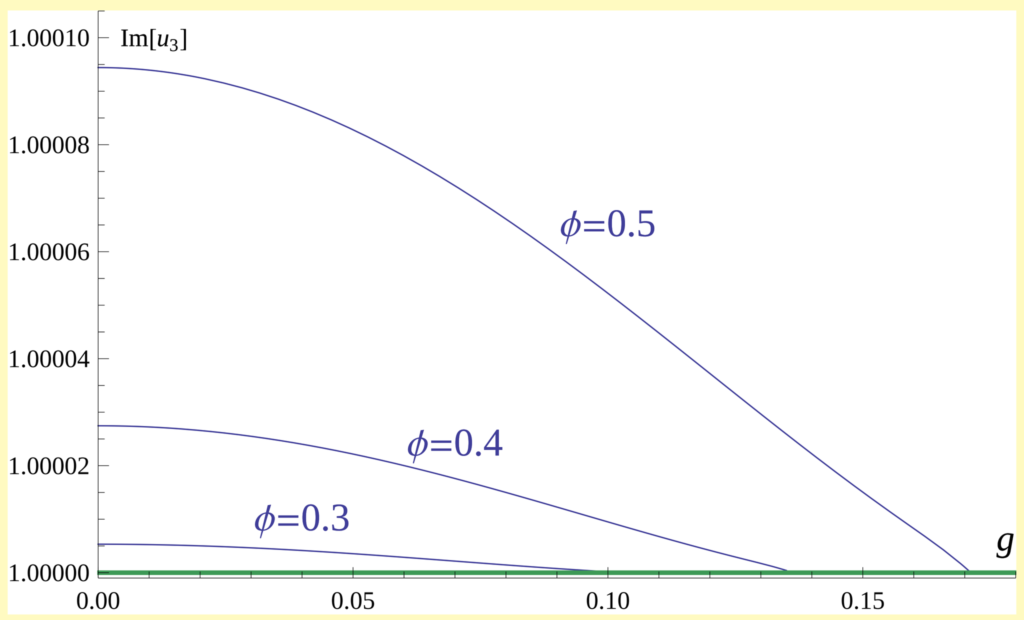
Beyond one-loop

All-loop Bethe Ansatz and for twisted theory relations can be used to find u_j and E.

Numerical solutions seem to exists only for

$$\phi\gtrsim\phi_{\text{critical}}(g)\,.$$





Even perturbative energy behaves badly

$$E_L=\sum_{n=0}^{\infty}E_L^{(2n)}(\phi)\,g^{2n}\,,$$

and one has, for instance

$$E_6^{(12)}=-\tfrac{2187}{1024}\phi^{-6}-\tfrac{3645}{8192}\phi^{-4}+\tfrac{189783}{1310720}\phi^{-2}+O(\phi^0).$$

Wrapping and Mirror TBA

All-loop Bethe Ansatz is **asymptotic**, i.e. does not account for **wrapping effects**.

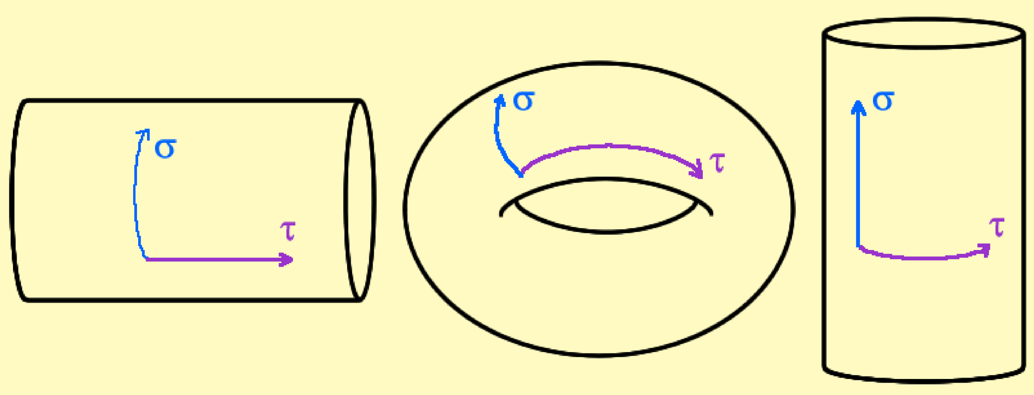
In the $AdS_5\times S^5$ **string theory picture** (holographically dual) they arise because the worldsheet is a circumference J **cylinder**.

Periodic space (σ) is hard for integrability. But **finite temperature** (periodic τ) is doable by **Thermodynamical Bethe Ansatz** (TBA).

Idea: imagine theory defined on a torus

$$H=\int_0^J d\sigma\;\mathcal{H}(p,x,\partial_\sigma x),\quad (\sigma,\tau)\in[0,J]\times[0,R]\,,$$
$$Z(J,R)=\int\mathcal{D}p\mathcal{D}x\,e^{\int_0^R d\tau\int_0^T d\sigma\,(ip\partial_\tau x-\mathcal{H})}\,.$$

Decompactify σ or τ and go from string theory to new **mirror theory** by a **double Wick-rotation** $\tau\to-i\tau$, $\sigma\to i\sigma$ and exchange $\sigma\leftrightarrow\tau$.



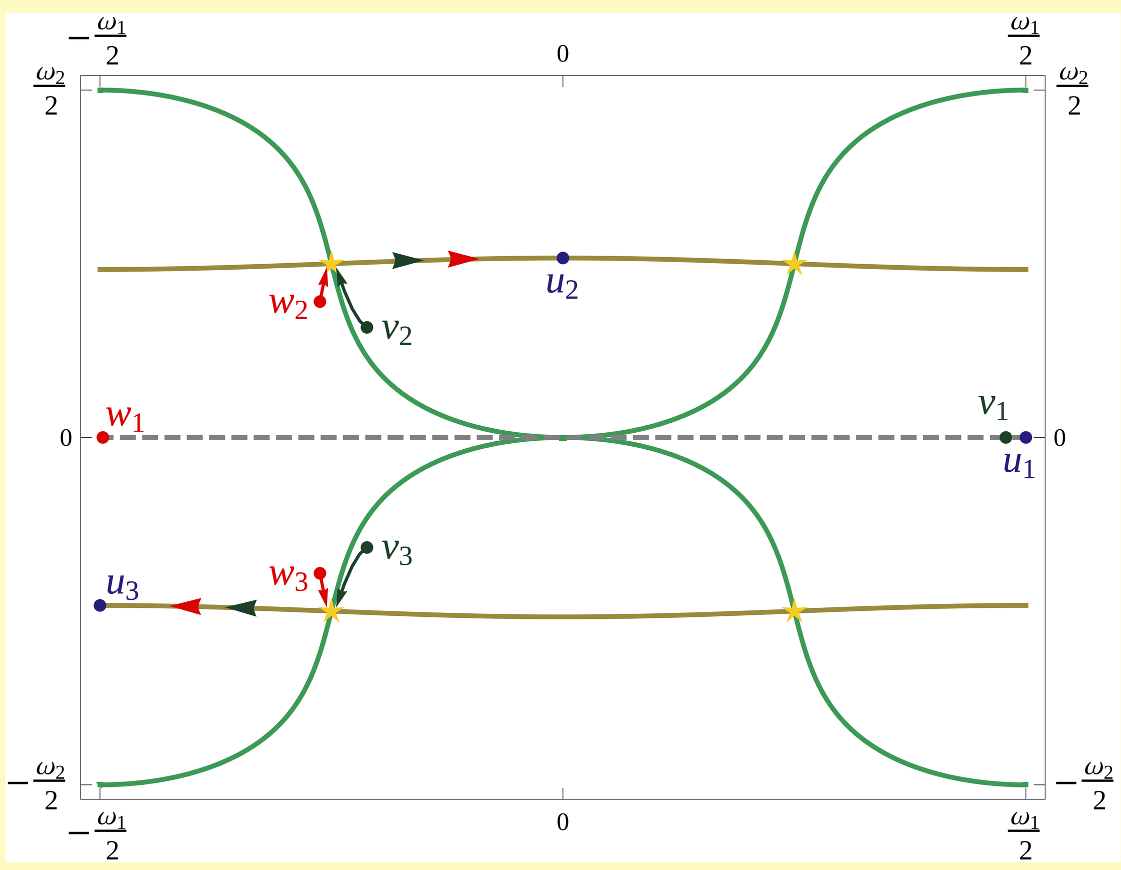
Wrapping effect are described by **thermal bath of mirror particles** (Y-functions).

From ground state equations and analyticity structure of Y-functions, we have **TBA equations for excited states** (contour deformation trick), and energies are found by

$$Y_{1*}(u_k)=-1,\quad E=-\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{\mathcal{C}}\log(1+Y_Q)\,d\tilde{p}_Q\,.$$

TBA for exceptional rapidities

Dispersion relations are uniformized on **rapidity torus**, common to string and mirror theory:



Green lines are the boundary of the **mirror plane**, golden lines denote the **string physical region** and the dashed line is $v\in\mathbb{R}$.

Twisted rapidities w_j and v_j live in the string region.

All-loop Bethe Ansatz suggests that when $\phi\to\phi_{\text{crit.}}(g)$ they **approach the branching points** of the string plane (yellow stars $\star$).

When $\phi\to0$ we conjecture that motion continues to rapidities the one-loop rapidities

$$u_1=0\,,\quad u_2=i\,,\quad u_3=-i\,.$$

We can use two strategies: one is to **write TBA equations for the twisted theory**. The energy must be regular when $\phi\to0$, and give the dimensions of $\mathcal{N}=4$ operators. The **singular contributions are cured by wrapping corrections** and one finds e.g.

$$E_6^{(12)}=-\frac{567}{128}\zeta(9)+\frac{189}{64}\zeta(5)+\frac{243}{128}\zeta(3)-\frac{84753}{1024}+O(\phi^2)\,,$$

and generally the energy appears to be regular also at higher loops.

Alternatively we can postulate that **rapidities u_j are exact** at any value of g. This yields a **rigid analytic structure** (double zeros and double poles) that permits to prove without solving the whole TBA system that quantization conditions $Y_{1*}(u_j)=-1$ hold for all values of g.

Furthermore, the energy $E_6^{(12)}$ agrees with the previous computation, and in general

$$E_L=L-M+\sqrt{1+4g^2}+\sqrt{4+4g^2}-3/2^{L-2}g^L+O(g^{L+2})\,,$$

agrees with Bethe Ansatz. Since rapidities are exact, the **finite coupling spectrum is easier to find.**

Furthermore, one can consider other excitations on top of the exceptional ones. Since these remain fixed this is **a closed sector of the theory** from the TBA point of view.