Eidgenössische Technische Hochschule Zürich
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# From Feynman integrals to the Hopf algebra of multiple polylogarithms

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Integrability in Gauge and String Theory
ETH, Zurich, 22/08/2012

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  - ➡ One loop:
    - ✓ Integral basis (boxes, triangles, bubbles)
    - ✓ Essentially solved
  - ➡ Two loops:
    - ◉ Two-loop amplitudes in general not known.
    - ◉ No two-loop integral basis known.

# Multi-loop computations

- Why are multi-loop computations so difficult..?
- Quantities are divergent:
  - ➡ UV & IR divergences.
- Two-loop integrals are generically polylogarithms of weight 4 in many external physical parameters.
  - ➡ multiple polylogarithms.
  - ➡ need to evaluate these functions numerically in a fast and efficient way, including all the branch cuts, etc.
- In other words, polylogarithms and their generalizations are everywhere!
  - ➡ Need to understand these functions!

# The life-cycle of a loop computation

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    - ⋆ Elliptic functions.
- In this talk: will concentrate exclusively on polylogarithms.

# The life-cycle of a loop computation

- Recursive definition of multiple polylogarithms:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1}\, G(a_2,\ldots,a_n;t) \quad \Big| \quad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t}\, \mathrm{Li}_{n-1}(t)$$

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- All the special functions physicists defined are just special cases thereof:
  - ➡ (Classical) polylogarithms: $\mathrm{Li}_n(z)=-G(0,\ldots,0,1;z)$
  - ➡ Harmonic polylogarithms: $a_i\in\{-1,0,1\}$
  - ➡ 2d harmonic polylogarithms: e.g., $a_i\in\{0,1,a\}$
  - ➡ Cyclotomic harmonic polylogarithms: roots of unity.

# The life-cycle of a loop computation

- Even if an amplitude is simple, it might be that our approach to the problem leads to a difficult answer.

- The polylogarithms satisfy various complicated functional equations.

  ➡ The simplicity of the answer might be hidden behind a swath of functional equations.

$$-\mathrm{Li}_2(z) - \ln z \ln(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

- In other words we need to 'control' the functional equations among polylogarithms.

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- Mathematicians have discovered very far reaching algebraic structures underlying polylogarithms.

- Obvious question: can this be useful for physics..?
  ➡ Yes! ... but let's motivate this by an example!

# The ‘classical’ example

- The ‘classical’ example of this is the six-point amplitude in N=4 Super Yang-Mills.
- By evaluating the individual diagrams one arrives at a very complicated combination of multiple polylogarithms (17 pages),

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \tag{H.1}$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+$$
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$$\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+$$
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$$\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+$$

[Del Duca, CD, Smirnov]

# The ‘classical’ example

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

[Goncharov, Spradlin, Vergu ,Volovich]

$$x_i^{\pm} = u_i x^{\pm}, \quad x^{\pm} = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}, \quad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3.$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right) \qquad J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

# Number theory and Loop integrals

- Could Feynman integrals be simpler than we thought...?
- Long term goal: get to the simple answer (the function) without the 'divide and conquer' strategy.
- In the mean time: gather data, and try to find a way to get the simple answer out of the 'divide and conquer' approach.

# Number theory and Loop integrals

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- Outline:
  - ➡ The Hopf algebra of multiple polylogarithms: combinatorics vs. functional equations.
  - ➡ Some examples from physics.

# The Hopf algebra of polylogarithms

# Combinatorics vs. functional equations

# Combinatorics of polylogarithms

- We usually think of functional equations as complicated relations among special functions arising from complicated changes of variables in some integrals.

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- Mathematicians conjecture that all the functional equations among polylogarithms follow from a simple algebraic structure.
- In other words: All functional equations are pure combinatorics!
  - ➡ You do not even need to know the integral in order to derive the relations among them!

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- Mathematical construction quickly gets pretty involved.
  - ➡ I will spend only three slides on the technical details.
  - ➡ After that, I will only concentrate on applications and examples.

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If we iterate,

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the order in which we do this is immaterial, because

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- As long as we sum over all possibilities, it does not matter which way we iterate, and always arrive at the same result.

# Hopf algebras

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  - that is at the same time a coalgebra
  - such that the product and coproduct are compatible

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  - ➡ and with an additional structure, the antipode (which we will not use in the following).
- Goncharov showed that multiple polylogarithms form a Hopf algebra with coproduct

$$\Delta(I(a_0; a_1, \ldots, a_n; a_{n+1}))$$

$$= \sum_{0=i_1<i_2<\ldots<i_k<i_{k+1}=n} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \left[\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})\right]$$

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- Imagine a two-loop multi-scale integral that evaluates to 1000's of $\mathrm{Li}_4$'s.
  - ➡ Can the expression be simplified?

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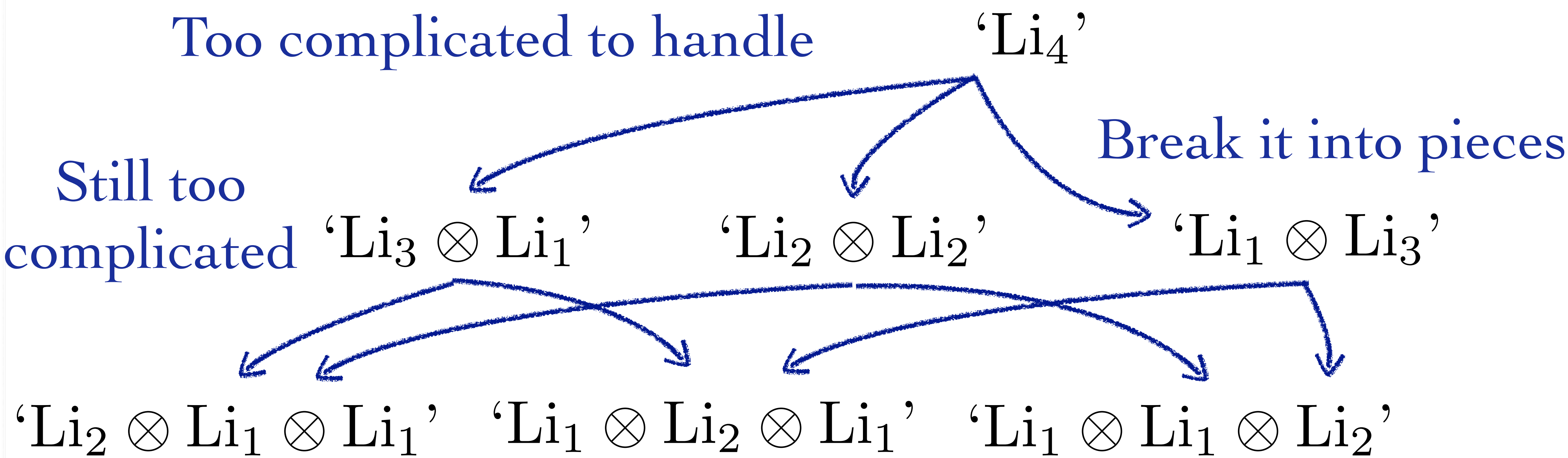

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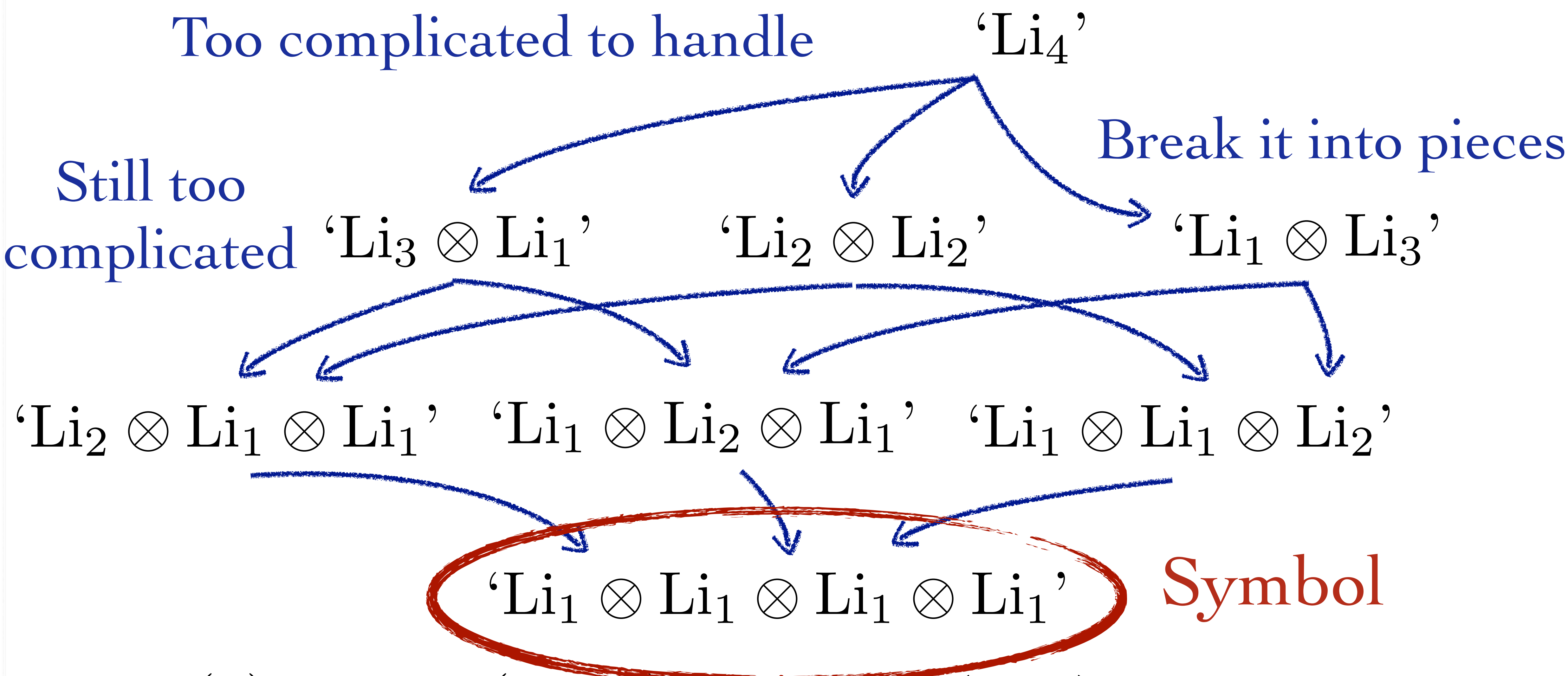


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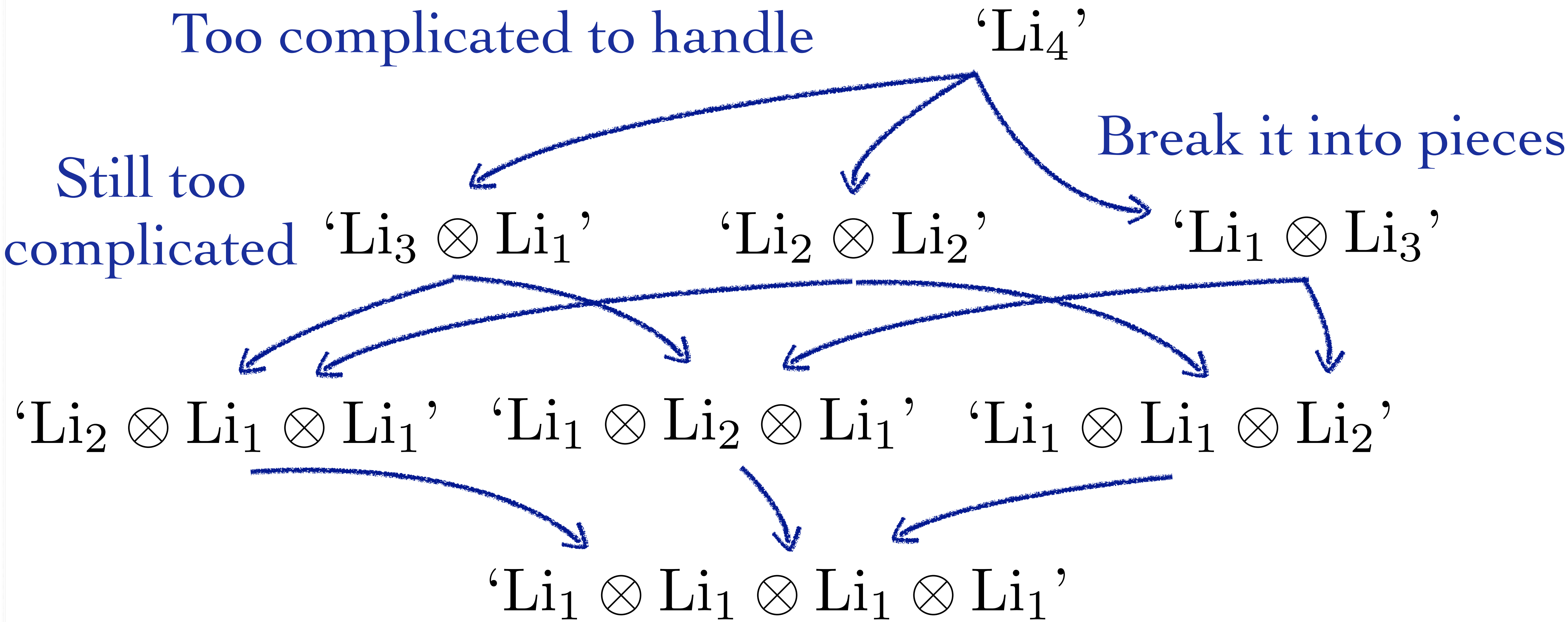


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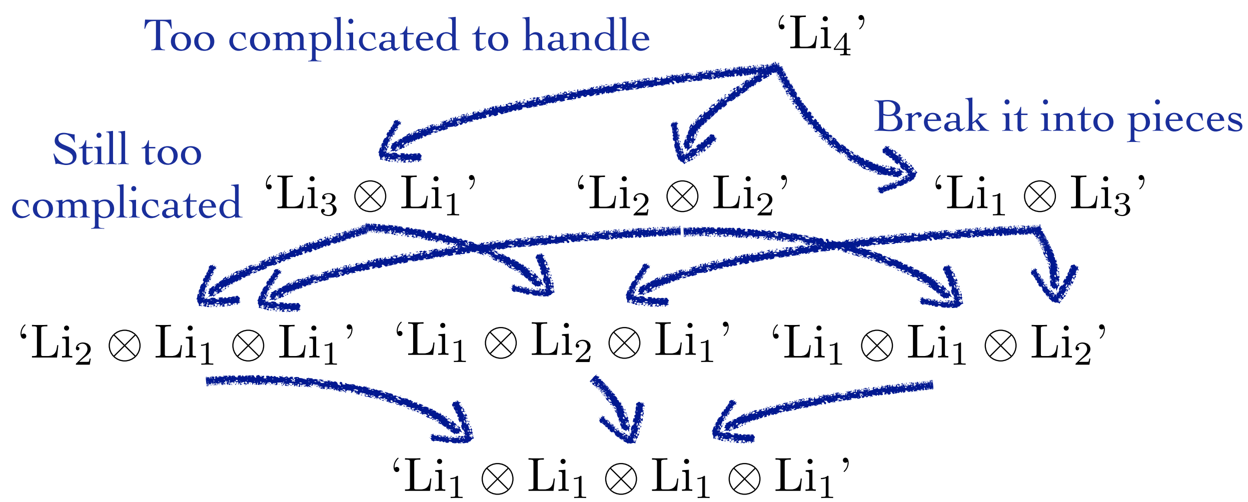


$\mathrm{Li}_1(z) = -\log(1-z)$ $\qquad$ $\log(a \cdot b) = \log a + \log b$

- At the end of this procedure, we have broken everything into little pieces (logarithms = symbol), for which all identities are known.
- We then need to reassemble the pieces to find the simplified expression (This is the most difficult step!)
- At each step information is lost, but in a controlled way:
  - ➡ Can be recovered by going back up one step at the time.

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- On the other hand, from $\zeta_4 = \frac{2}{5}\zeta_2^2$ we get

$$\Delta(\zeta_4) = \frac{2}{5}\Delta(\zeta_2)^2 = \frac{2}{5}[1 \otimes \zeta_2 + \zeta_2 \otimes 1]^2 = \frac{2}{5}[1 \otimes \zeta_2^2 + \zeta_2^2 \otimes 1 + 2\zeta_2 \otimes \zeta_2]$$

- So there is a contradiction, unless $\Delta(\zeta_{2n}) = 0$.
  - ➡ This is Goncharov’s original construction.

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- So there is a contradiction, unless $\Delta(\zeta_{2n}) = 0$.
  - ➡ This is Goncharov's original construction.
- But then, we have not gained much...

# Coproduct on zeta values

- In a recent paper on multiple zeta values, Francis Brown argues that one can also define

$$\Delta(\zeta_{2n}) = \zeta_{2n} \otimes 1$$

- This indeed solves the previous problem

$$\Delta(\zeta_4) = \frac{2}{5}\Delta(\zeta_2)^2 = \frac{2}{5}[\zeta_2 \otimes 1]^2 = \frac{2}{5}\,\zeta_2^2 \otimes 1 = \zeta_4 \otimes 1$$

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- We obtain a consistent way to include all the zeta values.
- I even argue that we can do better and define

$$\Delta(\pi) = \pi \otimes 1$$

- This will allow to include also $i\pi$ .

# Example: inversion relations

- Let us consider the inversion relations for (classical) polylogarithms:

$$\mathrm{Li}_n(1/z) = (-1)^{n+1}\,\mathrm{Li}_n(z) + \ldots$$

- Traditional approach:
  - ➡ Take the integral representation, and find a change of variable.
  - ➡ The integral has a branch cut, and develops an imaginary part.

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- Traditional approach:
  - ➡ Take the integral representation, and find a change of variable.
  - ➡ The integral has a branch cut, and develops an imaginary part.
- If my claim is correct, I should be able to get to this relation
  - ➡ in a purely algebraic/combinatorial way,
  - ➡ without even looking at the integral representation.

# Example: inversion relations

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- Weight 1: trivial

$$\mathrm{Li}_1\left(\frac{1}{x}\right) = -\ln\left(1-\frac{1}{x}\right) = -\ln(1-x) + \ln(-x) = -\ln(1-x) + \ln x - i\pi$$

with $x = x + i\,\varepsilon\,.$

# Example: inversion relations

- Weight 2:

$$\begin{aligned}\Delta_{1,1}\left[\mathrm{Li}_2\left(\frac{1}{x}\right)\right] &= -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\\ &= \ln(1-x)\otimes\ln x - \ln x\otimes\ln x + i\pi\otimes\ln x\\ &= \Delta_{1,1}\left[-\mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x + i\pi\ln x\right].\end{aligned}$$

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$$= \ln(1-x)\otimes\ln x - \ln x\otimes\ln x + i\pi\otimes\ln x$$
$$= \Delta_{1,1}\left[-\mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x + i\pi\ln x\right].$$

- This fixes the inversion relation, up to some zeta value.
  - ➡ At each step we loose a zeta value, they are indecomposable ('primitive').

$$\mathrm{Li}_2\left(\frac{1}{x}\right) = -\mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x + i\pi\ln x + c\,\pi^2$$

and $c$ = 1/3 from $x$=1.

# Example: inversion relations

- Weight 3:

$$\begin{aligned}\Delta_{1,1,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right)\right] &= -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\\ &= -\ln(1-x)\otimes\ln x\otimes\ln x+\ln x\otimes\ln x\otimes\ln x-i\pi\otimes\ln x\otimes\ln x\\ &= \Delta_{1,1,1}\left[\mathrm{Li}_3(x)+\frac{1}{6}\ln^3 x-\frac{i\pi}{2}\ln^2 x\right].\end{aligned}$$

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- Weight 3:

$$\Delta_{1,1,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right)\right] = -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)$$
$$= -\ln(1-x)\otimes\ln x\otimes\ln x + \ln x\otimes\ln x\otimes\ln x - i\pi\otimes\ln x\otimes\ln x$$
$$= \Delta_{1,1,1}\left[\mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x\right].$$

- At this stage however we have lost everything proportional to zeta values.

  ➡ Go one step up!

$$\Delta_{2,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right) - \left(\mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x\right)\right]$$
$$= \left[-\mathrm{Li}_2\left(\frac{1}{x}\right) - \mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x - i\pi\ln x\right]\otimes\ln x$$
$$= -\frac{1}{3}\pi^2\otimes\ln x = \Delta_{2,1}\left(-\frac{\pi^2}{3}\ln x\right)$$

# Example: inversion relations

- Finally:

  and $\alpha = \beta = 0$ from $x$=1.

- We could now go on like this and derive the inversion relations for arbitrary weight.

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- Finally:

$$\mathrm{Li}_3\left(\frac{1}{x}\right) = \mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x - \frac{\pi^2}{3}\ln x + \alpha\zeta_3 + \beta\, i\pi^3$$

and $\alpha = \beta = 0$ from $x$=1.

- We could now go on like this and derive the inversion relations for arbitrary weight.

➡ No painful manipulation of the integral representation at any step!

# Example: inversion relations

$$G(-z,-z,1-z,1-z;y) = \mathrm{Li}_3(1-x)\log(1-z) + \mathrm{Li}_3(1-z)\log(1-x) + \mathrm{Li}_4\left(1-\frac{1}{x}\right) + \mathrm{Li}_4(1-x)$$

$$- \mathrm{Li}_4(x) - \mathrm{Li}_3(1-x)\log(x) + \mathrm{Li}_4\left(1-\frac{1}{z}\right) + \mathrm{Li}_4(1-z) - \mathrm{Li}_4(z) - \mathrm{Li}_3(1-z)\log(z) + \frac{1}{4}\log^2(1-x)$$

$$\log^2(1-z) + \pi^2\left(-\frac{1}{6}\log(1-x)\log(1-z) + \frac{\log^2(x)}{12} + \frac{\log^2(z)}{12}\right) + \zeta(3)\log(x) - \zeta(3)\log(1-x) +$$

$$\frac{\log^4(x)}{24} - \frac{1}{6}\log(1-x)\log^3(x) - \zeta(3)\log(1-z) + \zeta(3)\log(z) + \frac{\log^4(z)}{24} - \frac{1}{6}\log(1-z)\log^3(z) + \frac{7\pi^4}{360},$$

with x+y+z=1, 0 < x,y,z < 1.

# The Hopf algebra of polylogarithms

- Goncharov's Hopf algebra, combined with Brown's treatment of even zeta values, gives an effective tool to deal with functional equations among multiple polylogarithms.
- All **functional equations** among multiple polylogarithms are pure **combinatorics**!

# The Hopf algebra of polylogarithms

- Goncharov's Hopf algebra, combined with Brown's treatment of even zeta values, gives an effective tool to deal with functional equations among multiple polylogarithms.
- All **functional equations** among multiple polylogarithms are pure **combinatorics**!

- It turns out that the coproduct knows even more!
  - ➡ The second factor knows about derivatives:

$$\Delta\left(\frac{\partial}{\partial x_k}F_w\right) = \left(\mathrm{id}\otimes\frac{\partial}{\partial x_k}\right)\Delta(F_w)$$

  - ➡ The first factor knows about discontinuities:

$$\Delta\left(\mathcal{M}_{x_k=a}F_w\right) = \left(\mathcal{M}_{x_k=a}\otimes\mathrm{id}\right)\Delta(F_w)$$

  - ➡ cf. $\Delta(\pi) = \pi\otimes 1$!

# Some examples from physics

# Hopf algebras meet Feynman integrals

# Pure Mathematics vs. Physics

- Multiple polylogarithms are everywhere in Feynman integrals and scattering amplitudes.
  - ➡ Need to 'control' these functions and the relations they satisfy.
- Understanding the underlying mathematics opens new possibilities in the world of loop computations!
  - ➡ Simplify complicated expressions.
  - ➡ Get symbol by other means (differential equations, OPE, educated guessing,...), then reconstruct the function.
  - ➡ In some cases: can even determine the space of functions to all loop orders a priori!
  - ➡ Can help for numerical evaluation of these functions.

[Buehler, Caron-Huot, Del Duca, Dixon, Drummond, CD, Ferro, Gaiotto, Goncharov, He, Henn, Maldacena, Pennington, Sever, Viera, ...]

# Pure Mathematics vs. Physics

- In the following, I will very briefly discuss two examples.

- The two-loop helicity amplitudes for H+3gluons.
  - ➡ Substantial simplification of the result.

- The 6-point remainder function in the Regge limit.
  - ➡ Knowledge of the space of functions allows us to go to 10 loops without much effort!

# Some examples from physics

# Helicity amplitudes for H + 3 gluons

# Higgs + 3 gluons

- Gehrmann, Jaquier, Glover and Koukoutsakis have recently computed the two-loop helicity amplitudes for a Higgs boson + 3 gluons
  - ➡ in the decay region

$$H \to g^+ g^+ g^+ \qquad H \to g^+ g^+ g^-$$

  - ➡ and the scattering region

$$g^+ g^+ \to g^+ H \qquad g^+ g^+ \to g^- H \qquad g^+ g^- \to g^+ H$$

- Kinematics (in the decay region):

$$x_1 = \frac{s_{12}}{m_H^2}, \qquad x_2 = \frac{s_{23}}{m_H^2}, \qquad x_3 = \frac{s_{31}}{m_H^2}$$

$$0 < x_i < 1 \quad \text{and} \quad x_1 + x_2 + x_3 = 1$$

# Higgs + 3 gluons

- The result was expressed in terms of complicated combinations of ‘2d harmonic polylogarithms’.
  - ➡ Symmetries completely lost (e.g. Bose symmetry).
  - ➡ Very long and complicated.
  - ➡ Numerical evaluation of complicated special functions.
  - ➡ Analytic continuation from decay to scattering region very complicated.

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  - ➡ Analytic continuation from decay to scattering region very complicated.
- Brandhuber, Gang and Travaglini observed that the symbol of the leading color weight 4 part (after subtracting the one-loop squared) is equal to the symbol of the form factor of 3 gluons in N=4 Super Yang-Mills.
  - ➡ A simpler representation of the Higgs amplitudes in terms of classical polylogarithms only should exist.

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[Brandhuber, Gang, Travaglini]

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$$\Delta_{2,1,1}\left[\overline{A}^{(2)}_{\alpha,\ \text{weight } 4} - \mathcal{R}^{(2)}_3\right] = -\frac{1}{6}\,\pi^2 \otimes \Delta_{1,1}\left[A^{(1)}_\alpha\right] = \Delta_{2,1,1}\left[-\frac{\pi^2}{6}\,A^{(1)}_\alpha\right]$$

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$$\overline{A}^{(2)}_{\alpha,\text{ weight }4} - \mathcal{R}^{(2)}_3 + \frac{\pi^2}{6}A^{(1)}_\alpha + \frac{1}{4}\zeta_3 B^{(1)}_\alpha = -0.03382260105347\ldots = -\frac{\pi^4}{2880}$$

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- We can of course do the same for all other color structures.

# Higgs + 3 gluons

$$
\begin{aligned}
\overline{A}_{\alpha}^{(2)} &= \mathcal{R}_3^{(2)} - \frac{\pi^2}{6} A_{\alpha}^{(1)} - \frac{1}{4}\zeta_3 B_{\alpha}^{(1)} - \frac{\pi^4}{2880} \\
&\frac{11}{6}\Bigg\{ \Lambda_3\left(-\frac{x_1x_3}{x_2}\right) + \Lambda_3\left(-\frac{x_2x_3}{x_1}\right) + \Lambda_3\left(-\frac{x_1x_2}{x_3}\right) - \sum_{i=1}^{3} \mathrm{Li}_3\left(1-\frac{1}{x_i}\right) \\
&- \Lambda_3\left(-\frac{x_1}{x_2}\right) - \Lambda_3\left(-\frac{x_2}{x_1}\right) - \Lambda_3\left(-\frac{x_1}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_1}\right) - \Lambda_3\left(-\frac{x_2}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_2}\right) \\
&+ \frac{1}{2}\ln(x_1\,x_2\,x_3)\,A_{\alpha}^{(1)} + \frac{7}{2}\sum_{i=1}^{3}\left[\mathrm{Li}_2\left(1-x_i\right)\,\ln x_i\right] + \frac{3}{4}\ln x_1\,\ln x_2\,\ln x_3 + \frac{1}{6}\ln^3\left(x_1x_2x_3\right) \\
&- \frac{5}{16}\pi^2\ln(x_1x_2x_3) - \frac{3}{8}\zeta_3 + i\pi\,A_{\alpha}^{(1)} + \frac{i\pi^3}{16} - \frac{1}{3}\sum_{i=1}^{3}\ln^3 x_i \Bigg\} \\
&+ \frac{1}{36}\sum_{i=1}^{3}\left[\frac{P_1(x_i,x_{i-1},x_{i+1})}{x_{i-1}^2x_{i+1}^2}\mathrm{Li}_2(1-x_i) + \frac{P_2(x_i,x_{i-1},x_{i+1})}{x_i^2}\ln x_{i-1}\ln x_{i+1} + \frac{121}{4}\ln^2 x_i\right] \\
&+ \frac{P_3(x_1,x_2,x_3)}{144x_1^2x_2^2x_3^2}\pi^2 - \frac{121}{72}i\pi\ln(x_1x_2x_2) + \frac{11}{36}i\pi\left(x_1x_2+x_2x_3+x_3x_1\right) + \frac{185}{24}i\pi \\
&+ \frac{1}{72}\sum_{i=1}^{3}\frac{P_4(x_i,x_{i-1},x_{i+1})}{x_{i-1}x_{i+1}}\ln x_i - \frac{1}{72}\left(x_1x_2+x_3x_2+x_1x_3\right)^2 + \frac{247}{108}\left(x_1x_2+x_3x_2+x_1x_3\right) \\
&+ \frac{1321}{216},
\end{aligned}
$$

➡ Kummer' function

$$
\Lambda_n(z) = \int_0^z \mathrm{d}t\,\frac{\ln^{n-1}|t|}{1+t} = (n-1)!\sum_{k=0}^{n-1}\frac{(-1)^{n-k}}{k!}\ln^k|z|\,\mathrm{Li}_{n-k}(z)
$$

# Higgs + 3 gluons

$$\overline{D}_\alpha^{(2)} = -\zeta_3 + \frac{i\pi}{4} - \frac{1}{6}(x_1x_2 + x_3x_2 + x_1x_3) + \frac{67}{48} + \frac{P_5(x_1,x_2,x_3)}{72x_1^2x_2^2x_3^2}\pi^2$$
$$+ \frac{1}{12}\sum_{i=1}^{3}\left[\frac{P_6(x_i,x_{i-1},x_{i+1})}{x_{i-1}^2x_{i+1}^2}\,\mathrm{Li}_2(1-x_i) + \frac{P_7(x_i,x_{i-1},x_{i+1})}{x_i^2}\ln x_{i-1}\ln x_{i+1}\right. \tag{7.19}$$
$$\left. + \frac{P_8(x_i,x_{i-1},x_{i+1})}{2x_{i-1}x_{i+1}}\ln x_i\right]$$

$$\overline{E}_\alpha^{(2)} = -\frac{i\pi^3}{48} - \frac{i\pi}{3}A_\alpha^{(1)} - \frac{1}{12}\ln(x_1x_2x_3)(\ln x_1\,\ln x_2 + \ln x_1\,\ln x_3 + \ln x_2\,\ln x_3)$$
$$+ \frac{P_{13}(x_1,x_2,x_3)}{432} + \frac{7}{12}\ln x_1\,\ln x_2\,\ln x_3 - \frac{5}{48}\pi^2\ln(x_1x_2x_3) - \frac{29}{24}\zeta_3$$
$$+ \frac{11}{18}i\pi\,\ln(x_1x_2x_3) + \frac{P_{11}(x_1,x_2,x_3)}{288x_1^2x_2^2x_3^2}\pi^2 + \sum_{i=1}^{3}\left[\mathrm{Li}_3(x_i) - \frac{1}{3}\mathrm{Li}_3(1-x_i)\right.$$
$$+ \frac{1}{6}\mathrm{Li}_2(1-x_i)\,\ln x_i + \frac{1}{2}\ln(1-x_i)\,\ln^2 x_i + \frac{1}{6}\ln(x_1x_2x_3)\,\mathrm{Li}_2(1-x_i) \tag{7.20}$$
$$+ \frac{P_9(x_i,x_{i-1},x_{i+1})}{36x_{i-1}^2x_{i+1}^2}\mathrm{Li}_2(1-x_i) + \frac{P_{10}(x_i,x_{i-1},x_{i+1})}{36x_i^2}\ln x_{i-1}\,\ln x_{i+1}$$
$$\left. + \frac{11}{36}\ln^2 x_i + \frac{P_{12}(x_i,x_{i-1},x_{i+1})}{216x_{i-1}x_{i+1}}\ln x_i\right] - \frac{13}{36}i\pi\,(x_1x_2 + x_3x_2 + x_1x_3) - \frac{71}{18}i\pi\,,$$

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$$
\begin{aligned}
\overline{F}_{\alpha}^{(2)} &= -\frac{i\pi}{18}\ln(x_1x_2x_3) - \frac{11}{144}\pi^2 + \frac{1}{36}\sum_{i=1}^{3}\ln^2 x_i - \frac{5}{54}\ln(x_1x_2x_3) + \frac{5i\pi}{18} \\
&+ \frac{i\pi}{18}(x_1x_2 + x_2x_3 + x_3x_1) + \frac{5}{54}(x_1x_2 + x_3x_2 + x_1x_3) \\
&- \frac{1}{72}(x_1x_2 + x_3x_2 + x_1x_3)^2 - \frac{x_1x_2x_3}{18}\sum_{i=1}^{3}\frac{\ln x_i}{x_i}\,,
\end{aligned}
$$

- Originally, the expressions filled up more than 6 pages!
- Bose symmetry is now completely manifest.
- Only simple functions (classical polylogarithms) with simple arguments.
  - ➡ easy numerical evaluation.
- Similar results can be obtained for $H \to g^+ g^+ g^-$ .

# Some examples from physics

# The Regge limit of the 6 point remainder function

# The Regge limit

- In the Regge limit, we can approximate the amplitude by the expansion in the logarithms that are divergent as $u_1 \to 1$.

$$R|_{\mathrm{MRK}} = 2\pi i \sum_{\ell=2}^{\infty} \sum_{n=0}^{\ell-1} a^{\ell} \log^n(1-u_1) \left[ g_n^{(\ell)}(w,w^*) + 2\pi i \, h_n^{(\ell)}(w,w^*) \right]$$

[Bartels, Lipatov, Sabio Vera]

- The coefficients $g_n^{(\ell)}(w,w^*)$ for *n=l-1* and *n=l-2* can be computed, to any loop order, by the integral

$$\cos \pi\omega_{ab} + i\,\frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2+\frac{n^2}{4}} |w|^{2i\nu}\, \Phi_{\mathrm{Reg}}(\nu,n) \left(-\frac{1}{\sqrt{u_2\,u_3}}\right)^{\omega(\nu,n)}$$

$$E_{\nu,n} \;=\; -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1)$$

$$\Phi_{\mathrm{Reg}}^{(1)}(\nu,n) = -\frac{1}{2}E_{\nu,n}^2 - \frac{3}{8}\frac{n^2}{(\nu^2+\frac{n^2}{4})^2} - \zeta_2$$

# The Regge limit

- Based on general grounds, we can argue that, to all loop orders, the results are given by combination of so-called single-valued harmonic polylogarithms.
- These functions have been classified by F. C. Brown for all weights, and thus we now the space of functions to all loop orders!
- Example:

# The Regge limit

- Based on general grounds, we can argue that, to all loop orders, the results are given by combination of so-called single-valued harmonic polylogarithms.
- These functions have been classified by F. C. Brown for all weights, and thus we now the space of functions to all loop orders!
- Example:

$$
\begin{aligned}
L_2^- &= \frac{1}{4}\Big[ -2\,H_{1,0} + 2\,\overline{H}_{1,0} + 2\,H_0\,\overline{H}_1 - 2\,\overline{H}_0\,H_1 + 2\,H_2 - 2\,\overline{H}_2 \Big] \\
&= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log|z|^2(\log(1-z) - \log(1-\bar{z}))\,,
\end{aligned}
$$

# The Regge limit

$$\cos \pi\omega_{ab} + i\,\frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} |w|^{2i\nu}\, \Phi_{\text{Reg}}(\nu, n) \left(-\frac{1}{\sqrt{u_2\, u_3}}\right)^{\omega(\nu,n)}$$

- Instead of having to sum up the infinite tower of residues, just match the truncated sum to the Taylor expansion of the basis functions.
- In this way, by exploiting the a priori knowledge on the space of functions, we obtain a constructive way to compute **any loop order we like**!

# The Regge limit

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- Instead of having to sum up the infinite tower of residues, just match the truncated sum to the Taylor expansion of the basis functions.
- In this way, by exploiting the a priori knowledge on the space of functions, we obtain a constructive way to compute **any loop order we like**!

$$g_1^{(2)}(w,w^*) = \frac{1}{4}\,[L_1^+]^2 - \frac{1}{16}\,[L_0^-]^2\,,$$

$$g_0^{(2)}(w,w^*) = -L_3^+ + \frac{1}{6}\,\left[L_1^+\right]^3 + \frac{1}{8}[L_0^-]^2\, L_1^+$$

[Lipatov, Prygarin; Dixon, Drummond, Henn]

# The Regge limit

$$g_2^{(3)}(w,w^*) = -\frac{1}{8}L_3^+ + \frac{1}{12}\left[L_1^+\right]^3 ,$$

$$g_1^{(3)}(w,w^*) = \frac{1}{8}L_0^-\,L_{2,1}^- - \frac{5}{8}L_1^+\,L_3^+ + \frac{5}{48}[L_1^+]^4 + \frac{1}{16}[L_0^-]^2\,[L_1^+]^2 - \frac{5}{768}[L_0^-]^4$$
$$- \frac{\pi^2}{12}[L_1^+]^2 + \frac{\pi^2}{48}[L_0^-]^2 + \frac{1}{4}\zeta_3\,L_1^+ .$$

[Lipatov, Prygarin; Dixon, Drummond, Henn]

# The Regge limit

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[Lipatov, Prygarin; Dixon, Drummond, Henn]

$$g_3^{(4)}(w,w^*) = \frac{1}{48}[L_2^-]^2 + \frac{1}{48}[L_0^-]^2\,[L_1^+]^2 + \frac{7}{2304}[L_0^-]^4 + \frac{1}{48}[L_1^+]^4 - \frac{1}{16}L_0^-\,L_{2,1}^- - \frac{5}{48}L_1^+\,L_3^+ - \frac{1}{8}L_1^+\,\zeta_3 ,$$

$$g_2^{(4)}(w,w^*) = \frac{3}{64}[L_0^-]^2\,[L_1^+]^3 + \frac{1}{128}L_1^+\,[L_0^-]^4 - \frac{3}{32}L_3^+\,[L_0^-]^2 + \frac{1}{8}[L_0^-]^2\,\zeta_3 - \frac{1}{8}[L_1^+]^2\,\zeta_3 + \frac{3}{80}[L_1^+]^5 - \frac{\pi^2}{24}[L_1^+]^3 - \frac{1}{16}L_0^-\,L_{2,1}^-\,L_1^+ + \frac{13}{16}L_5^+ + \frac{3}{8}L_{3,1,1}^+ + \frac{1}{4}L_{2,2,1}^+ - \frac{5}{16}L_3^+\,[L_1^+]^2 + \frac{\pi^2}{16}L_3^+ ,$$

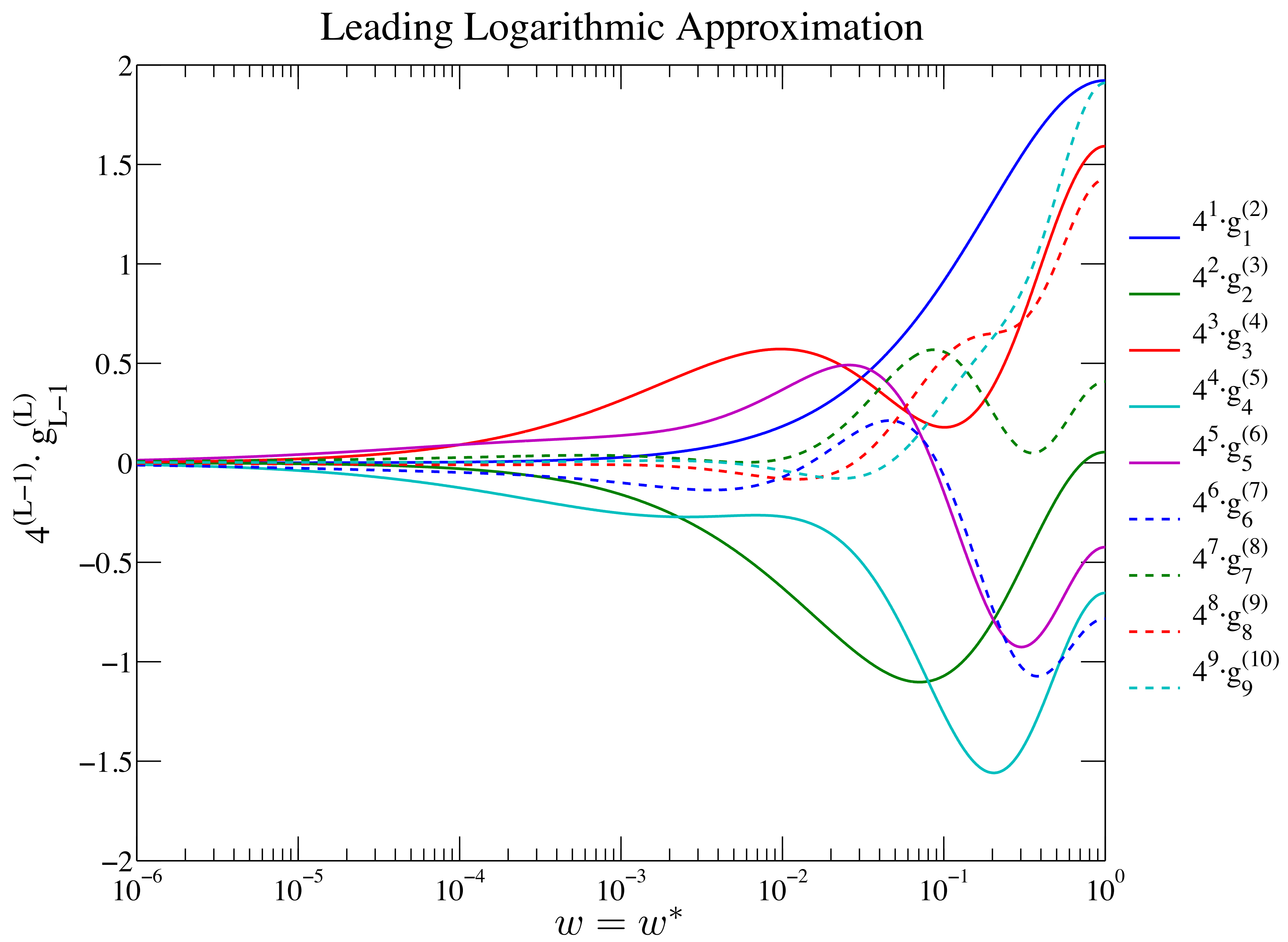
[Dixon, CD, Pennington]

# The Regge limit

$$
\begin{aligned}
g_4^{(5)}(w,w^*) \;=\;& \frac{1}{96}\,[L_0^-]^2\,[L_1^+]^3 + \frac{17}{9216}\,L_1^+\,[L_0^-]^4 - \frac{5}{384}\,L_3^+\,[L_0^-]^2 + \frac{1}{24}\,[L_0^-]^2\,\zeta_3 \\
& - \frac{1}{12}\,[L_1^+]^2\,\zeta_3 + \frac{1}{240}\,[L_1^+]^5 - \frac{1}{24}\,L_0^-\,L_{2,1}^-\,L_1^+ + \frac{43}{384}\,L_5^+ + \frac{1}{8}\,L_{3,1,1}^+ + \frac{1}{12}\,L_{2,2,1}^+ \\
& - \frac{1}{24}\,L_3^+\,[L_1^+]^2\,, \\
g_3^{(5)}(w,w^*) \;=\;& -\frac{1}{384}\,[L_2^-]^2\,[L_0^-]^2 + \frac{5}{64}\,[L_2^-]^2\,[L_1^+]^2 - \frac{\pi^2}{72}\,[L_2^-]^2 + \frac{1}{384}\,[L_0^-]^4\,[L_1^+]^2 - \frac{7}{48}\,\zeta_3^2 \\
& + \frac{5}{144}\,[L_0^-]^2\,[L_1^+]^4 - \frac{\pi^2}{72}\,[L_0^-]^2\,[L_1^+]^2 - \frac{31}{1152}\,L_{2,1}^-\,[L_0^-]^3 - \frac{11}{384}\,L_1^+\,L_3^+\,[L_0^-]^2 \\
& - \frac{7}{48}\,L_1^+\,[L_0^-]^2\,\zeta_3 + \frac{31}{69120}\,[L_0^-]^6 - \frac{7\pi^2}{3456}\,[L_0^-]^4 + \frac{7}{48}\,[L_{2,1}^-]^2 - \frac{31}{192}\,L_0^-\,L_{2,1}^-\,[L_1^+]^2 \\
& - \frac{65}{576}\,L_3^+\,[L_1^+]^3 - \frac{13}{96}\,[L_1^+]^3\,\zeta_3 + \frac{7}{720}\,[L_1^+]^6 - \frac{\pi^2}{72}\,[L_1^+]^4 + \frac{1}{48}\,[L_3^+]^2 + \frac{5}{96}\,L_4^-\,L_2^- \\
& - \frac{7}{24}\,L_2^-\,L_{2,1,1}^- + \frac{1}{192}\,L_0^-\,L_{4,1}^- + \frac{1}{16}\,L_0^-\,L_{3,2}^- + \frac{\pi^2}{24}\,L_0^-\,L_{2,1}^- + \frac{9}{16}\,L_0^-\,L_{2,1,1,1}^- \\
& + \frac{33}{64}\,L_5^+\,L_1^+ + \frac{5\pi^2}{72}\,L_1^+\,L_3^+ - \frac{7}{48}\,L_1^+\,L_{3,1,1}^+ + \frac{25}{32}\,L_1^+\,\zeta_5 + \frac{\pi^2}{12}\,L_1^+\,\zeta_3 - \frac{5}{32}\,L_3^+\,\zeta_3 \\
& ,
\end{aligned}
$$

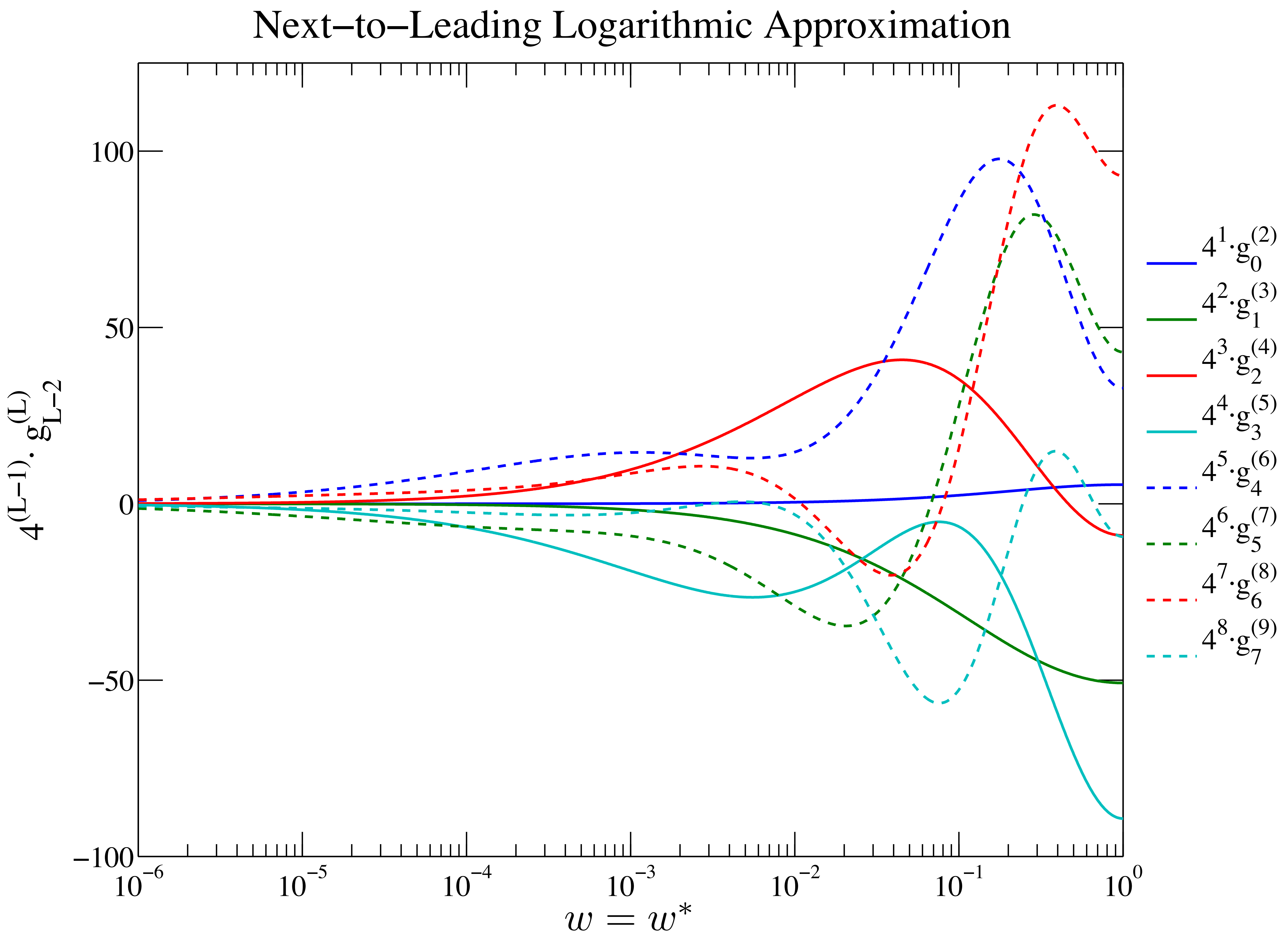
[Dixon, CD, Pennington]

# LLA and NLLA



Leading Logarithmic Approximation
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
$4^{(L-1)} \cdot g_{L-1}^{(L)}$
$10^{-6}$
$10^{-5}$
$10^{-4}$
$10^{-3}$
$10^{-2}$
$10^{-1}$
$10^{0}$
$w = w^*$
$4^1 \cdot g_1^{(2)}$
$4^2 \cdot g_2^{(3)}$
$4^3 \cdot g_3^{(4)}$
$4^4 \cdot g_4^{(5)}$
$4^5 \cdot g_5^{(6)}$
$4^6 \cdot g_6^{(7)}$
$4^7 \cdot g_7^{(8)}$
$4^8 \cdot g_8^{(9)}$
$4^9 \cdot g_9^{(10)}$

# LLA and NLLA



Next–to–Leading Logarithmic Approximation
$4^{(L-1)} \cdot g_{L-2}^{(L)}$
100
50
0
−50
−100
$10^{-6}$
$10^{-5}$
$10^{-4}$
$10^{-3}$
$10^{-2}$
$10^{-1}$
$10^{0}$
$w = w^*$
$4^1 \cdot g_0^{(2)}$
$4^2 \cdot g_1^{(3)}$
$4^3 \cdot g_2^{(4)}$
$4^4 \cdot g_3^{(5)}$
$4^5 \cdot g_4^{(6)}$
$4^6 \cdot g_5^{(7)}$
$4^7 \cdot g_6^{(8)}$
$4^8 \cdot g_7^{(9)}$

# Conclusion

- Very large classes of Feynman integrals and scattering amplitudes can be expressed in terms of multiple polylogarithms.
- Goncharov's Hopf algebra, combined with Brown's prescription for even zeta values, reduces functional equations among polylogarithms to purely combinatorial problems in the Hopf algebra.
- This opens many new ways to think about multi-loop computations.
- Open question: is there a coproduct on Feynman integrals/ scattering amplitudes that mimics the coproduct on the functions..?

# The CHAPLIN library [Buehler, CD]

- Loop amplitudes can often be expressed in terms of a special class of multiple polylogarithms, the so called *harmonic* polylogarithms. [Remiddi, Vermaseren]
- Numerical routines for these functions are needed, including
  - ➡ hplog: Fortran, real arguments only. [Gehrmann, Remiddi]
  - ➡ HPL: Mathematica [Maitre]
  - ➡ GiNaC [van Hameren, Vollinga, Weinzierl]
- CHAPLIN = Complex HArmonic PolyLogarithms In fortranN
- Based on a reduction of HPL's to a basis
  - ➡ only a few new functions appear up to weight 4 (= 2 loops)

[CD, Gangl, Rhodes]

# Reduction of HPLs

[CD, Gangl, Rhodes]

- Main idea of the reduction: HPLs have symbols with entries drawn from the set {x, 1-x, 1+x, 2}.
- Next construct a spanning set for all HPLs (up to weight 4) that generates all polylogarithms whose symbol has entries drawn from the set above.

$$\mathcal{B}_4^{(1)}(x) = \mathrm{Li}_4(x), \quad \mathcal{B}_4^{(2)}(x) = \mathrm{Li}_4(-x),$$

$$\mathcal{B}_4^{(3)}(x) = \mathrm{Li}_4(1-x), \quad \mathcal{B}_4^{(4)}(x) = \mathrm{Li}_4\left(\frac{1}{1+x}\right),$$

$$\mathcal{B}_4^{(5)}(x) = \mathrm{Li}_4\left(\frac{x}{x-1}\right), \quad \mathcal{B}_4^{(6)}(x) = \mathrm{Li}_4\left(\frac{x}{x+1}\right),$$

$$\mathcal{B}_4^{(7)}(x) = \mathrm{Li}_4\left(\frac{1+x}{2}\right), \quad \mathcal{B}_4^{(8)}(x) = \mathrm{Li}_4\left(\frac{1-x}{2}\right),$$

$$\mathcal{B}_4^{(9)}(x) = \mathrm{Li}_4\left(\frac{1-x}{1+x}\right), \quad \mathcal{B}_4^{(10)}(x) = \mathrm{Li}_4\left(\frac{x-1}{x+1}\right),$$

$$\mathcal{B}_4^{(11)}(x) = \mathrm{Li}_4\left(\frac{2x}{x+1}\right), \quad \mathcal{B}_4^{(12)}(x) = \mathrm{Li}_4\left(\frac{2x}{x-1}\right)$$

$$\mathcal{B}_4^{(13)}(x) = \mathrm{Li}_4\left(1-x^2\right), \quad \mathcal{B}_4^{(14)}(x) = \mathrm{Li}_4\left(\frac{x^2}{x^2-1}\right)$$

$$\mathcal{B}_4^{(15)}(x) = \mathrm{Li}_4\left(\frac{4x}{(x+1)^2}\right) \quad \mathcal{B}_4^{(16)}(x) = \mathrm{Li}_{2,2}(-1,x),$$

$$\mathcal{B}_4^{(17)}(x) = \mathrm{Li}_{2,2}\left(\frac{1}{2}, \frac{2x}{x+1}\right),$$

$$\mathcal{B}_4^{(18)}(x) = \mathrm{Li}_{2,2}\left(\frac{1}{2}, \frac{2x}{x-1}\right)$$

# Reduction of HPLs [CD, Gangl, Rhodes]

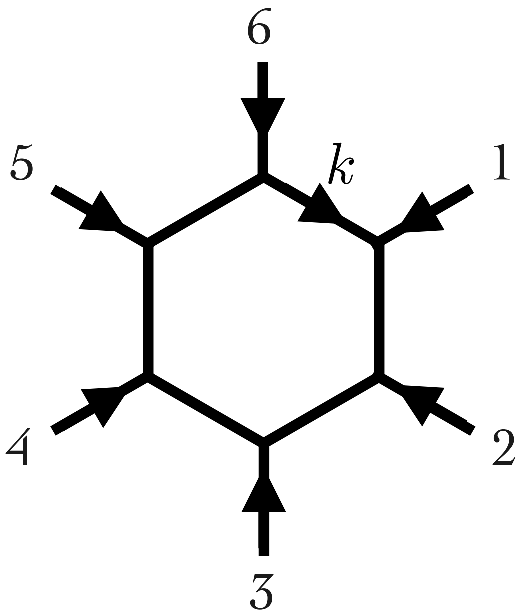
- Example:

$$
\begin{aligned}
H(0,0,1,-1;x) &= \mathrm{Li}_3(x)\log(1+x) + \frac{3}{4}\zeta_3\log(1+x) - \frac{1}{6}\log^4(1+x) + \frac{1}{3}\log 2\log^3(1+x) \\
&+ \frac{1}{6}\log x\log^3(1+x) + \frac{1}{3}\log^3 2\log(1+x) - \frac{1}{2}\log^2 2\log^2(1+x) + \frac{\pi^2}{6}\log^2(1+x) \\
&- \frac{\pi^2}{6}\log 2\log(1+x) + \frac{1}{2}\mathrm{Li}_4(-x) - \frac{3}{2}\mathrm{Li}_4(x) - \frac{1}{4}\mathrm{Li}_4\left(\frac{4x}{(x+1)^2}\right) - \mathrm{Li}_4\left(\frac{1}{1+x}\right) \\
&- \mathrm{Li}_4\left(\frac{x}{x+1}\right) + 2\mathrm{Li}_4\left(\frac{2x}{x+1}\right) - 2\mathrm{Li}_4\left(\frac{1+x}{2}\right) + 2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{90},
\end{aligned}
$$

- No new function is needed for the numerical evaluation in this case!
- In general, 3 new functions were needed (to express 118 HPLs).
- This reduction is what is implemented into the CHAPLIN library.
- For multiscale integrals more complicated functions appear.
- Same procedure can be applied there as well in principle.

# One-loop Hexagons in 6 dimensions

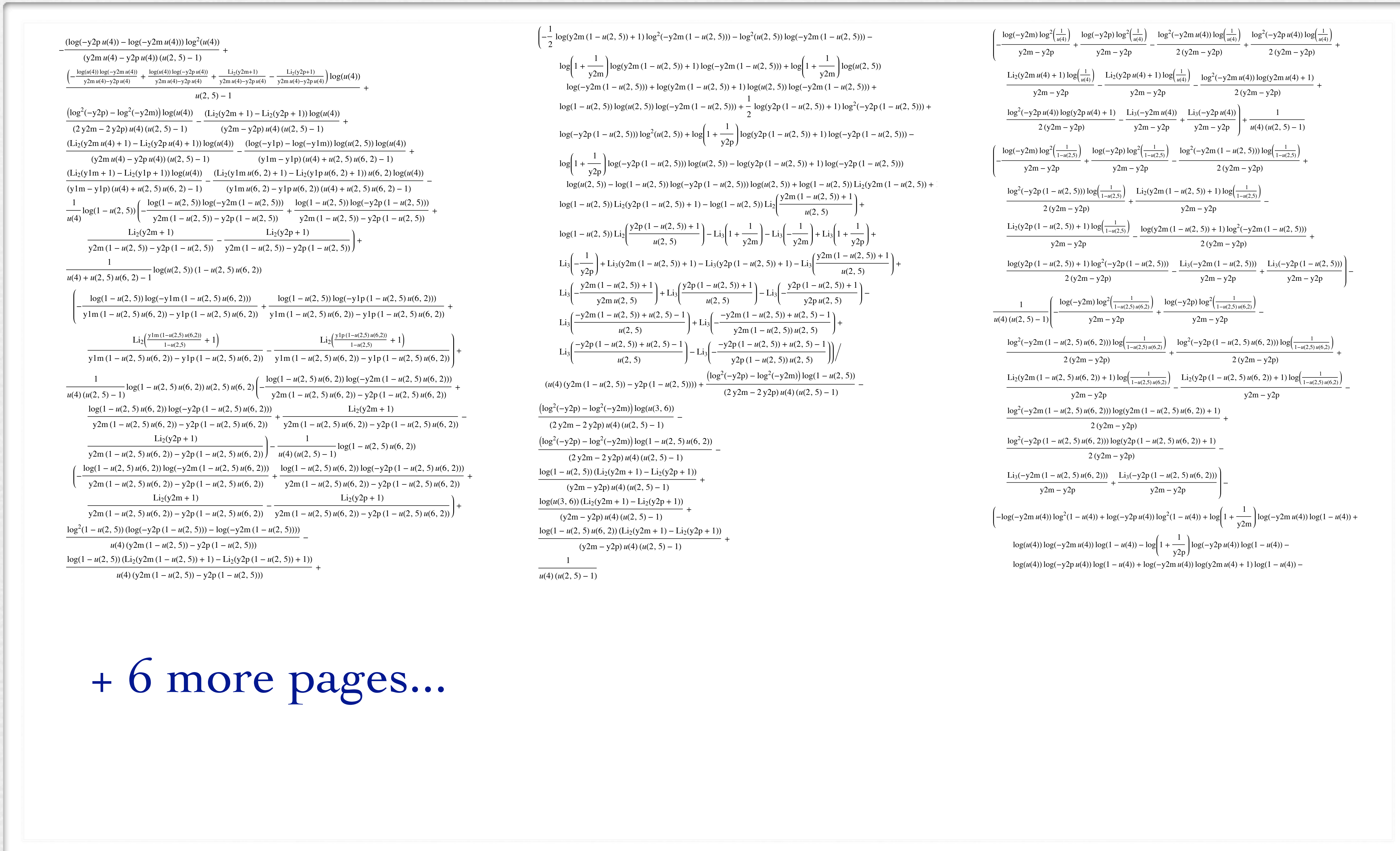
- The massless scalar one-loop hexagon integral in D=6 dimensions
  - ➡ is finite,
  - ➡ dual conformally invariant,
  - ➡ a weight 3 function.




$$I_6^{D=6} = \int \frac{d^6 k}{i\pi^3} \prod_{i=0}^{5} \frac{1}{D_i}\,,$$

$$D_0 = k^2 \quad \text{and} \quad D_i = (k+p_i)^2, \quad \text{for} \quad i = 1, \ldots, 5\,.$$

# One-loop Hexagons in 6 dimensions



+ 6 more pages...

# One-loop Hexagons in 6 dimensions

- After 'symbolizing' this result, it reduces to

$$\mathcal{I}_6(u_1,u_2,u_3) = \frac{1}{\sqrt{\Delta}}\left[-2\sum_{i=1}^{3} L_3(x_{i+},x_{i-}) + 2\zeta_2 J + \frac{1}{3}J^3\right]$$

$$L_3(x^+,x^-) = \sum_{k=0}^{2}\frac{(-1)^k}{(2k)!!}\ln^k(x^+ x^-)\left(\ell_{3-k}(x^+) - \ell_{3-k}(x^-)\right) ,$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n \mathrm{Li}_n(1/x)\right) ,$$

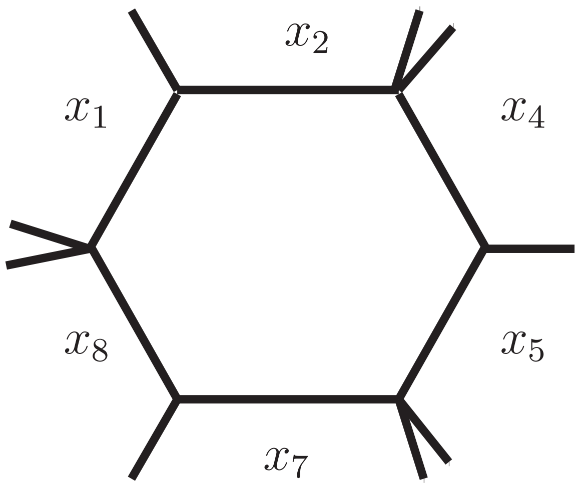
$$J = \sum_{i=1}^{3}\left(\ell_1(x_i^+) - \ell_1(x_i^-)\right) \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}$$

$$x_i^\pm = u_i\, x^\pm \qquad \Delta = (1-u_1-u_2-u_3)^2 - 4u_1u_2u_3$$

[Dixon, Drummond, Henn; Del Duca, CD, Smirnov]

# One-loop Hexagons in 6 dimensions

- This simplicity motivated the study of more complicated hexagons:




$$\Phi_9(u_1,\ldots,u_6) = \frac{1}{\sqrt{\Delta_9}} \sum_{i=1}^{4} \sum_{g\in S_3} \sigma(g)\, \mathcal{L}_3(x^+_{i,g}, x^-_{i,g})$$

$$\mathcal{L}_3(x^+,x^-) := \frac{1}{18}\left(\ell_1(x^+) - \ell_1(x^-)\right)^3 + L_3(x^+,x^-)$$

$$x_1^+ := \chi(1,4,7)\,,$$
$$x_1^- := \overline{\chi}(1,4,7)\,,$$
etc.

$$\chi(i,j,k) := -\frac{\langle 4\bar{7}\rangle\langle X_i X_k\rangle\langle X_j 17\rangle}{\langle 1\bar{7}\rangle\langle X_j X_k\rangle\langle X_i 47\rangle}$$

$$\overline{\chi}(i,j,k) := -\frac{\langle \bar{4}7\rangle\langle X_i X_k\rangle\langle X_j \bar{1}\cap\bar{7}\rangle}{\langle \bar{1}7\rangle\langle X_j X_k\rangle\langle X_i \bar{4}\cap\bar{7}\rangle}$$

[Del Duca, Dixon, Drummond, CD, Henn, Smirnov]

# One-loop Hexagons in 6 dimensions

$$x_1^+ := \chi(1,4,7)\,, \qquad x_1^- := \overline{\chi}(1,4,7)\,,$$

$$\chi(i,j,k) := -\frac{\langle 4\overline{7}\rangle\langle X_i X_k\rangle\langle X_j 17\rangle}{\langle 1\overline{7}\rangle\langle X_j X_k\rangle\langle X_i 47\rangle}$$

$$\begin{aligned}\Delta_9 \equiv\ & (1 - u_1 - u_2 - u_3 + u_4 u_1 u_2 + u_5 u_2 u_3 \\ & + u_6 u_3 u_1 - u_1 u_2 u_3 u_4 u_5 u_6)^2 \\ & - 4 u_1 u_2 u_3 (1-u_4)(1-u_5)(1-u_6).\end{aligned}$$

$$\begin{aligned}
x_1^+ &= \frac{2u_3(1-u_6)[1-u_3u_6-u_2(1-u_3u_5u_6)]-(1-u_3u_6)(g_1-\sqrt{\Delta_9})}{2u_3(1-u_6)[1-u_2-u_3(1-u_2u_5)u_6]},\\
x_2^+ &= \frac{2u_1u_3(1-u_6)[1-u_2u_4-u_3(1-u_2u_4u_5)]-(1-u_3)(g_6-\sqrt{\Delta_9})}{2u_1(1-u_6)[1-u_2u_4-u_3(1-u_2u_4u_5)]},\\
x_3^+ &= \frac{2u_3(1-u_6)[(1-u_2u_5)(1-u_3u_5)-u_1(1-u_5)]-(1-u_3u_5)(g_1-\sqrt{\Delta_9})}{2u_1u_3u_5(1-u_6)[1-u_2u_4-u_3(1-u_2u_4u_5)]},\\
x_4^+ &= -u_6\frac{2u_3(1-u_6)[1-u_5-u_1(1-u_2u_4u_5)(1-u_3u_5u_6)]+(1-u_3u_5u_6)(g_6-\sqrt{\Delta_9})}{2(1-u_6)[1-u_2-u_3(1-u_2u_5)u_6]}.
\end{aligned}$$