Baxter Q-operators and tau-function for quantum integrable spin chains

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This is based on the following papers.
Main text: [1] (also [2]) (with Alexandrov, Kazakov, Leurent, Zabrodin),
Appendix: [3].

1 Introduction

The Baxter Q-operators were introduced by Baxter when he solved the 8-vertex model. His method of the Q-operators is recognized as one of the most powerful tools in quantum integrable systems.

Our goals are

1. to construct Baxter Q-operators systematically
2. to write the T-operators (transfer matrices) in terms of the Q-operators: Wronskian-like determinant formulas
3. to establish functional relations among them: T-system, TQ-relations, QQ-relations

For these purposes, we consider an embedding of the quantum integrable system into the soliton theory. The key object is the master T-operator (τ-function in the soliton theory) (3.1), which is a sort of a generating function of the transfer matrices.

2 Cherednik-Bazhanov-Reshetikhin formula

Consider T-operators (transfer matrices) of a quantum integrable spin chain labeled by the Young diagram $\lambda = (\lambda_1, \lambda_2, \dots)$ for $gl(N)$.

$$T^{\lambda}(u) = \mathrm{Tr}_{\pi_\lambda} R_{L0}(u-\xi_L) \cdots R_{10}(u-\xi_1)(1^{\otimes L} \otimes \pi_\lambda(g)), \quad (2.1)$$

where $R_{j0}(u)$ is the R-matrix whose auxiliary space (denoted by '0') is an evaluation representation of $Y(gl(N))$ based on the tensor representation π_λ of $gl(N)$ labeled by λ and the quantum space (denoted by the lattice site 'j') is not specified; $g \in GL(N)$ is the boundary twist matrix (for the trigonometric case, it is a group-like element made from the Cartan subalgebra); $u \in \mathbb{C}$ is the spectral parameter; $\xi_j \in \mathbb{C}$ are imhomogenuities. The Cherednik-Bazhanov-Reshetikhin formula states that the T-operator for the general Young diagram can be written as a determinant over T-operators for Young diagrams with one row:

$$T^{\lambda}(u) = \left(\prod_{k=1}^{\lambda'_1 - 1} T^{\emptyset}(u-k) \right)^{-1} \det_{i,j=1,\dots,\lambda'_1} T^{(\lambda_i - i + j)}(u-j+1). \quad (2.2)$$

3 The master T-operator

Schur functions in the KP-time variables $t = \{t_1, t_2, t_3, \dots\}$ are defined by

$$\exp\left(\sum_{k=1}^{\infty} t_k z^{-k} \right) = \sum_{n=0}^{\infty} s_{(n)}(t) z^{-n},$$

$$s_\lambda(t) = \det_{1 \le i,j \le \lambda'_1} (s_{(\lambda_i - i + j)}).$$

The *master T-operator* (τ-function) is defined by

$$T(u, \mathbf{t}) = \sum_{\lambda} s_\lambda(\mathbf{t}) T^{\lambda}(u). \quad (3.1)$$

This is a generating function of the T-operators in the following sense:

$$T^{\lambda}(u) = s_{\lambda}(\tilde{\partial})T(u,\mathbf{t})\Big|_{\mathbf{t}=0}, \quad \tilde{\partial} = \{\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \dots\}.$$

The master T-operator (3.1) commutes for any u, t, and contains Baxter Q-operators and T-operators for all levels of the nested Bethe ansatz.
The master T-operator is a τ-function of

1. KP-hierarchy with respect to times $t_1, t_2, \cdots$,
2. MKP-hierarchy with respect to times $t_0, t_1, t_2, \cdots$.

Here $t_0 = u$ plays a role of the spectral parameter in the quantum integrable system. The statement (2) is equivalent to that *the coefficients* $T^{\lambda}(u)$ *of the Schur function expansion* (3.1) *obey the Cherednik-Bazhanov-Reshetikhin formula* (2.2).

4 Bilinear identity for the master T-operator

The master T-operator (3.1) obeys the bilinear identity

$$\oint_C e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{u-u'} T(u,\mathbf{t}-[z^{-1}])\, T(u',\mathbf{t}'+[z^{-1}])\, dz = 0 \qquad (4.1)$$

where $t + [z^{-1}] = \{t_1 + z^{-1}, t_2 + \frac{1}{2}z^{-2}, t_3 + \frac{1}{3}z^{-3}, \dots\}$,
$\xi(\mathbf{t}, z) = \sum_{n=1}^{\infty} t_n z^n$.
We can derive various bilinear equations by choosing u, u', t, t'.

Example 1 Let us consider the case $u' = u$, $t'_k = t_k - \frac{1}{k}(z_1^{-k} + z_2^{-k} + z_3^{-k})$,

$$e^{\xi(\mathbf{t}-\mathbf{t}',z)} = \frac{1}{(1-\frac{z}{z_1})(1-\frac{z}{z_2})(1-\frac{z}{z_3})},$$

Then (4.1) reduces to the discrete KP equation:

$$(z_2 - z_3)T\left(u, \mathbf{t} + [z_1^{-1}]\right) T\left(u, \mathbf{t} + [z_2^{-1}] + [z_3^{-1}]\right)$$
$$+ (z_3 - z_1)T\left(u, \mathbf{t} + [z_2^{-1}]\right) T\left(u, \mathbf{t} + [z_1^{-1}] + [z_3^{-1}]\right)$$
$$+ (z_1 - z_2)T\left(u, \mathbf{t} + [z_3^{-1}]\right) T\left(u, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]\right) = 0. \tag{4.2}$$

The equations in the hierarchy are obtained by expanding in negative powers of z_1, z_2, z_3.

Example 2 Let us consider the case $u' = u - 1$, $t'_k = t_k - \frac{1}{k}(z_1^{-k} + z_2^{-k})$,

$$ze^{\xi(\mathbf{t}-\mathbf{t}',z)} = \frac{z}{(1 - \frac{z}{z_1})(1 - \frac{z}{z_2})}.$$

Then (4.1) reduces to the discrete MKP equation:

$$z_2 T\left(u+1, \mathbf{t} + [z_1^{-1}]\right) T\left(u, \mathbf{t} + [z_2^{-1}]\right)$$
$$- z_1 T\left(u+1, \mathbf{t} + [z_2^{-1}]\right) T\left(u, \mathbf{t} + [z_1^{-1}]\right)$$
$$+ (z_1 - z_2)T\left(u+1, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]\right) T\Big(u, \mathbf{t}\Big) = 0. \tag{4.3}$$

The equations in the hierarchy are obtained by expanding in negative powers of z_1, z_2. This can be interpreted as the limit $z_3 \to 0$ of KP (4.2).

5 Bäcklund transformations for the master T-operator

Let us take any subset $\{i_1, i_2, \ldots, i_n\}$ of the set $\{1, 2, \ldots, N\}$. There are 2^N such sets. We define the *nested master T-operators* $T^{(i_1 \ldots i_n)}(u, \mathbf{t})$ recursively by taking the residue of the master T-operator (3.1).

$$T^{(i_1\dots i_n)}(u,\mathbf{t}) = \pm\mathrm{res}_{z_{i_n}=p_{i_n}}\left(z_{i_n}^{-u-1}e^{-\xi(\mathbf{t},z_{i_n})}T^{(i_1\dots i_{n-1})}(u+1,\mathbf{t}+[z_{i_n}^{-1}])\right), \tag{5.1}$$

where $\{p_1, p_2, \dots, p_N\}$ are the eigenvalues of the boundary twist matrix in (2.1) and $T^{\emptyset}(u,\mathbf{t}) = T(u,\mathbf{t})$. These define the undressing chain that terminates at the level N:

$$T(u,\mathbf{t}) \to T^{(i_1)}(u,\mathbf{t}) \to T^{(i_1 i_2)}(u,\mathbf{t}) \to \dots \to T^{(12\dots N)}(u,\mathbf{t}) \to 0$$

and satisfy the bilinear relations (Bäcklund transformations) in the same way as the master T-operator ((4.2),(4.3)):

$$\begin{aligned}\varepsilon_{ij}p_k^{-1}T^{(i_1\dots i_n ij)}(u,\mathbf{t})T^{(i_1\dots i_n k)}(u+1,\mathbf{t}) \\ + \varepsilon_{jk}p_i^{-1}T^{(i_1\dots i_n jk)}(u,\mathbf{t})T^{(i_1\dots i_n i)}(u+1,\mathbf{t}) \\ + \varepsilon_{ki}p_j^{-1}T^{(i_1\dots i_n ki)}(u,\mathbf{t})T^{(i_1\dots i_n j)}(u+1,\mathbf{t}) = 0,\end{aligned} \tag{5.2}$$

$$\begin{aligned}p_j^{-1}T^{(i_1\dots i_n i)}(u,\mathbf{t})T^{(i_1\dots i_n j)}(u+1,\mathbf{t}) - p_i^{-1}T^{(i_1\dots i_n j)}(u,\mathbf{t})T^{(i_1\dots i_n i)}(u+1,\mathbf{t}) \\ = \varepsilon_{ij}T^{(i_1\dots i_n ij)}(u,\mathbf{t})T^{(i_1\dots i_n)}(u+1,\mathbf{t}),\end{aligned} \tag{5.3}$$

where $i, j, k \in \{1, 2, \dots, N\} \setminus \{i_1, i_2, \dots, i_n\}$, $i \neq j, i \neq k, j \neq k$, $\varepsilon_{ij} = \pm 1$.

6 A general definition of the Baxter Q-operators

We define the *Baxter Q-operators* $Q_{(i_1\dots i_n)}(u)$ (up to the normalization) by the nested master T-operators as their restrictions to zero values of $\mathbf{t}$:

$$Q_{(i_1\dots i_n)}(u) = T^{(i_1\dots i_n)}(u, \mathbf{t}=0). \tag{6.1}$$

Then the functional relations among Q-operators (QQ-relations) follow from (5.3). The Bethe ansatz equations can be derived from the QQ-relations. This realizes an idea [4] that there are 2^N Q-operators. The Bäcklund transformations in [5] are realized on the level of operators systematically. If the quantum space in (2.1) is the fundamental representation, one can use [2, 1] the co-derivative on the group elements [6] to write the Q-operators. As further development,

• The generalization to the superalgebra case $gl(M|N)$ or the trigonometric case are also possible.

• The master T-operator will be realized as a sort of a column determinant over a function of L-operator for the Yangian or the quantum affine algebra (a generalization of a generating function of the characters).

Appendix: L-operators for the Baxter Q-operators

The Q-operators can also be defined as the trace of some monodromy matrices, which are defined as product of L-operators. In general, such L-operators are image of the universal R-matrix for q-oscillator representations of the Borel subalgebra of the quantum affine algebra (cf. [7] for $U_q(\hat{sl}(2))$ case). In [8, 10], we gave L-operators for the Q-operators for $U_q(\widehat{sl}(2|1))$. Here we mention construction of the L-operators for $U_q(\widehat{gl}(M|N))$ [3].

The (centerless) quantum affine superalgebra $U_q(\widehat{gl}(M|N))$ is defined by

$$
\begin{aligned}
&L_{ij}^{(0)} = \overline{L}_{ji}^{(0)} = 0, \quad \text{for} \quad 1 \le i < j \le M+N \\
&L_{ii}^{(0)}\overline{L}_{ii}^{(0)} = \overline{L}_{ii}^{(0)}L_{ii}^{(0)} = 1 \quad \text{for} \quad 1 \le i \le M+N, \\
&\mathbf{R}^{23}(x,y)\mathbf{L}^{13}(y)\mathbf{L}^{12}(x) = \mathbf{L}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x,y), \\
&\mathbf{R}^{23}(x,y)\overline{\mathbf{L}}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\overline{\mathbf{L}}^{13}(y)\mathbf{R}^{23}(x,y), \\
&\mathbf{R}^{23}(x,y)\mathbf{L}^{13}(y)\overline{\mathbf{L}}^{12}(x) = \overline{\mathbf{L}}^{12}(x)\mathbf{L}^{13}(y)\mathbf{R}^{23}(x,y), \quad x,y \in \mathbb{C},
\end{aligned}
$$

$$
\mathbf{L}(x) = \sum_{i,j} L_{ij}(x) \otimes E_{ij}, \quad \overline{\mathbf{L}}(x) = \sum_{i,j} \overline{L}_{ij}(x) \otimes E_{ij},
$$

$$L_{ij}(x)=\sum_{n=0}^{\infty}L_{ij}^{(n)}x^{-n},\quad \overline{L}_{ij}(x)=\sum_{n=0}^{\infty}\overline{L}_{ij}^{(n)}x^{n},$$

where $\mathbf{R}(x,y) = \mathbf{R} - \frac{x}{y}\overline{\mathbf{R}}$ is the R-matrix of the Perk-Schultz model; $\mathbf{R}$ and $\overline{\mathbf{R}}$ do not depend on the spectral parameter; E_{ij} is $(M+N)\times(M+N)$ matrix unit.

The quantum affine superalgebra $U_q(\hat{gl}(M|N))$ has a finite subalgebra $U_q(gl(M|N))$ defined by

$$\begin{aligned}
&L_{ij}=\overline{L}_{ji}=0 \quad \text{for} \quad 1\le i<j\le M+N\\
&L_{ii}\overline{L}_{ii}=\overline{L}_{ii}L_{ii}=1 \quad \text{for} \quad 1\le i\le M+N,\\
&\mathbf{R}^{23}\mathbf{L}^{13}\mathbf{L}^{12}=\mathbf{L}^{12}\mathbf{L}^{13}\mathbf{R}^{23},\\
&\mathbf{R}^{23}\overline{\mathbf{L}}^{13}\overline{\mathbf{L}}^{12}=\overline{\mathbf{L}}^{12}\overline{\mathbf{L}}^{13}\mathbf{R}^{23},\\
&\mathbf{R}^{23}\mathbf{L}^{13}\overline{\mathbf{L}}^{12}=\overline{\mathbf{L}}^{12}\mathbf{L}^{13}\mathbf{R}^{23},\\
&\mathbf{L}=\sum_{i,j}L_{ij}\otimes E_{ij},\quad \overline{\mathbf{L}}=\sum_{i,j}\overline{L}_{ij}\otimes E_{ij}.
\end{aligned}\tag{6.2}$$

There is an evaluation map from $U_q(\hat{gl}(M|N))$ to $U_q(gl(M|N))$ such that

$$\begin{aligned}
&\mathbf{L}(x)\mapsto \mathbf{L}-\overline{\mathbf{L}}x^{-1},\\
&\overline{\mathbf{L}}(x)\mapsto \overline{\mathbf{L}}-\mathbf{L}x.
\end{aligned}$$

The difference between $\mathbf{L}(x)$ and $\overline{\mathbf{L}}(x)$ is not very important under the evaluation map. We will consider only $\mathbf{L}(x)$ which generate the q-superYangian (a sort of a Borel subalgebra of $U_q(\hat{gl}(M|N))$).

Let us take any subset I of the set $\{1,2,\ldots,M+N\}$ and its complement set $\overline{I} := \{1,2,\ldots,M+N\}\setminus I$. There are 2^{M+N} choices of the subsets in this case. Corresponding to the set I, we consider 2^{M+N} kind of representations of the q-superYangian. For this purpose, we consider 2^{M+N} kind of contractions of $U_q(gl(M|N))$. Namely, let us consider an algebra whose condition (6.2) is replaced by

$$\begin{aligned} &L_{ii}\overline{L}_{ii} = \overline{L}_{ii}L_{ii} = 1 \quad \text{for} \quad i \in I, \\ &\overline{L}_{ii} = 0 \qquad \text{for} \quad i \in \overline{I}. \end{aligned}$$

Then one can obtain 2^{M+N} kind of algebraic solutions of the graded Yang-Baxter equation via the map $\mathbf{L}_I(x) = \mathbf{L} - \overline{\mathbf{L}}x^{-1}$. In addition, we consider subsidiary contractions for the non-diagonal elements. For example, suppose the set I has the form $I = \{1, 2, \ldots, n\}$ for $n > 0$, then we assume

$$\overline{L}_{ij} = 0 \qquad \text{for} \quad n < i < j \leq M + N.$$

Remark: A preliminary form of these contractions was discussed for $U_q(gl(3))$: [9]; $U_q(gl(2|1))$: [10].

We can construct some q-oscillator realizations of these contracted algebras. Then we obtain q-oscillator solutions of the graded Yang Baxter equation via $\mathbf{L}_I(x) = \mathbf{L} - \overline{\mathbf{L}}x^{-1}$. These L-operators are L-operators for the Q-operators. They also give q-oscillator representations of the q-superYangian. We remark that similar L-operators for $U_q(gl(3))$ were derived in [11]. We also remark that these L-operators reduce to L-operators similar to the ones in [12] in the rational limit $q \to 1$. Up to overall factors, these are image of the universal R-matrix $(\pi_1 \otimes \pi_2)\mathcal{R}$, where π_1 are q-oscillator representations and π_2 is the fundamental representation. Further development will be made for these L-operators for the case:

(1) π_1 are q-oscillator representations and π_2 are generic infinite dimensional representations.

(2) Both π_1 and π_2 are q-oscillator representations, where the factorization of the R-matrix for generic infinite dimensional representations with respect to these will occur for both auxiliary and quantum spaces.

(2) is a limit of (1) for the Kirillov-Reshetikhin modules (or their infinite dimensional analogues) of π_2. Similarly, (1) is also a limit of R-matrix whose π_1 are such modules. All these will be explained by the asymptotic representation theory of the quantum affine algebra [13]. Nevertheless, it is important to construct all these matrices explicitly.

There is also a different approach for related problems ([14] and references therein).

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