Integrable structure of modified melting crystal model

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Abstract

Our previous work on a hidden integrable structure of the melting crystal model (the $U(1)$ Nekrasov function) is extended to a modified crystal model. As in the previous case, "shift symmetries" of a quantum torus algebra plays a central role. With the aid of these algebraic relations, the partition function of the modified model is shown to be a tau function of the 2D Toda hierarchy. We conjecture that this tau function belongs to a class of solutions (the so called Toeplitz reduction) related to the Ablowitz-Ladik hierarchy.

1 Introduction

In a previous paper [1], we studied a hidden integrable structure in the melting crystal model (equivalently, the $U(1)$ Nekrasov function). Deforming the model by a charge variable s and external potentials with coupling constants $\boldsymbol{t} = (t_1, t_2, \cdots)$, we showed that the partition function coincides, up to simple factors, with a tau function of the 1D Toda hierarchy. A technical clue is a set of special algebraic relations (referred to as "shift symmetries") among the basis of a quantum torus algebra. With the aid of these relations, we could rewrite the partition function to a product of simple factors and the 1D Toda tau function.

In this report, we present similar results for a modified melting crystal model. In the context of topological string theory, this model is related to the resolved conifold, or local $\mathbf{CP}^1$ geometry of the type $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$, whereas the previous model corresponds to local $\mathbf{CP}^1$ geometry of the type $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$. As it turns out, a hidden integrable structure is a non-1D reduction of the 2D Toda hierarchy. We conjecture that this reduction will be the so called Toeplitz reduction [2], hence the relevant integrable hierarchy will be the Ablowitz-Ladik hierarchy.

2 Modified melting crystal model

2.1 Fermions

We use the same formulation of fermions as our previous work [1]:

• Fourier modes of 2D complex fermion fields

$$\psi(z) = \sum_{n\in\mathbf{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n\in\mathbf{Z}} \psi_n^* z^{-n}.$$

with anti-commutation relations

$$\psi_m\psi_n^* + \psi_m^*\psi_n = \delta_{m+n,0}, \quad \psi_m\psi_n + \psi_m\psi_n = \psi_m^*\psi_n^* + \psi_n^*\psi_m^* = 0.$$

• Ground and exited states

$$\langle s| = \langle -\infty| \cdots \psi_{s-1}^*\psi_s^*, \quad |s\rangle = \psi_{-s}\psi_{-s+1}\cdots|-\infty\rangle,$$
$$\langle \mu, s| = \langle -\infty| \cdots \psi_{\mu_2+s-1}^*\psi_{\mu_1+s}^*, \quad |\mu, s\rangle = \psi_{-\mu_1-s}\psi_{-\mu_2-s+1}\cdots|-\infty\rangle.$$

in the charge-s. The excited states are labelled by the set $\mathcal{P}$ of all partitions $\mu = (\mu_i)_{i=1}^\infty$, $\mu_1 \geq \mu_2 \geq \cdots \geq 0$, of arbitrary lengths.

• Special fermion bilinears

$$J_k = \sum_{n\in\mathbf{Z}} {:}\psi_{k-n}\psi_n^*{:}, \quad L_0 = \sum_{n\in\mathbf{Z}} n{:}\psi_{-n}\psi_n^*{:}, \quad W_0 = \sum_{n\in\mathbf{Z}} n^2{:}\psi_{-n}\psi_n^*{:}.$$

J_0, L_0 and W_0 are zero-modes of $U(1)$ current, Virasoro and $W^{(3)}$ algebras.

2.2 Partition function in fermionic form

Our previous melting crystal model [1] is defined by the partition function

$$Z(s, \boldsymbol{t}) = \langle s|G_+ q^{lW_0/2} Q^{L_0} e^{H(\boldsymbol{t})} G_-|s\rangle,$$

where

• q and Q are constant in the range $0 < |q| < 1$ and $0 < |Q| < 1$, and l is an integer.

• $H(\boldsymbol{t})$ is a linear combination $H(\boldsymbol{t}) = \sum_{k=1}^{\infty} t_k H_k$ of the special fermion bilinears

$$H_k = \sum_{n\in\mathbf{Z}} q^{kn}{:}\psi_{-n}\psi_n^*{:}, \quad k \in \mathbf{Z}.$$

• $G_\pm$ are the transfer operators

$$G_\pm = \exp\left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} J_{\pm k}\right)$$

of Okounkov and Reshetikhin [3].

We now modify this model as follows:

- Replace $H(\boldsymbol{t})$ with $H(\boldsymbol{t},\hat{\boldsymbol{t}}) = \sum_{k=1}^{\infty} t_k H_k + \sum_{k=1}^{\infty} \hat{t}_k H_{-k}$.
- Replace G_- with one of another pair of Okounkov and Pandharipande's transfer operators

$$G'_\pm = \exp\left(- \sum_{k=1}^{\infty} \frac{(-1)^k q^{k/2}}{k(1-q^k)} J_{\pm k} \right).$$

The partition function of the modified model is thereby defined as

$$\boxed{Z'(s,\boldsymbol{t},\hat{\boldsymbol{t}}) = \langle s|G_+ q^{lW_0/2} Q^{L_0} e^{H(\boldsymbol{t},\bar{\boldsymbol{t}})} G'_-|s\rangle}$$

2.3 Partition function as sum over partitions

According to Okounkov, Reshetikhin and Vafa [3], G_+ and G'_- act on the ground states $\langle s|$, $|s\rangle$ as

$$\langle s|G_+ = \sum_{\mu\in\mathcal{P}} \langle \mu,s| s_\mu(q^{-\rho}), \quad G'_-|s\rangle = \sum_{\mu\in\mathcal{P}} s_{{}^t\mu}(q^{-\rho})|\mu,s\rangle,$$

where $s_\mu(q^{-\rho})$'s are special values of the Schur functions at $q^{-\rho} = (q^{1/2}, q^{3/2}, \cdots, q^{n-1/2}, \cdots)$. These special values are known to have the hook length formula

$$s_\mu(q^{-\rho}) = \frac{q^{-\kappa(\mu)/4}}{\prod_{(i,j)\in\mu}(q^{-h(i,j)/2} - q^{h(i,j)/2})},$$

where $h(i,j)$ denotes the length of the hook with corner at the cell (i,j) of the Young diagram. ${}^t\mu$ denotes the conjugate (or transpose) of μ. Since the operators $q^{lW_0/2}$, Q^{L_0} and $e^{H(\boldsymbol{t},\hat{\boldsymbol{t}})}$ are diagonal with respect to $\langle\mu,s|$'s and $|\mu,s\rangle$'s, $Z'(s,\boldsymbol{t},\bar{\boldsymbol{t}})$ can be expanded to a single sum over $\mathcal{P}$ as

$$\boxed{Z'(s,\boldsymbol{t},\hat{\boldsymbol{t}}) = \sum_{\mu\in\mathcal{P}} s_\mu(q^{-\rho}) s_{{}^t\mu}(q^{-\rho}) q^{l\langle\mu,s|W_0|\mu,s\rangle/2} Q^{\langle\mu,s|L_0|\mu,s\rangle} e^{\langle\mu,s|H(\boldsymbol{t},\bar{\boldsymbol{t}})|\mu,s\rangle}}$$

The diagonal matrix elements of L_0 and W_o can be expressed as

$$\langle\mu,s|L_0|\mu,s\rangle = |\mu| + \frac{s(s+1)}{2},$$
$$\langle\mu,s|W_0|\mu,s\rangle = \kappa(\mu) + (2s+1)|\mu| + \frac{s(s+1)(2s+1)}{6},$$

where $|\mu| = \sum_{i\ge 1}\mu_i$, $\kappa(\mu) = \sum_{i\ge 1}\mu_i(\mu_i - 2i + 1)$. The diagonal matrix elements of $H(\boldsymbol{t},\hat{\boldsymbol{t}})$ give potentials of the form

$$\langle \mu, s|H(\boldsymbol{t},\bar{\boldsymbol{t}})|\mu, s\rangle = \sum_{k=1}^{\infty} t_k\Phi_k(\mu,s) + \sum_{k=1}^{\infty}\hat{t}_k\Phi_{-k}(\mu,s),$$
$$\Phi_k(\mu,s) = \sum_{i=1}^{\infty}(q^{k(s+\mu_i-i+1)} - q^{k(s-i+1)}) + \frac{q^k(1-q^{ks})}{1-q^k}.$$

When $\boldsymbol{t} = \hat{\boldsymbol{t}} = \mathbf{0}$ and $s = 0$, this partition function reduces to

$$\boxed{Z' = \sum_{\mu\in\mathcal{P}} s_\mu(q^{-\rho})s_{{}^t\mu}(q^{-\rho})q^{l\kappa(\mu)/2}(q^{l/2}Q)^{|\mu|}}$$

Up to a sign factor, this coincides with the sum derived by Brian and Pandharipande [4] from the Gromov-Witten theory of local curves. This sum was further studied by Caporaso et al [5] in the context of toric topological string theory. We need the coupling constants $\boldsymbol{t},\hat{\boldsymbol{t}}$ and the charge variable s to formulate an integrable structure.

3 Hidden integrable structure

3.1 Shift symmetries in quantum torus algebra

The fermion bilinears

$$V_m^{(k)} = q^{-km/2}\sum_{n\in\mathbf{Z}} q^{kn}{:}\psi_{n-m}\psi_n^*{:}, \quad k,m\in\mathbf{Z},$$

satisfy the commutation relations

$$\boxed{[V_m^{(k)}, V_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2})\left(V_{m+n}^{(k+l)} - \delta_{m+n,0}\frac{q^{k+l}}{1-q^{k+l}}\right)}$$

of (a central extension of) the quantum torus algebra. Note that the c-number term on the right hand side turns into $-m\delta_{m+n}$ as $k+l\to 0$.

The following algebraic relations, referred to as shift symmetries, play a central role in identifying a hidden integrable structure of the partition function:

(i) First Symmetries

$$G_-G_+\left(V_m^{(k)}-\delta_{m,0}\frac{q^k}{1-q^k}\right)(G_-G_+)^{-1}=(-1)^k\left(V_{m+k}^{(k)}-\delta_{m+k,0}\frac{q^k}{1-q^k}\right),$$
$$G'_-G'_+\left(V_m^{(-k)}-\delta_{m,0}\frac{1}{1-q^k}\right)(G'_-G'_+)^{-1}=V_{m+k}^{(-k)}-\delta_{m+k,0}\frac{1}{1-q^k}$$

for $k>0$, $m\in\mathbf{Z}$

(ii) Second symmetries

$$q^{W_0/2}V_m^{(k)}q^{-W_0/2}=V_m^{(k-m)}$$

for $k,m\in\mathbf{Z}$

3.2 Rewriting partition function to tau function

With the aid of the shift symmetries, we can rewrite the partition function to a tau function.

First step: Since $H_k=V_0^{(k)}$ and $J_k=V_k^{(0)}$, the shift symmetries with respect to $G_\pm$'s and $q^{W_0/2}$ yields the relation

$$G_+H_kG_+^{-1}=(-1)^kG_-^{-1}q^{-W_0/2}J_kq^{W_0/2}G_-+\frac{q^k}{1-q^k}.$$

This implies that

$$G_+\exp\left(\sum_{k=1}^\infty t_kH_k\right)G_+^{-1}=\exp\left(\sum_{k=1}^\infty\frac{q^kt_k}{1-q^k}\right)G_-^{-1}q^{-W_0/2}\exp\left(\sum_{k=1}^\infty(-1)^kt_kJ_k\right)q^{W_0/2}G_-.$$

Second step: In the same way, by the shift symmetries with respect to $G'_\pm$'s and $q^{W_0/2}$, we have the relation

$$G'_-{}^{-1}H_{-k}G'_-=G'_+q^{-W_0/2}J_{-k}q^{W_0/2}G'_+{}^{-1}+\frac{1}{1-q^k},$$

hence

$$G'_-{}^{-1}\exp\left(\sum_{k=1}^\infty\hat{t}_kH_{-k}\right)G'_-=\exp\left(\sum_{k=1}^\infty\frac{\hat{t}_k}{1-q^k}\right)G'_+q^{-W_0/2}\exp\left(\sum_{k=1}^\infty\hat{t}_kJ_{-k}\right)q^{W_0/2}G'_+{}^{-1}.$$

Third step: The foregoing calculations show that the operator in the definition of $Z'(s,\boldsymbol{t},\hat{\boldsymbol{t}})$ can be thus expressed as

$$G_+q^{lW_0/2}Q^{L_0}e^{H(\boldsymbol{t},\hat{\boldsymbol{t}})}G'_-=\exp\left(\sum_{k=1}^\infty\frac{q^kt_k+\hat{t}_k}{1-q^k}\right)G_-^{-1}q^{-W_0/2}\exp\left(\sum_{k=1}^\infty(-1)^kt_kJ_k\right)$$
$$\times q^{W_0/2}G_-G_+q^{lW_0/2}Q^{L_0}G'_-G'_+q^{-W_0/2}\exp\left(\sum_{k=1}^\infty\hat{t}_kJ_{-k}\right)q^{W_0/2}G'_+{}^{-1}.$$

The leftmost and rightmost operators in this expression act on $\langle s|$ and $|s\rangle$ as

$$\langle s|G_-^{-1}q^{-W_0/2} = q^{-s(s+1)(2s+1)/12}\langle s|, \quad q^{W_0/2}{G'_+}^{-1}|s\rangle = q^{s(s+1)(2s+1)/12}|s\rangle.$$

We thus find the following expression of the partition function:

$$\boxed{\begin{aligned} Z'(s,\boldsymbol{t},\hat{\boldsymbol{t}}) &= \exp\left(\sum_{k=1}^{\infty}\frac{q^k t_k + \hat{t}_k}{1-q^k}\right)\tau'(s,-t_1,t_2,-t_3,\cdots,-\hat{t}_1,-\hat{t}_2,-\hat{t}_3,\cdots), \\ \tau'(s,\boldsymbol{t},\hat{\boldsymbol{t}}) &= \langle s|\exp\left(\sum_{k=1}^{\infty}t_kJ_k\right)g'\exp\left(-\sum_{k=1}^{\infty}\hat{t}_kJ_{-k}\right)|s\rangle \end{aligned}}$$

where

$$g' = q^{W_0/2}G_-G_+q^{lW_0/2}Q^{L_0}G'_-G'_+q^{-W_0/2}.$$

$\tau'(s,\boldsymbol{t},\hat{\boldsymbol{t}})$ is a tau function of the 2D Toda hierarchy, in which t_k's and $\hat{t}_k$'s are two independent sets of time variables. It is well known that a tau functions remains to be a tau function after multiplied by an exponential function of a linear function of the time variables. Hence $Z'(s,\boldsymbol{t},\hat{\boldsymbol{t}})$ itself is also a tau function.

3.3 Comparison with previous model

The partition function

$$Z(s,\boldsymbol{t}) = \langle s|G_+q^{lW_0/2}Q^{L_0}e^{H(\boldsymbol{t})}G_-|s\rangle$$

of the previous model [1] can be rewritten as

$$\boxed{Z(s,\boldsymbol{t}) = \exp\left(\sum_{k=1}^{\infty}\frac{t_kq^k}{1-q^k}\right)q^{-s(s+1)(2s+1)/6}\tau(s,-t_1,t_2,-t_3,\cdots)}$$

where $\tau(s,\boldsymbol{t})$ is a tau function of the 1D Toda hierarchy. The 1D Toda hierarchy is a special case ("reduction" in the terminology of integrable systems) of the 2D Toda hierarchy in which the tau function depends on the two sets of time variables $\boldsymbol{t},\hat{\boldsymbol{t}}$ through their difference as

$$\tau(s,\boldsymbol{t},\hat{\boldsymbol{t}}) = \tau(s,\boldsymbol{t}-\hat{\boldsymbol{t}}).$$

The reduced function $\tau(s,\boldsymbol{t})$ becomes the tau function of the 1D Toda hierarchy. Actually, $\tau(s,\boldsymbol{t})$ in the expression of $Z(s,\boldsymbol{t})$ can have different expressions such as

$$\begin{aligned}\tau(s,\boldsymbol{t}) &= \langle s|\exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g|s\rangle \\ &= \langle s|\exp\left(\sum_{k=1}^{\infty}\frac{t_k}{2}J_k\right) g\exp\left(\sum_{k=1}^{\infty}\frac{t_k}{2}J_{-k}\right)|s\rangle \\ &= \langle s|g\exp\left(\sum_{k=1}^{\infty} t_k J_{-k}\right)|s\rangle\end{aligned}$$

where

$$g = q^{W_0/2}G_-G_+q^{lW_0/2}Q^{L_0}G_-G_+q^{W/2}.$$

This is a consequence of the intertwining relations

$$J_k g = gJ_{-k} \quad \text{for } k = 1, 2, \cdots$$

that can be derived from shift symmetries.

The tau function $\tau'(s,\boldsymbol{t},\hat{\boldsymbol{t}})$ of the modified model does not have this property, hence a tau function of the 2D Toda hierarchy in a genuine sense. Its status in the 2D Toda hierarchy, however, is still obscure.

3.4 Toeplitz reduction

In search for the status of $\tau'(s,\boldsymbol{t},\hat{\boldsymbol{t}})$, let us draw attention to the following fake intertwining relations

$$J_k g' = g' J_k \quad \text{for } k = \pm 1, \pm 2, \cdots$$

referred to as the "Toeplitz condition" in the literature of integrable systems. Though details are omitted, we can "derive" these relations from shift symmetries. Actually, these relations should not hold. If these relations were correct, we could move $J_\pm$'s in the tau function to the other side of the ground state expectation value and find that

$$\tau'(s,\boldsymbol{t},\hat{\boldsymbol{t}}) = \exp\left(\sum_{k=1}^{\infty} k t_k \hat{t}_k\right)\langle s|g'|s\rangle.$$

Namely, the tau function would turn out to be an almost trivial one. This is not the case.

This contradictory situation stems from potential inconsistency of shift symmetries. Careless use of shift symmetries can lead to wrong results. This inconsistency seems to be related to non-associativity of some operators on the fermionic Fock space. We have been unable to establish a fully consistent theory of shift symmetries.

In spite of apparent inconsistency, we are inspired by the fake intertwining relations to conjecture that the tau function $\tau'(s, \boldsymbol{t}, \hat{\boldsymbol{t}})$ belongs to the Toeplitz reduction of the 2D Toda hierarchy [2], hence a solution of the Ablowitz-Ladik hierarchy. This conjecture is also partially supported by the work of Brini on the resolve conifold [6]. Note that our partition function in the case of $l = 0$ is related to the partition function of topological strings on the resolved conifold. Another possible test towards this conjecture is to examine the thermodynamic limit as we have done for the previous model [7].

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