ANALYTIC SOLUTION OF BREMSSTRAHLUNG TBA

BASED ON* 1207.5489 *WITH A.SEVER

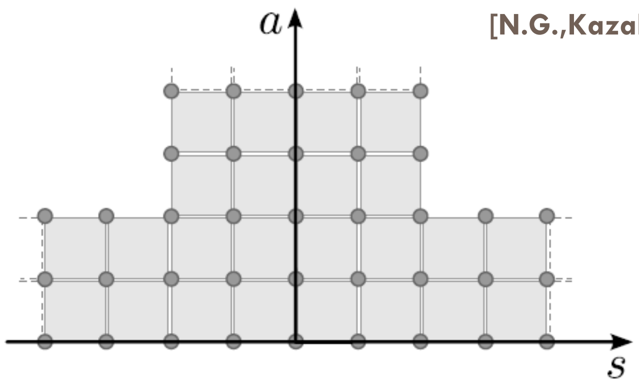
IGST2012 Zurich

Introduction

- Big progress in understanding of spectrum of N=4 SYM

  [Lipatov] [Faddeev,Korchemsky] [Minahan,Zarembo] [Beisert,Kristijanssen,Staudacher] [Bena,Roiban,Polchinski] [Kazakov,Marshakov, Minahan] [Zarembo, Frolov, Tseytlin] [Beisert,Kazakov,Sakai,Zarembo] [NG,Vieira] [Arutyunov,Frolov,Staudacher] [Staudacher, Beisert] [Janik] [Hernandez,Lopez] [Roiban, Tseytlin] [Beisert,Eden,Staudacher] [Ambjorn,Janik,Kristijanssen] [Arutyunov,Frolov] [Bajnok,Janik]

- Solution is given by Y-system (or Hirota) + simple analytical data



[N.G.,Kazakov,Vieira]

[Cavaglia, Fioravanti,Tateo; Bombardelli, Fioravanti,Tateo; N.G., Kazakov,Kozak,Vieira; Arutyunov, Frolov]

$$T_{a,s}(u+\tfrac{i}{2})T_{a,s}(u-\tfrac{i}{2}) = T_{a+1,s}(u)T_{a-1,s}(u) + T_{a,s+1}(u)T_{a,s-1}(u)$$

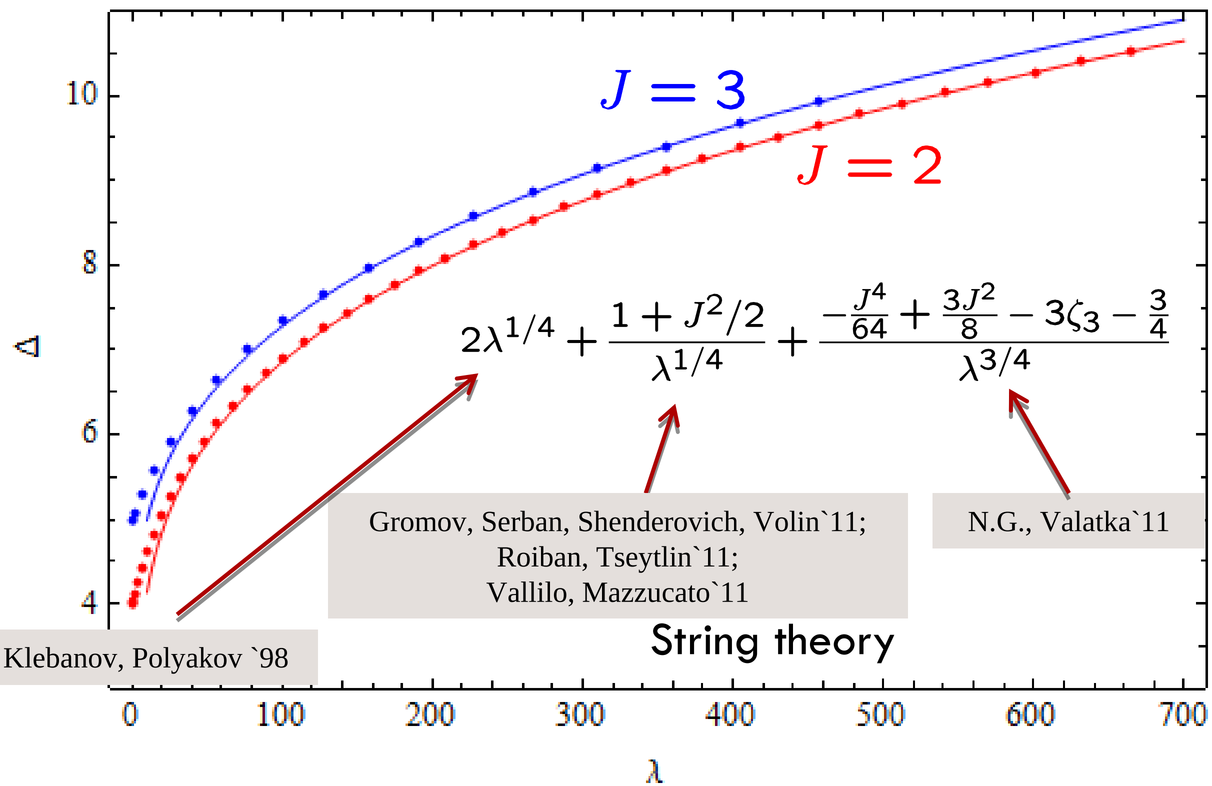
- Integrability of Hirota $\rightarrow$ finite set of Integral equations (FiNLIE)

  [N.G.,Kazakov, Leurent, Vieira]

Introduction

N.G., Kazakov, Vieira '09
Frolov `10

Numerics



Gubser, Klebanov, Polyakov `98

Gromov, Serban, Shenderovich, Volin`11;
Roiban, Tseytlin`11;
Vallilo, Mazzucato`11

N.G., Valatka`11

String theory

Agrees with weak coupling gauge theory up to 5 loop!

$$4+\frac{3\lambda}{4\pi^2}-\frac{3\lambda^2}{16\pi^4}+\frac{21\lambda^3}{256\pi^6}+\frac{3(-26+6\zeta_3-15\zeta_5)\lambda^4}{2048\pi^8}+\frac{3\left(158+72\zeta_3-54\zeta_3^2-90\zeta_5+315\zeta_7\right)\lambda^5}{32\,768\,\pi^{10}}+O[\lambda]^6$$

Fieamberti, Fantambrogio, Sieg, Zanon `08
Eden, Heslop, Korchemsky, Smirnov, Sokatchev `12

Bajnok, Janik, Lukowski `08
Bajnok, Hegedus, Janik, Lukowski `09
Arutyunov, Frolov, Suzuki `10

Introduction

Analytical results for Y-system are very rare:

•Asymptotic solution (Large volume/weak coupling) [N.G., Kazakov, Vieira]

•Strong coupling for spinning string states [N.G.; N.G.,Kazakov, Tsuboi]

Close to BPS one can hope to get analytical results for any coupling:

•Basso's slope function [Basso`12]

ABA

$$\gamma = S\frac{\Lambda}{J}\frac{I_{J+1}(\Lambda)}{I_J(\Lambda)} + \mathcal{O}(S^2)$$

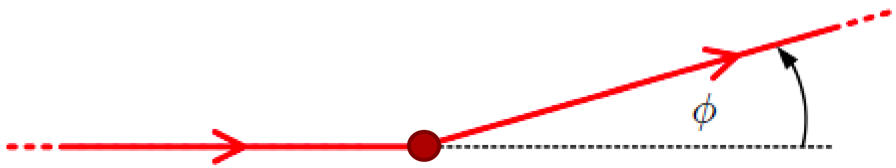
•Bremsstrahlung function [Correa, Henn, Maldacena, Sever `12]

TBA

$$B = \frac{1}{4\pi^4}\frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}$$

Introduction

[Drukker ; Corea, Maldacena, Sever]



$$W_L = \mathrm{P}\exp\int_{-\infty}^{0} dt\left(iA\cdot\dot{x}_q + \vec{\Phi}\cdot\vec{n}\,|\dot{x}_q|\right)\times Z^L\times \mathrm{P}\exp\int_{0}^{\infty} dt\left(iA\cdot\dot{x}_{\bar{q}} + \vec{\Phi}\cdot\vec{n}\,|\dot{x}_{\bar{q}}|\right)$$

$$\langle W\rangle = \left(\frac{\Lambda_{\mathrm{IR}}}{\Lambda_{\mathrm{UV}}}\right)^{\Gamma_{\mathrm{cusp}}}$$

We consider (in the notations from yesterday):

$$\theta = 0,\ \phi\to 0$$

The same consideration should be applicable to:

$$\text{any } \theta,\ \phi\to\theta$$

Bremsstrahlung TBA

"Simplified" TBA equations

[Corea, Maldacena, Sever]

$$\Phi - \Psi = \sum_{a=1}^{\infty} \pi \widehat{K}_a \mathbb{C}_a$$

$$\Phi + \Psi = \mathfrak{s} * \left[-2 \frac{\mathcal{X}_2}{1+\mathcal{Y}_2} + \sum_{a=1}^{\infty} \pi (\widehat{K}_a^+ - \widehat{K}_a^-) \mathbb{C}_a - \pi \delta(u) \mathbb{C}_1 \right]$$

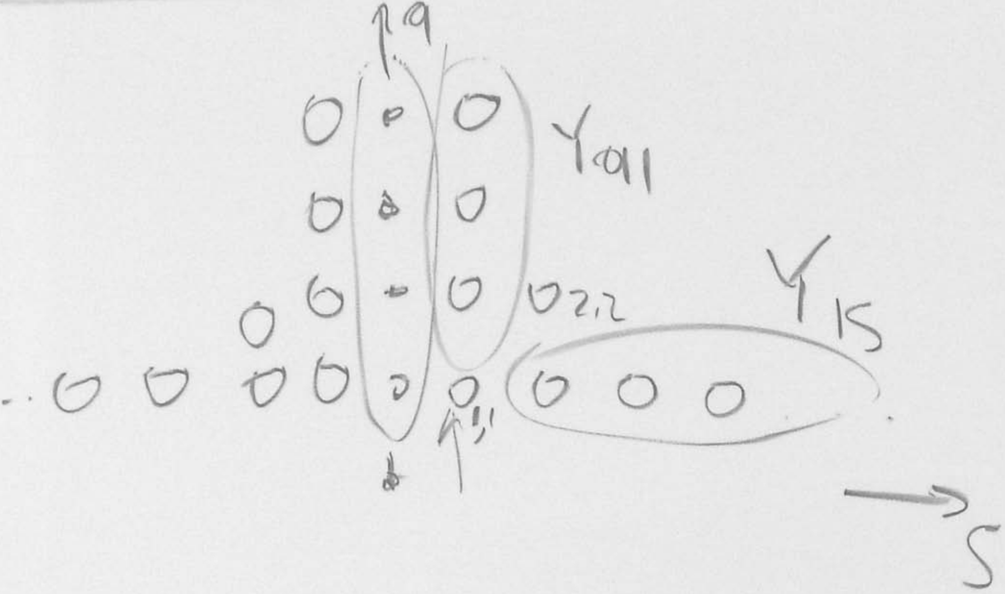
$$\log \mathbb{Y}_m = I_{m,n} \mathfrak{s} * \log \left(\frac{\mathbb{Y}_n}{1+\mathbb{Y}_n} \right) + \delta_{m,2} \mathfrak{s} \hat{*} \left[\log \frac{\Phi}{\Psi} + \phi^2 (\Phi - \Psi) \right] + \phi^2 \pi \mathfrak{s}(u) \mathbb{C}_m$$

$$\mathbb{Y}_s \equiv \mathcal{Y}_s (1 + \phi^2 \mathcal{X}_s)$$

$$\mathbb{C}_a = (-1)^a a^2 F_a \left(\sqrt{1 + \frac{a^2}{16 g^2}} - \frac{a}{4g} \right)^{2+2L} e^{\Delta_a}$$

$$\Delta_a = \left[\frac{1}{2} K_a \hat{*} \log \frac{\Psi}{\Phi} + \frac{1}{2} \widetilde{K}_a \hat{*} \log (\Psi \Phi) + \sum_{b=2}^{\infty} \widetilde{K}_{ab} * \log(1 + \mathcal{Y}_b) - \log a \right] \Bigg|_{u=0}$$

$$\log F_a = \widetilde{K}_a \hat{*} \log \frac{\sinh(2\pi u)}{2\pi u} \Bigg|_{u=0}$$



$\uparrow q$
Y_{q1}
Y_{15}
$\to S$

$$Y_{a0} = \frac{1}{4}\varphi^4 \frac{c_a^2}{u^2} + O(\varphi^6)$$

$$Y_{11} = -1 - \varphi^2 \Psi$$

$$1/Y_{22} = -1 - \varphi^2 \Phi$$

$$Y_{a1} = y_a \left(1 - \frac{1}{2}\varphi^2 X_a\right)$$

$$1/Y_{1s} = y_s \left(1 + \frac{1}{2}\varphi^2 X_s\right)$$

$$K_a = \frac{2a}{\pi(a^2+4u^2)}$$

$$\hat{K}_a = \sqrt{\frac{4g^2-u^2}{4g^2+a^2/4}}\, K_a \qquad \hat{*} \to \int_{-2g}^{2g}$$

$$\tilde{K}_a = \left[\quad\right]^{-1} K_a$$

$$S(u) = \frac{1}{2\cosh(\pi u)}$$

Lightweight FiNLIE

$$\log \mathbb{Y}_m = I_{m,n}\,\mathfrak{s} * \log\left(\frac{\mathbb{Y}_n}{1+\mathbb{Y}_n}\right) + \delta_{m,2}\,\mathfrak{s}\hat{*}\left[\log\frac{\Phi}{\Psi} + \phi^2(\Phi-\Psi)\right] + \phi^2\pi\,\mathfrak{s}(u)\mathbb{C}_m$$

Lightweight FiNLIE:

[Kazakov, Gromov IGST2010]

$$\frac{1}{\mathbb{Y}_m(u+i/2)\mathbb{Y}_m(u-i/2)} = (1+1/\mathbb{Y}_{m+1}(u))(1+1/\mathbb{Y}_{m-1}(u))$$

Solved by:

$$\mathbb{Y}_m(u) = \frac{\mathcal{T}_m(u+i/2)\mathcal{T}_m(u-i/2)}{\mathcal{T}_{m+1}(u)\mathcal{T}_{m-1}(u)} - 1$$

$$\begin{aligned} \mathcal{T}_1(u) &= \mathcal{T}_1(u) \\ \mathcal{T}_2(u) &= \mathcal{T}_1(u-i/2)+\mathcal{T}_1(u+i/2) \\ \mathcal{T}_3(u) &= \mathcal{T}_1(u-i)+\mathcal{T}_1(u)+\mathcal{T}_1(u-i) \\ &\dots \end{aligned}$$

Ansatz:

$$\mathcal{T}_1 = 1 + K_1\hat{*}\rho \quad \Rightarrow \quad \mathcal{T}_m = m + K_m\hat{*}\rho$$

So far all except m=2 are satisfied:

$$\frac{\Phi}{\Psi} = \frac{\mathcal{T}_1(u+i/2+i0)\mathcal{T}_1(u-i/2-i0)}{\mathcal{T}_1(u+i/2-i0)\mathcal{T}_1(u-i/2+i0)} = \frac{(1+K_1^+\hat{*}\rho-\frac{1}{2}\rho)(1+K_1^-\hat{*}\rho-\frac{1}{2}\rho)}{(1+K_1^+\hat{*}\rho+\frac{1}{2}\rho)(1+K_1^-\hat{*}\rho+\frac{1}{2}\rho)}$$

Lightweight FiNLIE

$$\log \mathbb{Y}_m \;=\; I_{m,n}\,\mathfrak{s} * \log\left(\frac{\mathbb{Y}_n}{1+\mathbb{Y}_n}\right) + \delta_{m,2}\,\mathfrak{s}\hat{*}\left[\log\frac{\Phi}{\Psi} + \phi^2(\Phi-\Psi)\right] + \phi^2\pi\,\mathfrak{s}(u)\mathbb{C}_m$$

$$\mathbb{Y}_m(u) = \frac{\mathcal{T}_m(u+i/2)\mathcal{T}_m(u-i/2)}{\mathcal{T}_{m+1}(u)\mathcal{T}_{m-1}(u)} - 1$$

$$\mathcal{T}_1 = 1 + K_1\hat{*}\rho \;\;\rightarrow\;\; 1 + K_1\hat{*}\rho + \phi^2\tau_1$$

$$\tau_1 = [u^2 - 1/12] + K_1\hat{*}\varrho + \sum_{n=1}^{\infty}[b_n K_{n-1}(u) + b_{-n}K_{n-1}(u)]$$

From m=2 equation:

$$\Phi - \Psi = \frac{\tau_1^+\rho - (1+K_1^+\hat{*}\rho)\varrho}{(1+K_1^+\hat{*}\rho)^2 - \frac{1}{4}\rho^2} + \frac{\tau_1^-\rho - (1+K_1^-\hat{*}\rho)\varrho}{(1+K_1^-\hat{*}\rho)^2 - \frac{1}{4}\rho^2}$$

$$\frac{\pi\mathbb{C}_m}{c_m} = 4\frac{b_m - b_{m-2}}{c_m^2 - c_{m-2}^2} - 4\frac{b_m - b_{m+2}}{c_m^2 - c_{m+2}^2}$$

“Simplified” TBA equations

[Corea, Maldacena, Sever]

$$\Phi - \Psi = \sum_{a=1}^{\infty} \pi \widehat{K}_a \, \mathbb{C}_a$$

$$\Phi + \Psi = \mathfrak{s} * \left[-2 \frac{\mathcal{X}_2}{1+\mathcal{Y}_2} + \sum_{a=1}^{\infty} \pi (\widehat{K}_a^+ - \widehat{K}_a^-) \, \mathbb{C}_a - \pi \delta(u) \, \mathbb{C}_1 \right]$$

$$\log \mathbb{Y}_m = I_{m,n} \, \mathfrak{s} * \log \left(\frac{\mathbb{Y}_n}{1+\mathbb{Y}_n} \right) + \delta_{m,2} \, \mathfrak{s} \hat{*} \left[\log \frac{\Phi}{\Psi} + \phi^2 (\Phi - \Psi) \right] + \phi^2 \pi \, \mathfrak{s}(u) \mathbb{C}_m$$

$$\mathbb{Y}_s \equiv \mathcal{Y}_s (1 + \phi^2 \mathcal{X}_s)$$

$$\mathbb{C}_a = (-1)^a a^2 \, F_a \left(\sqrt{1 + \frac{a^2}{16 g^2}} - \frac{a}{4g} \right)^{2+2L} e^{\Delta_a}$$

$$\Delta_a = \left[\frac{1}{2} K_a \hat{*} \log \frac{\Psi}{\Phi} + \frac{1}{2} \widetilde{K}_a \hat{*} \log(\Psi \Phi) + \sum_{b=2}^{\infty} \widetilde{K}_{ab} * \log(1 + \mathcal{Y}_b) - \log a \right] \Bigg|_{u=0}$$

$$\log F_a = \widetilde{K}_a \hat{*} \log \frac{\sinh(2\pi u)}{2\pi u} \Bigg|_{u=0}$$

Lightweight FiNLIE

$$\Delta_a = \left[\frac{1}{2}K_a \hat{*} \log\frac{\Psi}{\Phi} + \frac{1}{2}\widetilde{K}_a \hat{*} \log(\Psi\Phi) + \sum_{b=2}^{\infty} \widetilde{K}_{ab} * \log(1+\mathcal{Y}_b) - \log a\right]\Bigg|_{u=0}$$

$$\Delta_a = \frac{1}{2}\widetilde{K}_a \hat{*} \log\frac{\Psi\Phi\,\mathcal{T}_2^2}{\mathcal{T}_1^{-+}\mathcal{T}_1^{+-}\mathcal{T}_1^{--}\mathcal{T}_1^{++}} + \log\frac{\mathcal{T}_a}{a}\Bigg|_{u=0}$$

Equation for the density can be interpreted differently:

$$\frac{\Phi}{\Psi} = \frac{\mathcal{T}_1(u+i/2+i0)\mathcal{T}_1(u-i/2-i0)}{\mathcal{T}_1(u+i/2-i0)\mathcal{T}_1(u-i/2+i0)} = \frac{(1+\not{K}_1^+ \hat{*}\rho - \frac{1}{2}\rho)(1+\not{K}_1^- \hat{*}\rho - \frac{1}{2}\rho)}{(1+\not{K}_1^+ \hat{*}\rho + \frac{1}{2}\rho)(1+\not{K}_1^- \hat{*}\rho + \frac{1}{2}\rho)}$$

$$\eta \equiv \frac{\Psi\,\mathcal{T}_2}{\mathcal{T}_1^{-+}\,\mathcal{T}_1^{+-}} = \frac{\Phi\,\mathcal{T}_2}{\mathcal{T}_1^{--}\,\mathcal{T}_1^{++}}$$

$$\Delta_a = \widetilde{K}_a \hat{*} \log\eta + \log\frac{c_a}{a}$$

Lightweight FiNLIE

Write everything in terms of rho and eta

$$\eta = \Psi \frac{\mathcal{T}_1^{\{-\}} + \mathcal{T}_1^{\{+\}}}{(\mathcal{T}_1^{\{-\}} + \frac{1}{2}\rho)(\mathcal{T}_1^{\{+\}} + \frac{1}{2}\rho)} = \Phi \frac{\mathcal{T}_1^{\{-\}} + \mathcal{T}_1^{\{+\}}}{(\mathcal{T}_1^{\{-\}} - \frac{1}{2}\rho)(\mathcal{T}_1^{\{+\}} - \frac{1}{2}\rho)}$$

PV notation: $$\mathcal{T}_1^{\{\pm\}} \equiv \frac{1}{2}\left(\mathcal{T}_1^{\pm_+} + \mathcal{T}^{\pm_-}\right) = 1 + \fint_{-2g}^{2g} dv\, \rho(v)\, K_1(u - v \pm \tfrac{i}{2})$$

We can exclude “fermions” from all equations. In particular:

$$\boxed{\Psi - \Phi = \rho\,\eta\,, \qquad \Psi + \Phi = \eta \frac{\frac{1}{2}\rho^2 + 2\mathcal{T}_1^{\{-\}}\mathcal{T}_1^{\{+\}}}{\mathcal{T}_1^{\{-\}} + \mathcal{T}_1^{\{+\}}}}$$

FiNLIE summary

$$\rho = -\frac{1}{\eta}\sum_{a=1}^{\infty} \pi\, \mathbb{C}_a \widehat{K}_a \quad \textbf{(F1)}$$

$$\eta = \frac{\mathcal{T}_1^{\{-\}} + \mathcal{T}_1^{\{+\}}}{\frac{1}{2}\rho^2 + 2\mathcal{T}_1^{\{-\}}\mathcal{T}_1^{\{+\}}} \times \mathfrak{s} * \left[-2\frac{\mathcal{X}_2}{1+\mathcal{Y}_2} + \pi(\widehat{K}_a^+ - \widehat{K}_a^-)\,\mathbb{C}_a - \pi\delta(u)\,\mathbb{C}_1\right] \quad \textbf{(F2)}$$

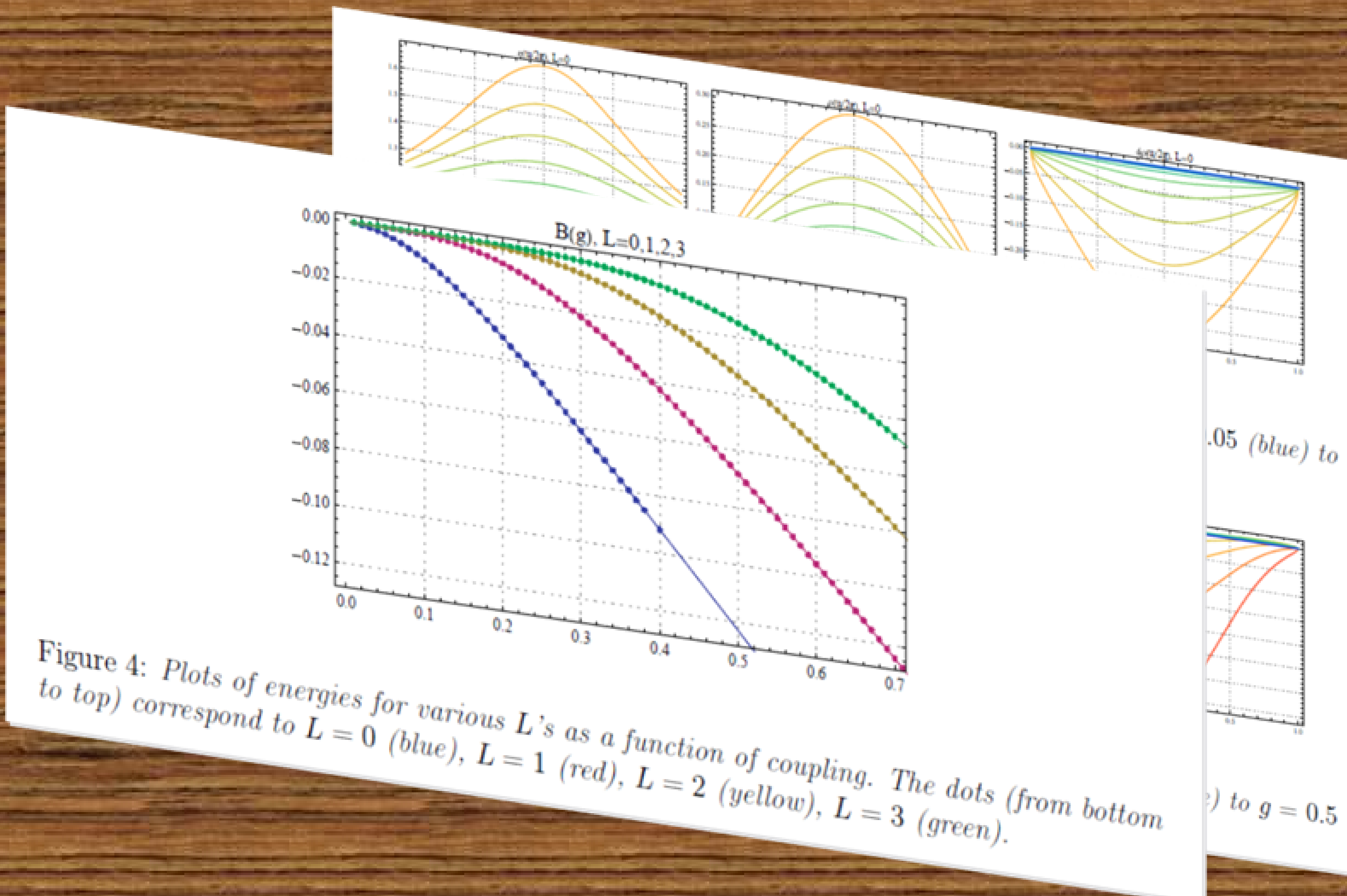
$$\varrho = \rho\frac{\tau_1^{\{+\}}\left(\frac{\rho^2}{4} - \mathcal{T}_1^{\{-\}2}\right) + \tau_1^{\{-\}}\left(\frac{\rho^2}{4} - \mathcal{T}_1^{\{+\}2}\right) - \frac{\eta}{4}\left(\frac{\rho^2}{4} - \mathcal{T}_1^{\{-\}\,2}\right)\left(\frac{\rho^2}{4} - \mathcal{T}_1^{\{+\}2}\right)}{\left(\mathcal{T}_1^{\{-\}} + \mathcal{T}_1^{\{+\}}\right)\left(\frac{\rho^2}{4} - \mathcal{T}_1^{\{-\}}\mathcal{T}_1^{\{+\}}\right)} \quad \textbf{(F3)}$$

$$\mathbb{C}_a = (-1)^a a\, c_a \left(\sqrt{1+\frac{a^2}{16g^2}} - \frac{a}{4g}\right)^{2+2L} \exp\left[\widetilde{K}_a \hat{*} \log\left(\eta\frac{\sinh(2\pi u)}{2\pi u}\right)\right] \quad \textbf{(F4)}$$

$$\frac{\mathcal{X}_2}{1+\mathcal{Y}_2} = -\frac{\mathcal{T}_3\mathcal{T}_1}{\mathcal{T}_1^{++}\mathcal{T}_1^{--}}\left(\frac{\tau_3}{\mathcal{T}_3} + \frac{\tau_1}{\mathcal{T}_1} - \frac{\tau_2^+}{\mathcal{T}_2^+} - \frac{\tau_2^-}{\mathcal{T}_2^-}\right) \qquad \mathcal{T}_m = m + K_m \hat{*} \rho\,, \qquad \tau_m = -\frac{m^3}{12} + mu^2 + K_m \hat{*} \varrho + \sum_{n=-\infty}^{\infty} b_n K_{m-n}$$

$$c_m = \mathcal{T}_m(0) \qquad b_{a+2} - b_a = (c_{a+2}^2 - c_a^2) \times \begin{cases} \sum_{n=1}^{\infty} \frac{\pi\mathbb{C}_{2n-1}}{4c_{2n-1}} + \sum_{n=a/2+1}^{\infty} \frac{\pi\mathbb{C}_{2n}}{4c_{2n}} & , \quad a \in 2\mathbb{Z} \\ -\sum_{n=0}^{a/2-1/2} \frac{\pi\mathbb{C}_{2n+1}}{4c_{2n+1}} & , \quad a \in 2\mathbb{Z}+1 \end{cases}$$

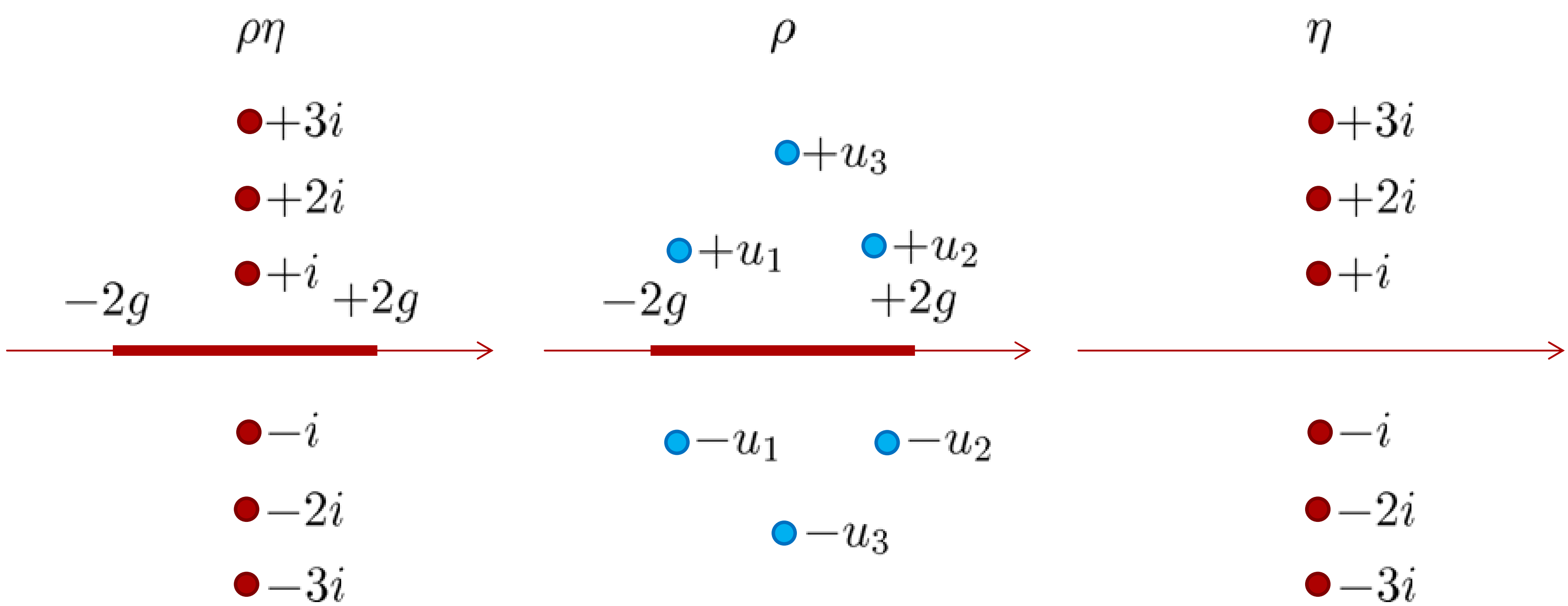
Numerics


B(g), L=0,1,2,3
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Figure 4: Plots of energies for various L's as a function of coupling. The dots (from bottom to top) correspond to L = 0 (blue), L = 1 (red), L = 2 (yellow), L = 3 (green).
.05 (blue) to
) to g = 0.5

Part 2 – Analytical solution

Analyticity ansatz

$$\rho\,\eta = -\pi \sum_{a=1}^{\infty} \widehat{K}_a \mathbb{C}_a \qquad \widehat{K}_a(u) = \sqrt{\frac{4g^2-u^2}{4g^2+a^2/4}}\,\frac{2a}{\pi(a^2+4u^2)}$$



$$\eta(u) = \prod_{a=1}^{\infty} \frac{u^2 - \boxed{u_a^2}}{u^2+a^2} = 1 + \sum_a \boxed{e_a} \left(\frac{1}{u-ia} - \frac{1}{u+ia} \right)$$

Miracle 1

Take the most ugly equation:

$$\eta = \frac{\mathcal{T}_1^{\{-\}} + \mathcal{T}_1^{\{+\}}}{\frac{1}{2}\rho^2 + 2\mathcal{T}_1^{\{-\}}\mathcal{T}_1^{\{+\}}} \times \mathfrak{s} * \left[-2\frac{\mathcal{X}_2}{1+\mathcal{Y}_2} + \pi(\widehat{K}_a^+ - \widehat{K}_a^-)\,\mathbb{C}_a - \pi\delta(u)\,\mathbb{C}_1 \right] \qquad \mathbf{(F2)}$$

$$\frac{\mathcal{X}_2}{1+\mathcal{Y}_2} = -\frac{\mathcal{T}_3\mathcal{T}_1}{\mathcal{T}_1^{++}\mathcal{T}_1^{--}}\left(\frac{\tau_3}{\mathcal{T}_3} + \frac{\tau_1}{\mathcal{T}_1} - \frac{\tau_2^+}{\mathcal{T}_2^+} - \frac{\tau_2^-}{\mathcal{T}_2^-}\right) \qquad \mathcal{T}_m = m + K_m \hat{*} \rho\,, \qquad \tau_m = -\frac{m^3}{12} + mu^2 + K_m \hat{*} \varrho + \sum_{n=-\infty}^{\infty} b_n K_{m-n}$$

$$c_m = \mathcal{T}_m(0) \qquad b_{a+2} - b_a = (c_{a+2}^2 - c_a^2) \times \begin{cases} \sum\limits_{n=1}^{\infty} \frac{\pi\mathbb{C}_{2n-1}}{4c_{2n-1}} + \sum\limits_{n=a/2+1}^{\infty} \frac{\pi\mathbb{C}_{2n}}{4c_{2n}} & , \ a \in 2\mathbb{Z} \\ -\sum\limits_{n=0}^{a/2-1/2} \frac{\pi\mathbb{C}_{2n+1}}{4c_{2n+1}} & , \ a \in 2\mathbb{Z}+1 \end{cases}$$

Plug the ansatz:

$$\eta(u) = 1 + \sum_a e_a \left(\frac{1}{u-ia} - \frac{1}{u+ia} \right)$$

Super simple answer:

$$e_a = -\frac{\pi\mathbb{C}_a}{c_a}$$

FiNLIE summary

$$\rho = -\frac{1}{\eta}\sum_{a=1}^{\infty}\pi\,\mathbb{C}_a\,\widehat{K}_a \qquad \textbf{(F1)}$$

$$\eta = \frac{\mathcal{T}_1^{\{-\}}+\mathcal{T}_1^{\{+\}}}{\frac{1}{2}\rho^2+2\mathcal{T}_1^{\{-\}}\mathcal{T}_1^{\{+\}}}\times\mathfrak{s}*\left[-2\frac{\mathcal{X}_2}{1+\mathcal{Y}_2}+\pi(\widehat{K}_a^{+}-\widehat{K}_a^{-})\,\mathbb{C}_a-\pi\delta(u)\,\mathbb{C}_1\right] \qquad \textbf{(F2)}$$

$$\varrho = \rho\frac{\mathcal{T}_1^{\{+\}}\left(\frac{\rho^2}{4}-\mathcal{T}_1^{\{-\}2}\right)+\mathcal{T}_1^{\{-\}}\left(\frac{\rho^2}{4}-\mathcal{T}_1^{\{+\}2}\right)-\frac{\eta}{4}\left(\frac{\rho^2}{4}-\mathcal{T}_1^{\{-\}2}\right)\left(\frac{\rho^2}{4}-\mathcal{T}_1^{\{+\}2}\right)}{\left(\mathcal{T}_1^{\{-\}}+\mathcal{T}_1^{\{+\}}\right)\left(\frac{\rho^2}{4}-\mathcal{T}_1^{\{-\}}\mathcal{T}_1^{\{+\}}\right)} \qquad \textbf{(F3)}$$

$$\mathbb{C}_a = (-1)^a a\,c_a\left(\sqrt{1+\frac{a^2}{16g^2}}-\frac{a}{4g}\right)^{2+2L}\exp\left[\widetilde{K}_a\hat{*}\log\left(\eta\frac{\sinh(2\pi u)}{2\pi u}\right)\right] \qquad \textbf{(F4)}$$

$$\frac{\mathcal{X}_2}{1+\mathcal{Y}_2} = -\frac{\mathcal{T}_3\mathcal{T}_1}{\mathcal{T}_1^{++}\mathcal{T}_1^{--}}\left(\frac{\mathcal{T}_3}{\mathcal{T}_3}+\frac{\mathcal{T}_1}{\mathcal{T}_1}-\frac{\mathcal{T}_2^{+}}{\mathcal{T}_2^{+}}-\frac{\mathcal{T}_2^{-}}{\mathcal{T}_2^{-}}\right) \qquad \mathcal{T}_m = m+K_m\hat{*}\rho\,, \qquad \mathcal{T}_m = -\frac{m^3}{12}+mu^2+K_m\hat{*}\varrho+\sum_{n=-\infty}^{\infty}b_nK_{m-n}$$

$$c_m = \mathcal{T}_m(0) \qquad b_{a+2}-b_a = (c_{a+2}^2-c_a^2)\times\begin{cases}\sum_{n=1}^{\infty}\frac{\pi\mathbb{C}_{2n-1}}{4c_{2n-1}}+\sum_{n=a/2+1}^{\infty}\frac{\pi\mathbb{C}_{2n}}{4c_{2n}}\,, & a\in 2\mathbb{Z}\\ -\sum_{n=0}^{a/2-1/2}\frac{\pi\mathbb{C}_{2n+1}}{4c_{2n+1}}\,, & a\in 2\mathbb{Z}+1\end{cases}$$

Miracle 2

Take the most ugly equation:

$$\mathbb{C}_a = (-1)^a a\, c_a \left(\sqrt{1+\frac{a^2}{16g^2}} - \frac{a}{4g}\right)^{2+2L} \exp\left[\widetilde{K}_a \hat{*} \log\left(\eta \frac{\sinh(2\pi u)}{2\pi u}\right)\right] \qquad \textbf{(F4)}$$

Plug the ansatz:

$$\eta(u) = \prod_{a=1}^{\infty} \frac{u^2 - u_a^2}{u^2 + a^2}$$

The boundary dressing phase – simply cancels all poles

Integrate:

$$\mathbb{C}_a = (-1)^a a c_a \left(\sqrt{1+\frac{a^2}{16g^2}} - \frac{a}{4g}\right)^{2+2L} \left[(-1)^a \frac{\mathbb{C}_a}{a\, c_a} \prod_{k=1}^{\infty} \frac{x_k^2 - \frac{1}{y_a^2}}{x_k^2 - y_a^2}\right]$$

$$x_k = x(u_k)$$
$$y_a = x(ia/2)$$

Massive cancellations we get a Bethe-like equation!!:

$$1 = \left(\frac{i}{y_a}\right)^{2+2L} \left[\prod_{k=1}^{\infty} \frac{x_k^2 - \frac{1}{y_a^2}}{x_k^2 - y_a^2}\right]$$

Effective Baxter equation

$$1 = \left(\frac{i}{y_a}\right)^{2+2L} \left[\prod_{k=1}^{\infty} \frac{x_k^2 - \frac{1}{y_a^2}}{x_k^2 - y_a^2}\right]$$

The corresponding Baxter equation:

$$\mathbf{T}(x) \equiv x^{L+1}\mathbf{Q}(x) + \frac{(-1)^L}{x^{L+1}}\mathbf{Q}(1/x) \ , \qquad \mathbf{Q}(x) \equiv \prod_{k=1}^{\infty} \frac{x_k^2 - x^2}{x_k^2}$$

Due to the BAE it has the properties:

$$\mathbf{T}(x) = (-1)^L \mathbf{T}(1/x) \ , \qquad \mathbf{T}(y_a) = 0 \qquad\qquad y_a = x(ia/2)$$

This is the Baxter like equation for the “crossing” type of shift of the spectral parameter Curiously this is exactly the same type of shifts like in the Y-system for classical minimal surfaces (Thermodynamic Bobble Ansatz)!

L=0 case:

$$\mathbf{T}(x) = C_1 \sinh\left[2\pi g\left(x + \frac{1}{x}\right)\right] = C_1 \sum_{r=-\infty}^{\infty} I_{2r+1}(4\pi g) x^{2x+1}$$

$$\mathbf{Q}(0) = 1. \quad \Rightarrow \quad C_1 = \frac{1}{I_1(4\pi g)} \quad \Rightarrow \quad Q(x) = \frac{1}{I_1}\sum_{r=0}^{\infty} I_{2r+1} x^{2r}$$

Energy for general L

In general:

$$\mathbf{T}(y) = y^{L+1}\mathbf{Q}(y) + \frac{(-1)^L}{y^{L+1}}\mathbf{Q}(1/y) = \sinh(2\pi u_y)P(y)$$

$$= \underbrace{\cdots + \frac{(-1)^L}{y^{L+1}}}_{Q(1/y)} + \underbrace{\frac{0}{y^{L-1}} + \cdots + 0 \times y^{L-1}}_{\text{GAP}} + \underbrace{y^{L+1} + \ldots}_{Q(y)}$$

Solving a linear system for the coefficients of the polynomial P:

$$P_L(x) = \frac{1}{\det \mathcal{M}_L} \begin{vmatrix} I_{-1} & I_1 & \ldots & I_{2L-3} & I_{2L-1} \\ I_{-3} & I_{-1} & \ldots & I_{2L-5} & I_{2L-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{1-2L} & I_{3-2L} & \ldots & I_{-1} & I_1 \\ 1/x^L & 1/x^{L-2} & \ldots & x^{L-2} & x^L \end{vmatrix}$$

$$\mathcal{M}_L = \begin{pmatrix} I_{-1} & I_1 & \ldots & I_{2L-3} & I_{2L-1} \\ I_{-3} & I_{-1} & \ldots & I_{2L-5} & I_{2L-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{1-2L} & I_{3-2L} & \ldots & I_{-1} & I_1 \\ I_{-1-2L} & I_{1-2L} & \ldots & I_{-3} & I_{-1} \end{pmatrix}$$

Final result:

$$B_L(g) = g^2 \left(-\frac{\det \mathcal{M}_{L+2}^{(2,1)}}{\det \mathcal{M}_{L+2}^{(1,1)}} + 2\frac{\det \mathcal{M}_{L+1}^{(2,1)}}{\det \mathcal{M}_{L+1}^{(1,1)}} - \frac{\det \mathcal{M}_{L}^{(2,1)}}{\det \mathcal{M}_{L}^{(1,1)}} \right)$$

$\mathcal{M}_L^{(a,b)}$ is the matrix obtained by deleting the a^{th} row and b^{th} column of $\mathcal{M}_L$.

Examples

$$B_{L=0} = g^2 \left(1 - \frac{I_3(4\pi g)}{I_1(4\pi g)}\right) \qquad B_{L=1} = g^2 \frac{I_1 I_3 + I_1 I_5 - 2I_3^2}{I_1^2 - I_1 I_3}$$

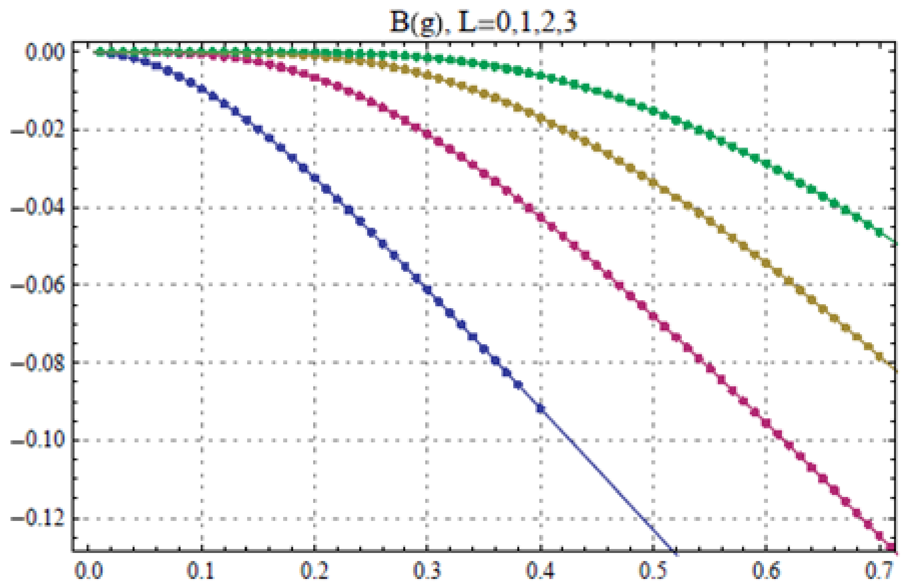


Figure 4: *Plots of energies for various L's as a function of coupling. The dots (from bottom to top) correspond to* $L = 0$ *(blue),* $L = 1$ *(red),* $L = 2$ *(yellow),* $L = 3$ *(green).*

Classical Limit

Classical Worldsheet Solution

Ansatz:

$$\underbrace{y_1 + iy_2 = e^{i\kappa\tau}\sqrt{1+r^2(\sigma)}\ , \qquad y_3 + iy_4 = r(\sigma)e^{i\varphi(\sigma)}}_{AdS_3} \qquad \underbrace{x_1 + ix_2 = e^{i\tau\gamma}\sqrt{1-\rho^2(\sigma)}\ , \qquad x_3 + ix_4 = \rho(\sigma)e^{if(\sigma)}}_{S^3}$$

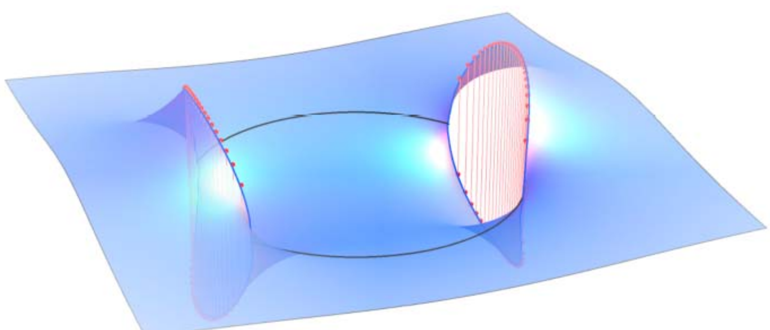
Solution is parametric form:

$$L = 4g\left[\mathbb{K}\left(\omega^2\right) - \mathbb{E}\left(\omega^2\right)\right]$$

$$E = L + g(\theta^2 - \phi^2)\frac{1-\omega^2}{2\mathbb{E}(\omega^2)} \qquad \text{or} \qquad B^{\rm WS} = g\frac{1-\omega^2}{2\mathbb{E}(\omega^2)}$$

These equations precisely coincide with SU(2) folded string! $J_1 = L,\ J_2 = 0$

The algebraic curve is known:



That is GM curve! [Vicedo]

See also [Janik, Laskos-Grabowski]

Classical Limit From TBA

$$\mathcal{M}_L = \begin{pmatrix} I_{-1} & I_1 & \dots & I_{2L-3} & I_{2L-1} \\ I_{-3} & I_{-1} & \dots & I_{2L-5} & I_{2L-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{1-2L} & I_{3-2L} & \dots & I_{-1} & I_1 \\ I_{-1-2L} & I_{1-2L} & \dots & I_{-3} & I_{-1} \end{pmatrix}$$

Using that:

$$I_n = (-1)^{n+1} \oint \frac{dx}{2\pi i} e^{-2\pi g(x+1/x)} x^{n-1}$$

We rewrite the matrix determinant as

$$\det \mathcal{M}_{L-1} = \oint \prod_k^L \frac{dx_k}{2\pi i} e^{-2\pi g(x_k+1/x_k)} \begin{vmatrix} x_1^{-2} & x_1^0 & \dots & x_1^{2L-6} & x_1^{2L-4} \\ x_2^{-4} & x_2^{-2} & \dots & x_2^{2L-8} & x_2^{2L-6} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{L-1}^{2-2L} & x_{L-1}^{4-2L} & \dots & x_{L-1}^{-2} & x_{L-1}^0 \\ x_L^{-2L} & x_L^{2-2L} & \dots & x_L^{-4} & x_L^{-2} \end{vmatrix}$$

$$\det \mathcal{M}_{L-1} = \oint \prod_k^L \frac{dx_k}{2\pi i\, x_k^{2L}} \frac{\Delta^2(x_i^2)}{L!} e^{-2\pi g \sum_k^L (x_k+1/x_k)}$$

O(-2) matrix model!

Classical Limit From TBA

Classical limit = saddle point approximation

$$-\pi g \frac{x_k^2-1}{x_k^2} + \sum_{j\neq k}\left(\frac{1}{x_k-x_j}+\frac{1}{x_k+x_j}\right) - \frac{L}{x_k} = 0$$

Introducing quansimomenta

$$p(x) \equiv \frac{L}{g}\frac{x}{x^2-1} - \frac{2L}{g}\frac{x^2}{x^2-1}G_L^{\mathrm{cl}}(x) \qquad G_L^{\mathrm{cl}}(x) \equiv \frac{1}{2L}\sum_{k=1}^{L}\left(\frac{1}{x-x_k}+\frac{1}{x+x_k}\right)$$

Saddle point equation becomes:

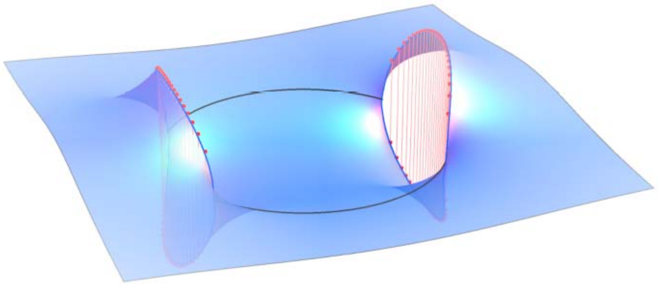
$$p(x_k+i0) + p(x_k-i0) = -2\pi$$

Solution is simple (see KMMZ):

$$p(x) = \pi + 2\mathbb{K}\sqrt{\frac{1-x^2e^{2i\phi}}{-x^2+e^{2i\phi}}}\left(\frac{2ix\sin\phi}{x^2-1}-1\right) + \frac{4\mathbb{E}}{\cos\phi}F_1 - 4\cos\phi\,\mathbb{K}\,F_2$$

$$F_1 = i\mathbb{F}\left[\sin^{-1}\sqrt{\frac{(e^{2i\phi}+1)(e^{i\phi}-x)}{(e^{2i\phi}-1)(e^{i\phi}+x)}};-\tan^2\phi\right]$$

$$F_2 = i\mathbb{E}\left[\sin^{-1}\sqrt{\frac{(e^{2i\phi}+1)(e^{i\phi}-x)}{(e^{2i\phi}-1)(e^{i\phi}+x)}};-\tan^2\phi\right]$$



Conclusions

- Generalize to $\theta \sim \varphi$ [Fedor, how is the progress?]
- Consider general excitations on top
- Relation to the Bubble Y-system?
- How much of this can be used for full TBA?
  additional analyticity is expected – further simplification of FiNLIE
- Understand the Amit's question about X^L vs Z^L
  does it replace the vacuum by a simpler one? With trivial curve?
- What is an interpretation of O(-2) matrix model
  Localization of WS theory? Why 2Lx2L matrices? Non-planar corrections vs. string theory quantization.