From spin chains to sigma models

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'A haven of geometry in an ocean of algebra'
Unknown author

arXiv:1104.1419 and arXiv:1206.2777

Nonlinear σ-models

- Entered physics via low-energy QCD with the work of M.Gell-Mann and M.Levy (1960)
- Describe the scattering of Goldstone bosons in 4D, for example π-mesons in the case $\frac{SU(2)\times SU(2)}{SU(2)}$ $(u, d$ quarks)
- Renormalizable in 2D

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  Therefore provides a method for the exploration of target-space geometry
- The θ-term / the WZNW term $\Rightarrow$ Quantization of amplitudes
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From a spin chain to a σ-model

- QFT requires regularization preserving the symmetries
- Natural to discretize both $\boldsymbol{\Sigma}$ and $\boldsymbol{\mathcal{M}}$ $\Rightarrow$ spin chain
- The ferromagnetic vacuum $\Rightarrow$ Nonrelativistic models a-la the continuous Heisenberg ferromagnet
- The antiferromagnetic vacuum = the ground state, spinons = particles

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From a spin chain to a σ-model. II.

- F.D.M.Haldane, 1983
  $SU(2)$, representation of spin $s = \frac{m}{2}$ on $\mathrm{Sym}(V_{\mathrm{fund}}^{\otimes m})$ with Hamiltonian
  $$\mathcal{H} = \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$
- Long-range correlations of the spin chain in the large s limit are described by the σ-model with $\mathcal{M} = S^2$ and the topological term
  $$\Omega = \frac{\theta}{2\pi i} \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2}, \qquad \theta = \pi m$$

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The goal / result

- The goal is to construct the spin chain with target space $\frac{U(N)}{U(n_1)\times\cdots\times U(n_m)}$.

The Hamiltonian

$$H = \sum_{i=1}^{L} \sum_{k=1}^{m-1} d_k \, \vec{S}_i \cdot \vec{S}_{i+k}$$

where $d_k = \sqrt{\frac{m-k}{k}}$

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The path integral for the spin chain

- Build a path integral for the spin chain partition function → convenient for the semiclassical and continuum limits.
- The construction of coherent states for simple groups Berezin, Perelomov, mid-70's
- Consider representation V. Take $|w\rangle \in V$ and form an orbit $G\,|w\rangle$.
- Example: $SU(2) \Rightarrow \mathbf{CP}^1$:

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$$|w\rangle = z_1^m \Rightarrow \phi_v(z) = \left(\bar{v}_1 z_1 + \bar{v}_2 z_2\right)^m, \; {}_{v \in \mathbf{CP}^1}$$

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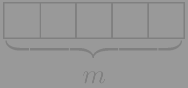

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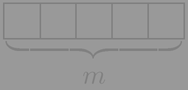

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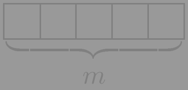

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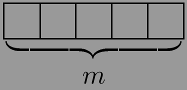

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The path integral for the spin chain. 2.

- The path integral for $\mathrm{tr}\left(e^{-\beta H}\right)$ is built by splitting the 'time' segment of length β into an infinite number of pieces ($K \to \infty$):

$$\mathrm{tr}\left(e^{-\beta H}\right)=\mathrm{tr}\ \lim\left(1-\frac{\beta}{K}H\right)^{K}=$$

$$=\lim\int\prod_{i=1}^{K-1}d\mu(z_i,\bar{z}_i)\ \ \tau(q,\bar{z}_{K-1})\tau(z_{K-1},\bar{z}_{K-2})\ldots\tau(z_2,\bar{z}_1)\tau(z_1,\bar{y})\times$$

$$\times\ \frac{(\phi_{\bar{y}},\phi_{\bar{z}_1})(\phi_{\bar{z}_1},\phi_{\bar{z}_2})\cdots(\phi_{\bar{z}_{K-2}},\phi_{\bar{z}_{K-1}})(\phi_{\bar{z}_{K-1}},\phi_{\bar{q}})}{(\phi_{\bar{y}},\phi_{\bar{q}})(\phi_{\bar{z}_1},\phi_{\bar{z}_1})\cdots(\phi_{\bar{z}_{K-1}},\phi_{\bar{z}_{K-1}})}$$

where

$$\tau(z_{k+1},\bar{z}_k)=1-\frac{\beta}{K}\,\mathcal{H}(z_{k+1},\bar{z}_k),\ \ (\phi_{\bar{z}_k},\phi_{\bar{z}_{k+1}})=(z_k\circ\bar{z}_{k+1})^{m}$$

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The path integral for the spin chain. 3.

- Aiming at an expression of the following form:

$$\mathcal{Z} = \int \prod_{t\in[0,1]} d\mu(z(t),\bar{z}(t)) \; \exp\left(-\mathcal{S}\right),$$

where $z \in \mathbf{CP}^{N-1}$ and

$$S = m \int_0^1 dt \sum_i \left(i\frac{\dot{z}_i \circ \bar{z}_i}{z_i \circ \bar{z}_i} + \beta \left| \frac{z_i \circ \bar{z}_{i+1}}{z_i \circ \bar{z}_i} \right|^2 \right)$$

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The path integral for the spin chain. 4.

- Let us have a closer look at the action

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- The kinetic term

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- The Kähler current, i.e. $j : dj = \omega$ — the Fubini-Study form (1D WZNW term)

The path integral for the spin chain. 4.

- The potential term

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- The angle between two vectors in $\mathbf{C}^{N+1}$

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- We see therefore that the whole object is geometric!

Elements of symplectic geometry

- Phase space $\mathcal{N}$ is a symplectic manifold.
- ω is a non-degenerate closed 2-form: $d\omega = 0$
- $G \circlearrowright \mathcal{N}$, the moment map $\mu : \mathcal{N} \to \mathfrak{g}$
- Equivariance:
  $$\mu(g \circ x) = Ad_g \, \mu(x) \equiv g\mu(x)g^{-1}$$
- Hamiltonians: $d\mu_a = i_{X_a}\omega, \quad a \in \mathfrak{g}$
- Simplest example: angular momentum

$$\mathcal{N} = \mathbb{R}^6, \, G = SO(3), \, \omega = d\vec{r} \wedge d\vec{p}, \, \vec{L} = \vec{r} \times \vec{p}.$$

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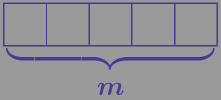
- Let V be a representation of $U(N)$ with highest weight $\vec{\lambda} = (\lambda_1, \cdots, \lambda_N)$.
- It can be built on the space of sections of a holomorphic fiber bundle

$$L_\lambda = \mathcal{O}_1(\lambda_1) \otimes \cdots \otimes \mathcal{O}_N(\lambda_N) \to U(N)/U(1)^N$$

- A generalization of the fact that for $SU(N)$  leads to symmetric polynomials of degree m in N variables (viewed as sections of $\mathcal{O}(m)$)

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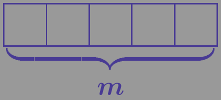

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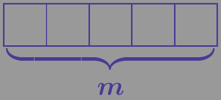

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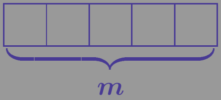
$$L_\lambda = \mathcal{O}_1(\lambda_1) \otimes \cdots \otimes \mathcal{O}_N(\lambda_N) \to U(N)/U(1)^N$$

- A generalization of the fact that for $SU(N)$


leads to symmetric polynomials of degree m in N variables (viewed as sections of $\mathcal{O}(m)$)

The A.Borel-A.Weil-R.Bott theorem. 2.

- Simple way to deal with it: use the embedding

$$i: \ \mathcal{F}_N \hookrightarrow \underbrace{\mathbf{CP}^{N-1} \times \cdots \times \mathbf{CP}^{N-1}}_{N \text{ times}}.$$

$\mathcal{F}_N$ is the space of N orthogonal lines in $\mathbf{C}^N$

- The first Chern class of the line bundle:

$$c_1(L_\lambda) = \mathrm{i}^*(\tilde{L}_\lambda) = \mathrm{i}^*(\sum_{i=1}^{N} \lambda_i \omega_i) = \sum_{i=1}^{N} \lambda_i \Omega_i$$

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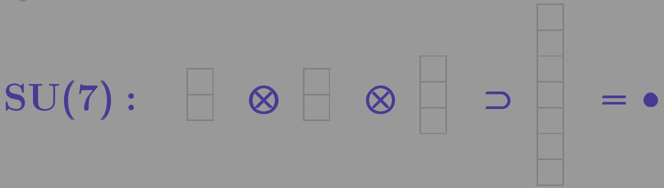
The semiclassical picture of the antiferromagnetic vacuum

- Getting a singlet from a tensor product of representations



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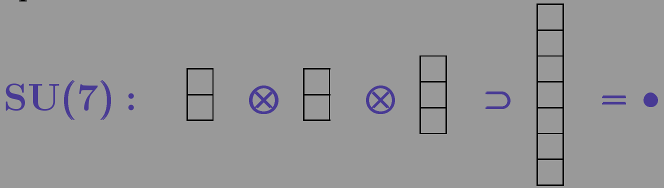

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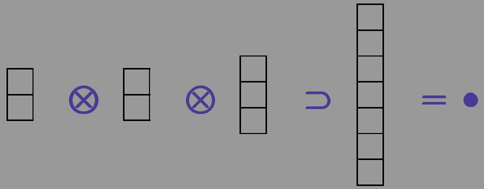
$\mathbf{SU(7):}$



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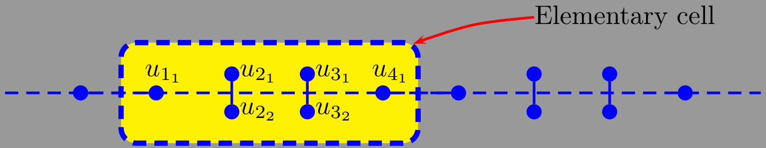
A picture of the spin chain



The spin chain for the coset $\frac{U(6)}{U(2)\times U(2)\times U(1)\times U(1)}$

On the choice of Hamiltonian: an example

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- The minimum $\mathcal{H}_{i,i+1} = 0$: when $z_i \circ \bar{z}_{i+1} = 0$ — the AF 'vacuum'. The space of solutions is $\mathbf{CP}^1$, but it is a Lagrangian submanifold inside $\mathbf{CP}^1 \times \mathbf{CP}^1$
- The equation above is equivalent to the statement that the moment map $\mu = \frac{\bar{z}_i \otimes z_i}{\bar{z}_i \circ z_i} + \frac{\bar{z}_{i+1} \otimes z_{i+1}}{\bar{z}_{i+1} \circ z_{i+1}} - 1$ is zero: $\mu = 0$.
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One more property of the moment map

- $\mu^{-1}(0)$ is invariant under $U(N)$ (recall that $\mu(g \circ x) = g\,\mu(x)\,g^{-1}$) $\Rightarrow$ splits into orbits.
- When a nondegeneracy condition is fulfilled,

$$\mu^{-1}(0) \text{ is a single orbit} \iff \mu^{-1}(0) \text{ is a Lagrangian submanifold}$$

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The general setup

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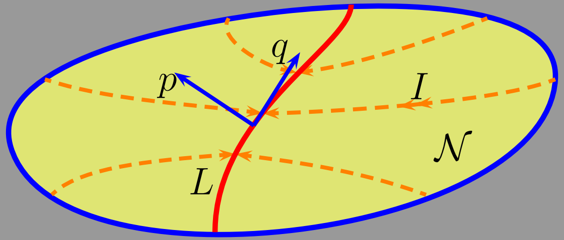

- Consider a function $\boldsymbol{I}$ which has a minimum on a Lagrangian submanifold $\boldsymbol{L} \subset \boldsymbol{\mathcal{N}}$.
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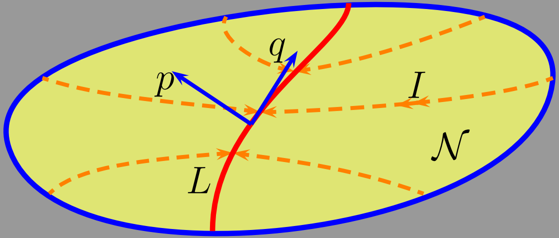

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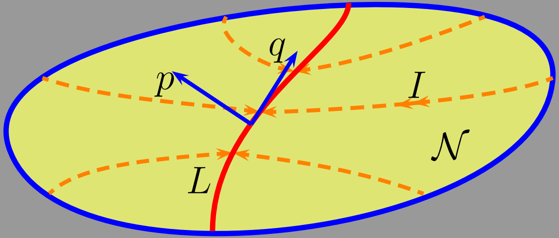

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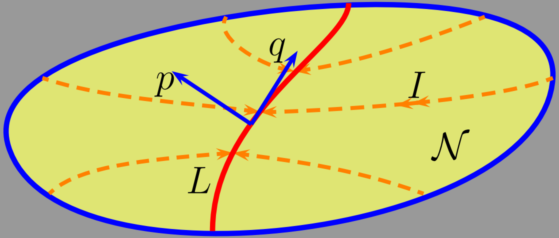

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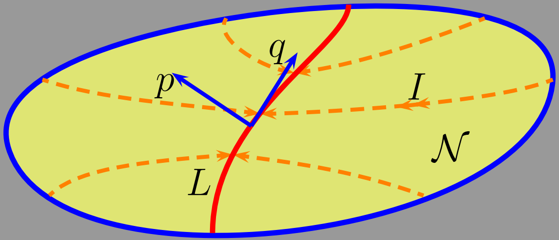

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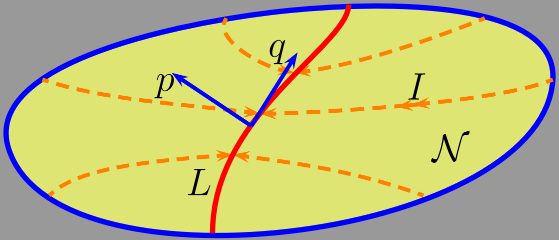

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The metric

- The metric in the normal directions to L: the Hessian $h_{ij} = \frac{\partial^2 I}{\partial x^i \, \partial x^j}$

The metric on L

$$g_{ij} = \omega_{im} \cdot \left[\left(\frac{\partial^2 I}{\partial x^2} \right)^{-1} \right]^{mn} \cdot \omega_{nj} = \omega_{im} \, h^{mn} \, \omega_{nj}$$

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Why is the metric well-defined?

- Take a symmetric *degenerate* matrix $\boldsymbol{B}, \ \mathbf{det}\boldsymbol{B} = \mathbf{0}$
- If $\boldsymbol{u}, \boldsymbol{v} \perp \mathbf{ker}\boldsymbol{B}$, then the matrix element $\langle \boldsymbol{u} | \boldsymbol{B}^{-1} | \boldsymbol{v} \rangle$ makes sense: $\boldsymbol{B}^{-1} | \boldsymbol{v} \rangle \equiv | \boldsymbol{w} \rangle \ : \ \boldsymbol{B} | \boldsymbol{w} \rangle = | \boldsymbol{v} \rangle$
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- Ingredients for the θ-term (the elements of $H^2(\mathcal{F}_{n_1,\cdots,n_m})$): $r_k = i^*\big(c_1(\mathcal{O}_{G_{n_k}}(1))\big), \quad \sum_{k=1}^m r_k = 0$.
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- Hence $\theta = \frac{2\pi}{m}$. Permuting the sites of the spin chain changes the θ-term in $\mathbf{H}^2(\mathcal{M}, \mathbf{Z}_m)$!
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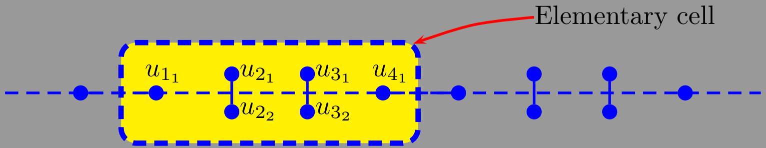
The picture revisited



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The Hamiltonian

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where $d_k = \sqrt{\frac{m-k}{k}}$

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where $d_k = \sqrt{\frac{m-k}{k}}$

- Depends on m but not on the partition $n_1, \cdots, n_m$

The result revisited

- The goal was to construct the spin chain with target space $\frac{U(N)}{U(n_1)\times\cdots\times U(n_m)}$.

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Questions / Answers

- Constructed a spin chain for a σ-model with target space a flag manifold.
- Universal expressions for the metric and θ-term.
- Is there a **mod m** periodicity of the mass gap?
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Thank you!