# The quark anti-quark potential in $\mathcal{N} = 4$ SYM from a TBA equation

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Based on arXives: 1202.4455, 1203.1019 and 1203.1913

In collaboration with: J. Henn, J. Maldacena and A. Sever

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  2. WL sets open boundaries: determine the reflection matrix $R^a{}_b$
  3. Global rotation of one of the $R^a{}_b$ to introduce cusp angles
  4. TBA to incorporate finite size effects
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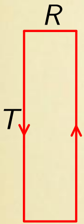
This is valid in the planar limit of $\mathcal{N} = 4$ SYM and for any value of the 't Hooft coupling $\lambda$.

# Introduction

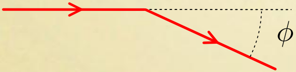
- Quark-antiquark potential




for $T \gg R$

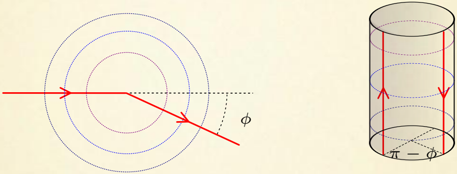
$$e^{-V_{q\bar{q}}(R)T} = \langle \mathrm{Tr} \left[ P e^{i \oint A.dx} \right] \rangle$$

- Cusp anomalous dimension [Polyakov 80]




$$e^{-\Gamma_{\mathrm{cusp}}(\phi) \log(\frac{\Lambda_{\mathrm{IR}}}{\Lambda_{\mathrm{UV}}})} = \langle \mathrm{Tr} \left[ P e^{i \oint A.dx} \right] \rangle$$

- $\Gamma_{\rm cusp}(\phi)$ gives the **quark anti-quark potential** on $S^3$ for a configuration which is separated by an angle $\delta = \pi - \phi$.




Plane to cylinder map ($\log r = t$)

$$\langle W \rangle \simeq e^{-\log(\frac{\Lambda_{IR}}{\Lambda_{UV}})\Gamma_{\rm cusp}} = e^{-T\Gamma_{\rm cusp}} \quad \Rightarrow \quad \Gamma_{\rm cusp} = V_{q\bar{q}}$$

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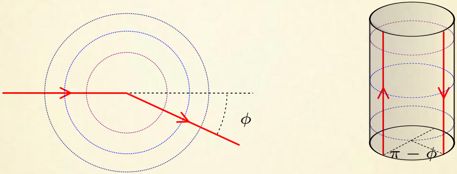


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- In $\mathcal{N} = 4$ SYM, the locally susy Wilson loop also has a coupling to the scalars, specified by $\vec{n}$

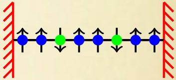
$$W \sim \mathrm{Tr}\left[ P e^{i\oint A\cdot dx + \oint |dx| \vec{n}\cdot\vec{\Phi}} \right]$$

We can take $\vec{n}$ and $\vec{n}'$ for the 2 lines of the cusp. This introduces an internal cusp angle $\cos\theta = \vec{n}\cdot\vec{n}'$.

# Open chain spectral problem

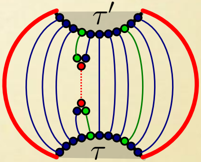
- WL with fields inserted regarded as open spin chain states

$$\mathrm{Tr}\left[P\mathcal{O}(\tau)e^{\oint d\tau(iA_\tau+\vec{n}\vec{\Phi})}\right] = \longrightarrow\!\!\!-(\overbrace{ZZYZZYZZ}^{\mathcal{O}(\tau)})\longrightarrow\!\!\!- \quad \longleftrightarrow$$



- Computing $\langle W[\mathcal{O}(\tau)\mathcal{O}(\tau')]\rangle$ perturbatively leads to a mixing problem which is equivalent to some open spin chain spectral problem,

$$\langle W[\mathcal{O}_A^{ren}(\tau)\mathcal{O}_B^{ren}(\tau')]\rangle = \frac{\delta_{AB}}{|\tau-\tau'|^{2\Delta_A}}$$




- For instance, to 1-loop in an $su(2)$ sector, an integrable open XXX chain is obtained [Drukker,Kawamoto]

- This problem is argued to be integrable to all-loop order: Find all-loop reflection matrix & check BYB is satisfied

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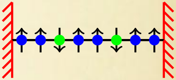

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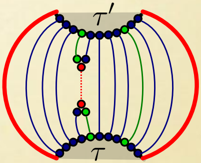


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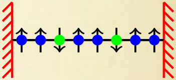

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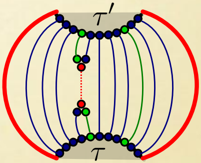


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# Wilson loop reflection matrix

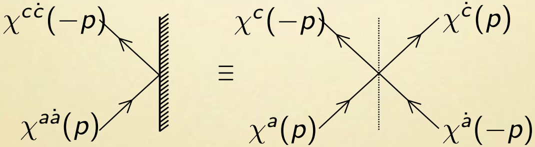
- Magnons are fundametals of $SU(2|2)_L \times SU(2|2)_R$

$$\longrightarrow (ZZZ\chi^{a\dot{a}}ZZZ) \longrightarrow$$

$a$ is a fund. of $SU(2|2)_L$
$\dot{a}$ is a fund. of $SU(2|2)_R$

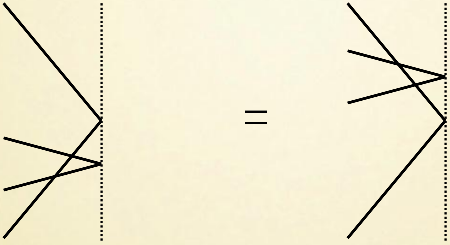
- Reflection matrix is fixed with the boundary/vacuum symm.
$SU(2|2)_D = SU(2|2)^2 \cap OSp(4^*|4)$ [Correa, Young], [Correa, Regelskis, Young] [Correa, Maldacena, Sever] &[Drukker]

1 Bulk magnon $\equiv$ 1 pair of magnons of $SU(2|2)_D$




$$R^{a\dot{a}}_{c\dot{c}}(p) = \frac{1}{\sigma_B(p)} \frac{1}{\sigma(p,-p)} \hat{S}^{a\dot{a}}_{c\dot{c}}(p,-p)$$

# Boundary Yang-Baxter condition



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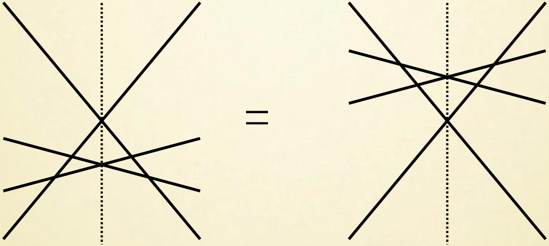
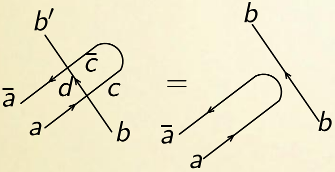


Up to some overall scalar factors, the boundary Yang-Baxter condition looks like a succession of bulk scattering factors

Thus, bulk Yang-Baxter condition ensures boundary Yang-Baxter condition for the Wilson loop reflection matrix

# Wilson loop boundary dressing phase

- Crossing symmetry constrains the unknown function [Janik 06]
  **Crossing:** particle $(E, p)$ $\leftrightarrow$ anti-particle $(-E, -p)$




scatt. with a singlet is trivial[Beisert]

$$S^{cd}_{ab}(p,q)\mathcal{C}_{c\bar{c}}S^{\bar{c}b'}_{\bar{a}'d}(\bar{p},q) = \mathcal{C}_{a\bar{a}}\delta_{bb'}$$

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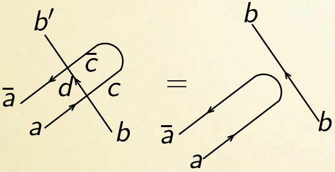


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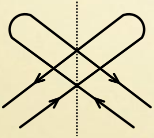
- There is also a boundary crossing condition



$$R(p)\cdot S(p,-\bar{p})\cdot R(\bar{p}) = \mathbb{1}$$

This imposes a condition on $\sigma_B$ $\quad (16\pi^2 g^2 = \lambda)$

$$\sigma_B(p)\sigma_B(\bar{p}) = \frac{x^- + \frac{1}{x^-}}{x^+ + \frac{1}{x^+}} \qquad x^\pm := x(u \pm \tfrac{i}{2}), \quad x(u) + \frac{1}{x(u)} = \frac{u}{g}$$

- The solution to this crossing equation is not unique
- Applying a method proposed for the bulk dressing factor [Volin] & [Volin,Vieira], we found the following solution [Correa,Maldacena,Sever] & [Drukker]

$$\sigma_B = e^{i\chi(x^+)-i\chi(x^-)}$$

$$\chi(x) = \Phi(x) = \oint \frac{dz}{2\pi}\frac{1}{z-x}\log\left\{\frac{\sinh[2\pi g(z+\frac{1}{z})]}{2\pi g(z+\frac{1}{z})}\right\}, \quad |x|>1$$

$$\chi(x) = \Phi(x) - i\log\left\{\frac{\sinh[2\pi g(x+\frac{1}{x})]}{2\pi g(x+\frac{1}{x})}\right\}, \quad |x|<1$$

which passes a few non-trivial checks

# Strong coupling dressing phase check

In the strong coupling limit, (when $x^{\pm} = e^{\pm i \frac{ip}{2}}$),the boundary scattering phase we have proposed:

$$R_0(p) = \frac{1}{\sigma_B(p)} \frac{1}{\sigma(p,-p)} = e^{i\delta_{\mathrm{R}}(p)}$$

goes as

$$\delta_{\mathrm{R}}(p) = -\frac{\sqrt{\lambda}}{\pi} \cos\frac{p}{2} \log\left(\frac{1-\sin\frac{p}{2}}{1+\sin\frac{p}{2}}\right) - \frac{2\sqrt{\lambda}}{\pi} \cos\frac{p}{2} \log\cos\frac{p}{2}$$

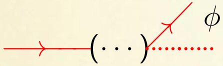
This coincides exactly with the classical string computation.

One computes the time delay $\Delta t$ suffered for magnon during the reflection. The time delay is related to the derivative of the reflection phase with respect to the energy [Jackiw,Woo 75]

$$\Delta t = \frac{\partial \delta}{\partial \epsilon}$$

Now that we know the reflection matrix, let's continue with the steps enumerated in the outline

3. Introduce cusps: globally rotate the right reflection matrix

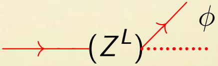



$$R^a_c(\phi) = m^a_b(\phi) R^b_c$$

4. Finite $L$ corrections $\to$ Thermodynamic Bethe ansatz

5. Focus on the ground state in the limit $L \to 0$

$\lim_{L\to 0}$



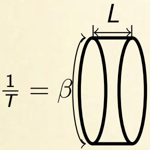

The limit $L \to 0$ of the Casimir energy gives the cusp anomalous dimension

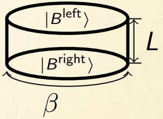
$$\Gamma_{\rm cusp} = \lim_{L\to 0} \mathcal{E}_0(L)$$

## Boundary Thermodynamic Bethe Ansatz

[Zamolodchikov 90]
[LeClair, Mussardo, Saleur, Skorik 95]

Physical strip $\left\{ \begin{array}{c} p \leftrightarrow i\tilde{E} \\ E \leftrightarrow i\tilde{p} \end{array} \right\}$ Mirror Theory




$$Z^{\text{open}}_{B_l,B_r} = \mathrm{Tr}_{\text{open}}[e^{-\beta H^{\text{open}}_{B_l,B_r}}] = \langle B_l | e^{-LH_{\text{closed}}} | B_r \rangle$$

- Analytic continuation of $R(p)$ gives the probability of emitting pairs of particles from the boundary state [Ghoshal, Zamolodchikov 93]

$$|B\rangle = \exp\left( \int_0^\infty \frac{d\tilde{p}}{2\pi} K^{a,b}(\tilde{p}) a^\dagger_a(-\tilde{p}) a^\dagger_b(\tilde{p}) \right) |0\rangle = \exp\left( \int_0^\infty \frac{d\tilde{p}}{2\pi} \cdots \right) |0\rangle$$

with $K^{a,b}(\tilde{p}) = \left[R^{-1}(\tilde{p})\right]^a_d C^{d,b}$ $\quad \tilde{p}$ has mirror kinematics

- In the $\beta \to \infty$ limit,

**(i)** Partition function $\to$ the ground state energy $Z^{\text{open}}_{B_l,B_r} \sim e^{-\beta \mathcal{E}_0(L)}$

**(ii)** Bethe Ansatz in the mirror theory becomes exact

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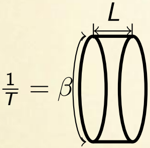


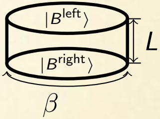


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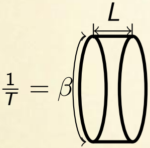


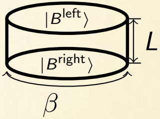


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# Partition Function in the mirror channel

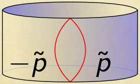
$$e^{-\beta \mathcal{E}_0(L)} \quad \sim \quad \langle B_l | e^{-LH_{\mathrm{closed}}} | B_r \rangle \quad \text{for } \beta \to \infty$$

- Still not straightforward. $|B\rangle$ is written as superpositions of $a^\dagger(\tilde{p})$ which are not eigenstates of $H_{\mathrm{closed}}$ (unless mirror S-matrix were trivial)

- Lüscher-type correction gives the leading finite size correction and can be obtained by regarding superpositions of $a^\dagger(\tilde{p})$ as eigenstates of $H_{\mathrm{closed}}$

The partition function is reduced to the overlap of the 2-particle, 4-particles,... components of $|B\rangle$

$$1 + \int_0^\infty \frac{d\tilde{p}}{2\pi} e^{-2L\tilde{E}(\tilde{p})}$$



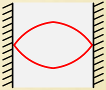

$$+ \cdots$$

This leads to [LeClair, Mussardo, Saleur, Skorik 95]

$$\mathcal{E}_0(L) \sim -\int_0^{\infty} \frac{d\tilde{p}}{2\pi} \log\left\{1 + e^{-2L\tilde{E}(\tilde{p})}\mathrm{Tr}[K(\tilde{p})\bar{K}(\tilde{p})]\right\}$$

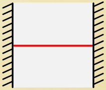
This can be expanded either as

$$\mathcal{E}_0(L) \sim -\int_0^{\infty} \frac{d\tilde{p}}{2\pi} e^{-2L\tilde{E}(\tilde{p})}\mathrm{Tr}[K(\tilde{p})\bar{K}(\tilde{p})] + \mathcal{O}(e^{-4L\tilde{E}(0)})$$



or as

$$\mathcal{E}_0(L) \sim -\frac{1}{2} e^{-L\tilde{E}(0)} \sqrt{\tilde{p}^2 \mathrm{Tr}[K(\tilde{p})\bar{K}(\tilde{p})]|_{\tilde{p}=0}} + \mathcal{O}(e^{-2L\tilde{E}(0)})$$



when $K\bar{K}$ has a double pole at $\tilde{p} = 0$

Our dressing phase $\sigma_B$ produces such pole, which we will see is crucial for getting the correct cusp anomalous dimension

# Boundary TBA derivation

- The mirror system is the same as the one obtained in the periodic case. [Arutyunov, Frolov], [Bombardelli, Fioravanti, Tateo] [Gromov, Kazakov, Kozak, Vieira]

  The difference is that now we overlap the Bethe eigenstates between the boundary states rather than tracing over them
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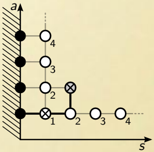
- Carrying mom. particles come in pairs with $(-\tilde{p}, \tilde{p})$
  $\Rightarrow$ $Y_{a,0}$ is needed for $u_4 > 0$ only ($\tilde{p} > 0$)

- Boundary state is invariant under $SU(2|2)_D$.

If roots $u_1, u_2, u_3$ appear, also $-u_7, -u_6, -u_5$ appear

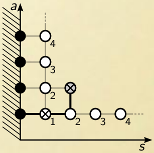



$\Rightarrow$ $Y_{a,-s}(u) = Y_{a,s}(-u)$

# Boundary TBA derivation

- The mirror system is the same as the one obtained in the periodic case. [Arutyunov, Frolov], [Bombardelli, Fioravanti, Tateo] [Gromov, Kazakov, Kozak, Vieira]

  The difference is that now we overlap the Bethe eigenstates between the boundary states rather than tracing over them
- Same mirror part. $\Rightarrow$ same Y-functions $Y_{a,s}$ $\left(\frac{\text{dens. of particles}}{\textit{dens. of holes}}\right)$
- Carrying mom. particles come in pairs with $(-\tilde{p}, \tilde{p})$
  $\Rightarrow$ $Y_{a,0}$ is needed for $u_4 > 0$ only $(\tilde{p} > 0)$
- Boundary state is invariant under $SU(2|2)_D$.




If roots $u_1, u_2, u_3$ appear, also $-u_7, -u_6, -u_5$ appear

$$\Rightarrow Y_{a,-s}(u) = Y_{a,s}(-u)$$

- Rotation: acts diagonally on the impurities of each level
  $\Rightarrow$ cusp angles $\phi$ and $\theta$ enter as chemical potentials for the $\neq$ magnon bound states
- Boundary dressing factor: $\sigma_B$ enter as a $u_4$ dependent chemical potential for the $Y_{a,0}$

# Ground state TBA equations

$$\log Y_{1,1} = i\theta + i\phi + K_{m-1} * \log \frac{1+\overline{Y}_{1,m}}{1+Y_{m,1}} + \mathcal{R}^{(01)}_{1\,a} * \log(1+Y_{a,0})$$

$$\log \overline{Y}_{2,2} = \ i\theta + i\phi + K_{m-1} * \log \frac{1+\overline{Y}_{1,m}}{1+Y_{m,1}} + \mathcal{B}^{(01)}_{1\,a} * \log(1+Y_{a,0})$$

$$\log \overline{Y}_{1,s} = 2i(s-1)\theta - K_{s-1,t-1} * \log(1+\overline{Y}_{1,t}) - K_{s-1} * \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}}$$

$$\log Y_{a,1} = i2(a-1)\phi - K_{a-1,b-1} * \log(1+Y_{b,1}) - K_{a-1} * \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}} +$$

$$+ \left[\mathcal{R}^{(01)}_{ab} + \mathcal{B}^{(01)}_{a-2,b}\right] * \log(1+Y_{b,0})$$

$$\log Y_{a,0} = -i2a\phi + \log[\sigma_B \bar{\sigma}_B] - 2L\tilde{E}_a(u) + \left[2\mathcal{S}_{a\,b} - \mathcal{R}^{(11)}_{a\,b} + \mathcal{B}^{(11)}_{a\,b}\right] * \log(1+Y_{b,0})$$

$$+ 2\left[\mathcal{R}^{(1\,0)}_{a\,b} + \mathcal{B}^{(1\,0)}_{a,b-2}\right] \underset{\text{sym}}{*} \log(1+Y_{b,1}) + 2\mathcal{R}^{(1\,0)}_{a\,1} \underset{\text{sym}}{*} \log(1+Y_{1,1}) - 2\mathcal{B}^{(1\,0)}_{a\,1} \underset{\text{sym}}{*} \log(1+\overline{Y}_{2,2})$$

- Same kernels as in periodic case TBA. $\overline{Y}_{a,s} = 1/Y_{a,s}$
- Apart from the folding symmetry and the boundary dressing factor $\sigma_B$, they are similar to the twisted boundary conditions TBA equations, [Arutyunov, deLeeuw, vanTongeren] , [Ahn, Bajnok, Bombardelli, Nepomechie] , [deLeeuw, vanTongeren]

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$$\log Y_{1,1} = i\theta + i\phi + K_{m-1} * \log \frac{1+\overline{Y}_{1,m}}{1+Y_{m,1}} + \mathcal{R}^{(01)}_{1\,a} * \log(1+Y_{a,0})$$

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- Recovering Lüscher: Throwing convolutions with $Y_{a,0}$

# Asymptotic solution to the TBA equations

$$Y_{1,1} = -\frac{\cos\theta}{\cos\phi}\,, \qquad Y_{1,s} = \frac{\sin[(s+1)\theta]\sin[(s-1)\theta]}{\sin^2\theta}$$

$$Y_{2,2} = -\frac{\cos\phi}{\cos\theta}\,, \qquad Y_{a,1} = \frac{\sin^2\phi}{\sin[(a+1)\phi]\sin[(a-1)\phi]}$$

$$Y_{a,0} = 4\sigma_B\bar{\sigma}_B\left(\frac{z^{[-a]}}{z^{[+a]}}\right)^{2L+2}(\cos\phi-\cos\theta)^2\,\frac{\sin^2 a\,\phi}{\sin^2\phi}$$

The ground state energy is

$$\mathcal{E}_0(L) = -\sum_{a=1}^{\infty}\int_0^{\infty}\frac{d\tilde{p}}{2\pi}\log\left(1+Y_{a,0}\right)$$

Since as $\tilde{p}\to 0$ we have $Y_{a,0}\sim\frac{G_a^2}{\tilde{p}^2}$,

$$\mathcal{E}_0(L) \sim -\frac{1}{2}\sum_{a=1}^{\infty}G_a$$

# Strong coupling check

- Large $L$ at strong coupling

$$\left(\frac{z^{[-a]}}{z^{[+a]}}\right)^{L+1}\Bigg|_{\tilde{p}=0} = e^{-(L+1)\tilde{E}_a}\Big|_{\tilde{p}=0} \sim e^{-\frac{aL}{2g}}$$

To leading order only $a = 1$ contributes

- Evaluating the dressing factors for $\tilde{p} \to 0$ and $g \to \infty$

$$\mathcal{E}_0(L) \sim (\cos\phi - \cos\theta)\frac{16g}{e^2}e^{-\frac{L}{2g}}$$

which exactly agrees with a classical string theory computation ($E - L$ of a string that stretches from the center to the boundary of $AdS_5$ and carries $L$ units of angular momentum in the $S^5$) [Correa, Maldacena, Sever]

# Weak coupling check

- For $g \ll 1$, $e^{-(L+1)\tilde{E}_a}$ is small for any $L$

$$e^{-(L+1)\tilde{E}_a} \sim \left(\frac{4g^2}{a^2+\tilde{p}^2}\right)^{(L+1)}$$

- The product of $\sigma_B$'s becomes (for $g \ll 1$ and $\tilde{p} \to 0$)

$$\sigma_B(\tilde{p})\bar{\sigma}_B(\tilde{p}) \sim \frac{a^2}{\tilde{p}^2}$$

- Collecting all contributions:

$$\begin{aligned}
\mathcal{E}_0(L) &\sim -4g^{2L+2}\frac{(\cos\phi-\cos\theta)}{\sin\phi}\sum_{a=1}^{\infty}(-1)^a\frac{\sin a\phi}{a^{2L+1}} \\
&\sim -g^{2L+2}\frac{(\cos\phi-\cos\theta)}{\sin\phi}\frac{(-1)^L(4\pi)^{2L+1}}{(2L+1)!}B_{2L+1}\left(\frac{\pi-\phi}{2\pi}\right)+\mathcal{O}(g^{4+2L})
\end{aligned}$$

$B$ is the Bernoulli polynomial

# Weak coupling check

If we take $L \to 0$ in above ground state energy, we should get the cusp anomalous dimension (at leading weak coupling order)

$$\mathcal{E}_0(0) = \Gamma_{\rm cusp} = V_{q\bar{q}} = 2g^2(\cos\phi - \cos\theta)\frac{\phi}{\sin\phi} + \mathcal{O}(g^4)$$

**In exact agreement with the weak coupling computation for the cusp anomalous dimension** [Drukker,Gross,Ooguri 99]

In the small $\phi$ limit, TBA equations simplify a bit. We solved them iteratively and analytically up to 3-loop order

$$\Gamma_{\rm cusp} = V_{q\bar{q}} = -\phi^2 \left[ \frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} + \frac{\lambda^3}{6144\pi^2} + \mathcal{O}(\lambda^4) \right]$$

- This is in perfect agreement with the weak coupling expansion of the exact small angles answer computed using localization results [Correa,Henn,Maldacena,Sever]

$$\Gamma_{\rm cusp} \simeq (\theta^2 - \phi^2) H(\lambda, N) \qquad \text{where}$$

$$\begin{aligned} H(\lambda, N) &= \frac{1}{2\pi^2} \lambda \partial_\lambda \log \left( \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \right) = \frac{\sqrt{\lambda}}{4\pi^2} \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \\ &= \frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} + \frac{\lambda^3}{6144\pi^2} + \mathcal{O}(\lambda^4) \end{aligned}$$

The complete $H(\lambda, N)$ has been recently obtained from a simplified TBA [Gromov,Sever]

# Conclusions

- We derived a set of TBA equations to compute $\Gamma_{\text{cusp}}(\phi, \theta, \lambda) = V_{q\bar{q}}$ potential exactly in the planar limit
- We checked they give the correct answer for arbitrary cusp angles at leading weak coupling orders
- In the strong coupling limit we checked they give the correct answer for a string with arbitrary cusp angles and large angular momentum $L$
- We checked they give the correct answer for small cusp angles up to 3-loop order weak coupling (now checked to all-loop [Gromov,Sever])

# Interesting limits to consider

- Small angles limit (or $\phi \simeq \theta$): TBA eqs. drastically simplify [Gromov,Sever]
- BES equation for $i\phi = \varphi \to \infty$
- $q\bar{q}$ potential in flat space for $\phi \to \pi$
- Ladders limit, when $\theta = i\vartheta$ for $\vartheta \to \infty$ while keeping $e^{\vartheta}\lambda$ fixed

## Also to consider:

- Solve the TBA eqs. numerically for any $\lambda$
- can be something similar done in ABJM? Maybe the small cusp angles limit can help to fix the unknown function $h(\lambda)$ in the ABJM dispersion relation

$$E(p) = \sqrt{1 + h(\lambda)\sin^2(\tfrac{p}{2})}$$