Three-point Functions of BMN Operators at Weak and Strong Coupling in the $SO(6)$ Sector

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Motivation

A recent work *[Bissi, Harmark and Orselli 2011]* has shown that the AdS/CFT prediction for the three-point function may disagree with the weak coupling calculation. This has necessitated a series of different tests of three-point function calculations to prove their consistency. Here we provide tests for the three-point functions in the $SO(6)$ sectors from

- Perturbation theory
- String field theory
- Integrability-assisted resummation conjecture

and show explicitly that they mutually agree in a non-trivial way.

Definitions

We consider two-magnon BMN operators

$$\mathcal{O}^J_{ij,n} = \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} \mathrm{tr}\left(\phi_i Z^l \phi_j Z^{J-l}\right) \psi_{n,l},$$

which fall into the three irreducible representations of $SO(4)$; we choose the symmetric one for which

$$\psi^S_{n,l} = \cos\frac{(2l+1)\pi n}{J+1}$$

We consider three operators: $\mathcal{O}_1 = \mathcal{O}^{J_1,12}_{n_1}$, $\mathcal{O}_2 = \mathcal{O}^{J_2,23}_{n_2}$, $\mathcal{O} = \mathcal{O}^{J,31}_{n}$, where n_1, n_2, n_3 are the magnon momenta, J_1, J_2, J_3 are their R-charges R_3, $J = J_1 + J_2$, $J_1 = Jy$, $J_2 = J(1-y)$.

We shall be looking for the quantity

$$C_{123} = \langle \bar{\mathcal{O}}_3 \mathcal{O}_1 \mathcal{O}_2 \rangle$$

as a function of y, J, n_1, n_2, n_3, and compare it at one loop in FT with ST in Penrose limit.

Problems on our way

Let us point out some of the obstacles that may be encountered on the way to three-point functions:

1. Double-trace admixture
2. Fermionic operators admixture
3. Magnon momentum nonconserving admixture

Happily enough, problems (1) and (2) are resolved by choosing the symmetric sector operators in $SO(6)$, and (3) is resolved by invoking the large-J limit.

String field theory calculation

In terms of the BMN basis $\{\alpha_m\}$ our operators look like

$$\mathcal{O}_m = \alpha^\dagger_m \alpha^\dagger_{-m}|0\rangle$$

The three-point function is related to the matrix element of the string field Hamiltonian as follows

$$\langle \bar{\mathcal{O}}_3 \mathcal{O}_1 \mathcal{O}_2 \rangle = \frac{4\pi}{-\Delta_3 + \Delta_1 + \Delta_2} \sqrt{\frac{J_1 J_2}{J}} H_{123}$$

where

$$\Delta_i = J_i + 2\sqrt{1 + \lambda' n_i^2},$$

and the matrix element is defined as

$$H_{123} = \langle 123|V\rangle.$$

Dobashi–Yoneya prefactor

We use the findings of [Grignani et al. 2006] to start with the Dobashi–Yoneya prefactor [Dobashi, Yoneya 2004] in the natural string basis $\{a_m^r\}$.

$$V = P\, e^{\frac{1}{2}\sum_{m,n} N^{rs}_{mn}\delta^{IJ} a_m^{rI\dagger} a_n^{sJ\dagger}}.$$

Here I, J are $SU(4)$ flavour indices, r, s run within 1, 2, 3 and refer to the first, second and third operator. The natural string basis is related to the BMN basis for $m > 0$ as follows

$$\alpha_m = \frac{a_m + ia_{-m}}{\sqrt{2}},\ \alpha_{-m} = \frac{a_m - ia_{-m}}{\sqrt{2}}$$

The Neumann matrices are given as

$$N^{rs}_{m,n} = \frac{1}{2\pi}\frac{(-1)^{r(m+1)+s(n+1)}}{x_s\omega_{rm} + x_r\omega_{sn}}\sqrt{\frac{x_r x_s(\omega_{rm}+\mu x_r)(\omega_{sn}+\mu x_s)s_{rm}s_{qn}}{\omega_{rm}\omega_{sn}}},$$

$$N^{rs}_{-m,-n} = -\frac{1}{2\pi}\frac{(-1)^{r(m+1)+s(n+1)}}{x_s\omega_{rm} + x_r\omega_{sn}}\sqrt{\frac{x_r x_s(\omega_{rm}-\mu x_r)(\omega_{sn}-\mu x_s)s_{rm}s_{qn}}{\omega_{rm}\omega_{sn}}},$$

where m, n are always meant positive, $s_{1m} = 1$, $s_{2m} = 1$, $s_{3m} = -2\sin(\pi m y)$, $x_1 = y$, $x_2 = 1 - y$, $x_3 = -1$, the frequencies are $\omega_{r,m} = \sqrt{m^2 + \mu^2 x_r^2}$,, and the expansion parameter is $\mu = \frac{1}{\sqrt{\lambda'}}$. The Dobashi-Yoneya prefactor we are using is the prefactor supported with positive modes only:

$$P = \sum_{m>0}\sum_{r,I}\frac{\omega_r}{\mu\alpha_r} a_m^{Ir\dagger} a_m^{Ir}.$$

String result

Due to the flavour structure of C_{123} the only combinations of terms from the exponent that could contribute are $N^{12}_{n_1 n_2} N^{23}_{n_2 n_3} N^{31}_{n_3 n_1}$. The leading order contribution is

$$C^0_{123} = \frac{1}{\pi^2} \frac{\sqrt{J}}{N} \frac{n_3^2 y^{3/2} (1-y)^{3/2} \sin^2(\pi n_3 y)}{(n_3^2 y^2 - n_1^2)(n_3^2(1-y)^2 - n_2^2)}$$

The next-order coefficient in the expansion

$$C_{123} = C^0_{123} \left(1 + \lambda' c^1_{123}\right),$$

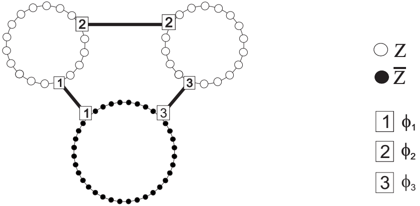
where $c^1_{123} \equiv \frac{C^1_{123}}{C^0_{123}}$ is

$$c^1_{123} = -\frac{1}{4} \left(\frac{n_1^2}{y^2} + \frac{n_2^2}{(1-y)^2} + n_3^2 \right).$$

Let us compare this calculation to the field theory calculation.

Leading Order

The tree-level diagram is shown below:



and evaluates in the leading order to

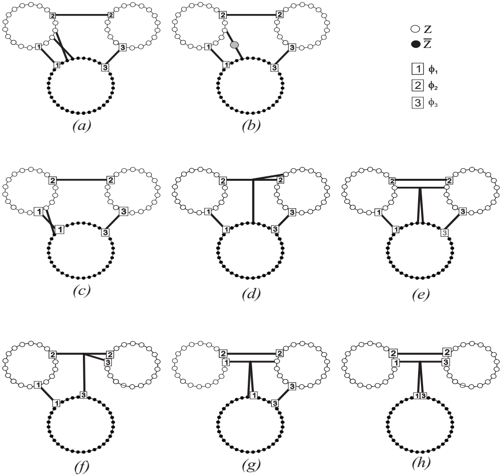
$$N\sqrt{J_1 J_2 J}\sum_{l_1, l_2} \cos\frac{\pi(2l_1+1)}{J_1+1}\cos\frac{\pi(2l_2+1)}{J_2+1}\cos\frac{\pi(2(l_1+l_2)+1)}{J+1},$$

which after the $1/J$ expansion and the due normalization of the operator to unity yields

$$C^0_{123} = \frac{1}{\pi^2}\frac{\sqrt{J}}{N}\frac{n_3^2 y^{3/2}(1-y)^{3/2}\sin^2(\pi n_3 y)}{(n_3^2 y^2 - n_1^2)(n_3^2(1-y)^2 - n_2^2)},$$

corresponding exactly to the ST result above.

At the one loop level we estimate all possible insertions of the interaction terms of the Hamiltonian $H_2 = \frac{\lambda}{8\pi^2}(I - P)$ depicted below:



One loop result from FT

After summation (details not shown here) we get

$$c^1_{123} = -\frac{1}{4}\left(\frac{n_1^2}{y^2} + \frac{n_2^2}{(1-y)^2} + n_3^2\right).$$

exactly as in the string theory above.

Consider a set of operators $\mathcal{O}_A$ normalized to unity

$$\langle \mathcal{O}_A(x)\bar{\mathcal{O}}_A(0)\rangle = \frac{1}{x^{2\Delta_A}}\,.$$

The space-time dependence of any three-point function is prescribed by conformal symmetry to be

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\bar{\mathcal{O}}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}\,.$$

The general expression for the structure constant arising from the EGSV procedure then is

$$N_c C_{123} = \sum_{\text{Root partitions}} \text{Cut} \times \text{Flip} \times \text{Norm} \times \text{Scalar products}\,.$$

The most natural generalization of the EGSV formula to a general group with Cartan matrix $M_{a_i b_j}$ follows from replacing the factors f, g, S in their expressions by their analogs in higher sectors

$$f(u_i,u_j) = 1 + \frac{iM_{a_i a_j}}{2(u_i-u_j)}\,,$$
$$g(u_i,u_j) = \frac{iM_{a_i a_j}}{2(u_i-u_j)}\,.$$
$$S(u,v) = \frac{f(u,v)}{f(v,u)}$$

The holonomy factors $a(u), d(u)$ retain their standard definitions for higher levels $a(u_j) = u_j + iV_{a_j}/2$, $d(u_j) = u_j - iV_{a_j}/2$, $e(u) = \frac{a(u)}{d(u)}$ so that the Bethe equations have the known form

$$\left(\frac{u_j - iV_{a_j}/2}{u_j + iV_{a_j}/2}\right)^L = \prod_{\substack{k=1\\k\neq j}}^{K} \frac{u_j - u_k - \frac{i}{2}M_{a_j a_k}}{u_j - u_k + \frac{i}{2}M_{a_j a_k}} \qquad (1)$$

Notations

Following EGSV we introduce useful shorthand notation for products of functions: for an arbitrary function $F(u, v)$ of two variables and for arbitrary sets α, $\bar{\alpha}$ of lengths K, $\bar{K}$, $\alpha = \{\alpha_i\}_K$, $\bar{\alpha} = \{\bar{\alpha}_i\}_{\bar{K}}$

$$
\begin{aligned}
F^{\alpha,\bar{\alpha}} &= \prod_{i,j} F^{\alpha_i,\bar{\alpha}_j}, \\
F^{\alpha,\alpha}_{<} &= \prod_{i<j} F^{\alpha_i,\alpha_j}, \\
F^{\alpha,\alpha}_{>} &= \prod_{i>j} F^{\alpha_i,\alpha_j}.
\end{aligned}
$$

For functions $G(u)$ of a single variable let us define

$$
\begin{aligned}
G^{\alpha} &= \prod_j F^{\alpha_j}, \\
G^{\alpha \pm i/2} &= \prod_j F^{\alpha_j \pm i/2}.
\end{aligned}
$$

Let us take three Bethe vectors u, v, w of lengths L_1, L_2, L_3, corresponding to the operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and split each of them into two pieces so that the rapidities are such that $\alpha \cup \bar{\alpha} = u$, $\beta \cup \bar{\beta} = v$, $\gamma \cup \bar{\gamma} = w$. The lengths $L_{\bar{\alpha}}$, L_{α}, $L_{\bar{\beta}}$, L_{β}, $L_{\bar{\gamma}}$, L_{γ} of these pieces are uniquely defined by the possible contraction structures:

$$
\begin{aligned}
L_{\alpha} &= L_{\bar{\beta}} = L_1 + L_2 - L_3, \\
L_{\beta} &= L_{\bar{\gamma}} = L_2 + L_3 - L_1, \\
L_{\gamma} &= L_{\bar{\alpha}} = L_3 + L_1 - L_2.
\end{aligned}
$$

Main Conjecture

The three-point function will look like

$$N_c C_{123} = \sum_{\substack{\alpha \cup \bar{\alpha} = u \\ \beta \cup \bar{\beta} = v \\ \gamma \cup \bar{\gamma} = w}} \sqrt{L_1 L_2 L_3}\ \mathtt{Cut}(\alpha, \bar{\alpha})\mathtt{Cut}(\beta, \bar{\beta})\mathtt{Cut}(\gamma, \bar{\gamma}) \times \mathtt{Flip}(\bar{\alpha})\mathtt{Flip}(\bar{\beta})\mathtt{Flip}(\bar{\gamma}) \times$$
$$\times \frac{1}{\sqrt{\mathtt{Norm}(u)\mathtt{Norm}(v)\mathtt{Norm}(w)}} \times \langle \alpha \bar{\beta} \rangle \langle \beta \bar{\gamma} \rangle \langle \gamma \bar{\alpha} \rangle \, .$$

We work in the "coordinate" normalization, where the $\mathtt{Cut}(\alpha, \bar{\alpha})$ factor is organized as

$$\mathtt{Cut}(\alpha, \bar{\alpha}) = \left(\frac{a^{\bar{\alpha}}}{d^{\bar{\alpha}}} \right)^{L_1} \frac{f^{\alpha \bar{\alpha}} f_<^{\bar{\alpha} \bar{\alpha}} f_<^{\alpha \alpha}}{f_<^{uu}} ,$$

the factors $\mathtt{Cut}(\beta, \bar{\beta})$ and $\mathtt{Cut}(\gamma, \bar{\gamma})$ being analogous to the expression above. The a, d, f, g factors are all defined in terms of Bethe Ansatz with higher-level states taken into account as well. In similar terms the flip factor may now be rewritten as

$$\mathtt{Flip}(\bar{\alpha}) = (e^{\bar{\alpha}})^{L}_{\bar{\alpha}} \frac{g^{\bar{\alpha} - i/2}}{g^{\bar{\alpha} + i/2}} \frac{f_>^{\bar{\alpha} \bar{\alpha}}}{f_<^{\bar{\alpha} \bar{\alpha}}} ,$$

analogous expressions work for $\mathtt{Flip}(\bar{\beta})$ and $\mathtt{Flip}(\bar{\gamma})$. The norm can also be generalized directly from eq. (5.2) in *[Escobedo 2010]* and in the coordinate normalization we get

$$\mathcal{N}(u) = d^u a^u f_>^{uu} f_<^{uu} \frac{1}{g^{u+i/2} g^{u-i/2}} \det(\partial_j \phi_k) ,$$

here $\partial_j = \frac{\partial}{\partial u_j}$ and the phases are the ratio of the left and right sides of the Bethe equations

$$e^{i\phi_j} = e(u_j)^{L_u} \prod_{k \neq j} S^{-1}(u_j, u_k).$$

Problem with Straightforward $SU(2) \to SO(6)$ Generalization

The remaining factor necessary to construct the correlator is the scalar product. Considering the expression for the scalar product of EGSV

$$\langle v|u\rangle = g_<^{uu} g_>^{vv} \frac{1}{d^u a^{v^*} g^{u+i/2} g^{v^*-i/2} f_<^{uu} f_>^{v^* v^*}} \times$$
$$\times \sum_{\alpha\cup\bar\alpha = u \beta\cup\bar\beta = v} (-1)^{P_\alpha + P_\gamma} (d^\alpha)^{L_V} (a^{\bar\alpha})^{L_V} (a^\gamma)^{L_V} (d^{\bar\gamma})^{L_V} \times$$
$$\times\, h^{\alpha\gamma} h^{\bar\gamma\alpha} h^{\alpha\bar\alpha} h^{\bar\gamma\gamma} \det t^{\alpha\gamma} t^{\bar\gamma\bar\alpha},$$

where $t(u) = g^2(u)/f(u)$, and trying to extend the definition towards the $SO(6)$ sector of the factor h as defined by EGSV

$$h(u) = \frac{f(u)}{g(u)} ,$$

the result one gets is not well defined. In fact h does not have a direct physical meaning unlike f and g which are taken directly from the R-matrix. A factor h defined as above would be meaningless since it would then contain division by zero.

Resolution

To circumvent this problem we formulate the $SO(6)$ norm conjecture via the recursive relation proposed in *[Escobedo 2010]*, eq.(A.5). This expression is completely regular and is formulated in terms of physically meaningful objects f, g, a, d, S, thus it makes full sense to conjecture that its validity extends towards a broader sector. The meaning of this formula goes beyond the original $SU(2)$ and is supposed to cover the full $SO(6)$

$$\langle v_1 \dots v_N | u_1 \dots u_N \rangle_N = \sum_n b_n \langle v_1 \dots \hat{v}_n \dots v_N | \hat{u}_1 \dots u_N \rangle_{N-1} - \\ - \sum_{n<m} c_{n,m} \langle u - 1 v_1 \dots \hat{v}_n \dots \hat{v}_m \dots v_N | \hat{u}_1 \dots u_N \rangle_{N-1},$$

where

$$b_n = \frac{g(u_1 - v_n)\left(\prod_{j\neq n}^N f(u_1 - v_j) \prod_{j<n}^N S(v_j, v_n) - \frac{e(u_1)}{e(v_n)} \prod_{j\neq n} f(v_j - u_1) \prod_{j>n} S(v_n, v_j)\right)}{g(u_1 + i/2) g(v_n - i/2) \prod_{j\neq 1} f(u_1 - u_j)},$$

and

$$c_{n,m} = \frac{e(u_1) g(u_1 - i/2) g(u_1 - v_n) g(u_1 - v_m) \prod_{j\neq n,m} f(v_j - u_1)}{g(u_1 + i/2) g(v_n - i/2) g(v_m - i/2) \prod_{j\neq 1} f(u_1 - u_j)} \times \\ \times \left(\frac{S(v_m, v_n)}{e(v_n)} \prod_{j>n} S(v_n, v_j) \prod_{j<m} S(v_j, v_m) + \frac{d(v_m)}{a(v_n)} \prod_{j>m} S(v_m, v_j) \prod_{j<n} S(v_j, v_n) \right).$$

This will be our working proposal, which shall be checked in a specific example in the next section.

Integrability against Perturbation Theory Test

Let us introduce our states as Bethe states. We shall denote an N-root state as

$$\langle u| = \{\{u_1, l_1\}, \dots \{u_N, l_N\}\}$$

where u_i denotes the value of the rapidity and l_i the level of Bethe Ansatz it belongs to. The states corresponding to those studied in the first part of the work are

$$\begin{aligned}
\mathcal{O}_1 \sim \langle u| &= \{\{0,1\}, \{\tfrac{1}{2}\cot\tfrac{\pi n_1}{J_1+2}, 2\}, \{-\tfrac{1}{2}\cot\tfrac{\pi n_1}{J_1+2}, 2\}\}, \\
\mathcal{O}_2 \sim \langle v| &= \{\{0,3\}, \{\tfrac{1}{2}\cot\tfrac{\pi n_2}{J_2+2}, 2\}, \{-\tfrac{1}{2}\cot\tfrac{\pi n_2}{J_2+2}, 2\}\}, \\
\mathcal{O}_3 \sim \langle w| &= \{\{\tfrac{1}{2}\cot\tfrac{\pi n_3}{J+1}, 2\}, \{-\tfrac{1}{2}\cot\tfrac{\pi n_3}{J+1}, 2\}\}.
\end{aligned}$$

The lengths of the states are $L_1 = J_1 + 2$, $L_2 = J_2 + 2$, $L_3 = J + 2$. The lengths of substates (or, alternatively, the number of contractions between each ith and jth states) are $L_{12} = 1$, $L_{23} = J_2 + 1$, $L_{31} = J_1 + 1$. Expansion in $1/J$ is presumed everywhere below.

The Example

The flip and cut factors together are

$$\mathtt{Cut}(\alpha,\bar\alpha)\mathtt{Cut}(\beta,\bar\beta)\mathtt{Cut}(\gamma,\bar\gamma)\times\mathtt{Flip}(\bar\alpha)\mathtt{Flip}(\bar\beta)\mathtt{Flip}(\bar\gamma)=-1\ ,$$

the norms yield

$$\mathtt{Norm}(u)\mathtt{Norm}(v)\mathtt{Norm}(w)=4J^2n_1^2n_2^2\pi^4\ ,$$

and the scalar products read

$$\langle\alpha\bar\beta\rangle\langle\beta\bar\gamma\rangle\langle\gamma\bar\alpha\rangle=\frac{n_1n_2\sin^2(\pi n_3 r)}{2(n_1-rn_3)(n_2+(1-r)n_3)}\ .$$

The other contributing partitions in the leading order are realized by simple transformations $n_1\to-n_1$, $n_2\to-n_2$. There are also partitions that contribute at higher orders in $1/J$, which we do not list here. Taking all the pieces together we get

$$N_cC_{123}=-\frac{n_3^2J^{1/2}(r(1-r))^{3/2}\sin^2(\pi n_3 r)}{(n_2^2-n_3^2(1-r)^2)(n_1^2-n_3^2(1-r)^2)}\,,$$

which corresponds exactly to the results from the first part obtained both from perturbation theory and string field theory.

Discussion

For the first time in the $SO(6)$ sector we have explicitly demonstrated that for the three-point functions

- SFT at strong coupling identical with perturbation theory at small coupling in the Frolov-Tseytlin limit at one loop.
- Integrability-assisted resummation a la Escobedo-Gromov-Sever-Vieira can be successfully generalized to the $SO(6)$ case and is shown to be identical with SFT and perturbation theory.

Given these correspondences, discussion can be raised:

- To which extent may these equalities be understood as cooincidences?
- How essential is the role of Frolov-Tseytlin limit? To which order will the equalities hold beyond it