Yangian symmetry of scattering amplitudes in planar $\mathcal{N} = 4$ Super Yang-Mills

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Plan of the talk

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  - Restricted kinematics
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- Summary and outlook

The symmetry of the S-matrix in planar $\mathcal{N} = 4$ SYM

- All the on-shell states in $\mathcal{N} = 4$ SYM can be combined into an on-shell superfield,

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\bar{\Gamma}^D + \frac{1}{4!}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D G^-,$$

  which depends on the Grassmann variable η^A, and a null momenta $p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$.

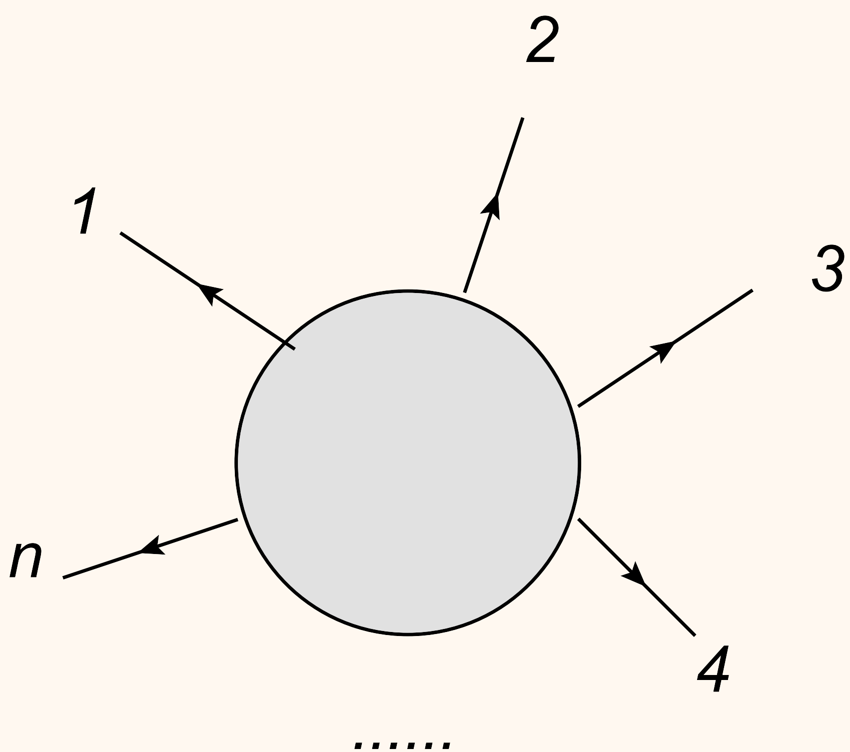
- All color-ordered amplitudes are packaged into a superamplitude $\mathcal{A}(\{\lambda_i, \bar{\lambda}_i, \eta_i\})$; it can be classified according to the Grassmann degree $4k + 8$,

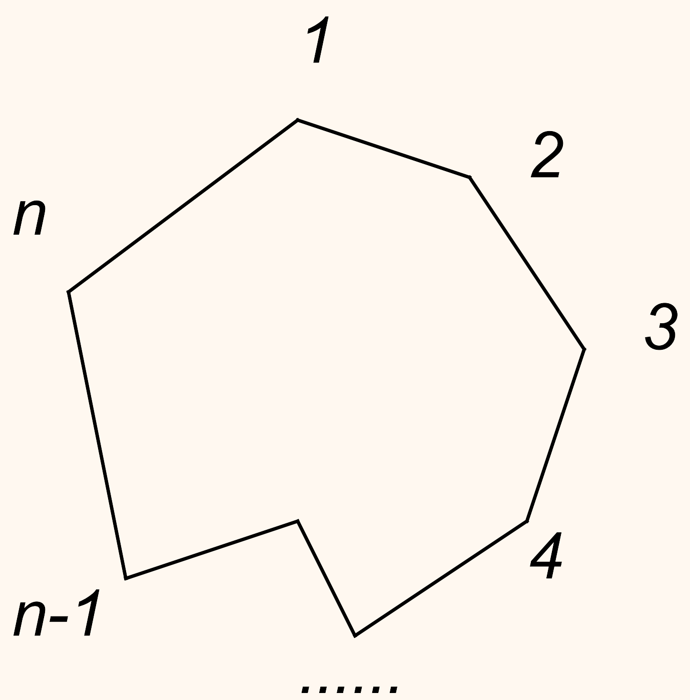
$$\mathcal{A}_n = \mathcal{A}_{n,\mathrm{MHV}} + \mathcal{A}_{n,\mathrm{NMHV}} + \cdots + \mathcal{A}_{n,\overline{\mathrm{MHV}}} = \frac{\delta^4(\sum_i \lambda_i \bar{\lambda}_i)\delta^{0|8}(\sum_i \lambda_i \eta_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_{k=0}^{n-3} A_{n,k}.$$

  where we strip off the MHV tree prefactor; $A_{n,k}$ denotes the N^{k}MHV amplitude.

- $\mathcal{N} = 4$ SYM is a *superconformal* field theory. By introducing a deformation of the free algebra, the tree-level S-matrix is invariant under this $\mathfrak{psu}(2,2|4)$ symmetry: $\{\mathfrak{q}^\alpha_A, \bar{\mathfrak{q}}^A_{\dot{\alpha}}, \mathfrak{p}_{\alpha\dot{\alpha}}, \mathfrak{m}_{\alpha\beta}, \bar{\mathfrak{m}}_{\dot{\alpha}\dot{\beta}}, \mathfrak{s}^A_\alpha, \bar{\mathfrak{s}}^{\dot{\alpha}}_A, \mathfrak{k}_{\alpha\dot{\alpha}}, \mathfrak{d}, \mathfrak{r}^A_B\}$ [Bargheer Beisert Galleas Loebbert McLoughlin 2009].

The symmetry of the S-matrix in planar $\mathcal{N}=4$ SYM





$$x_i^{\alpha\dot\alpha} - x_{i-1}^{\alpha\dot\alpha} = \lambda_i^\alpha \bar\lambda_i^{\dot\alpha}, \qquad \theta_i^{\alpha A} - \theta_{i-1}^{\alpha A} = \lambda_i^\alpha \eta_i^A.$$

Yangian symmetry of the S-matrix in planar $\mathcal{N}=4$ SYM

- In the planar limit, a *dual* conformal symmetry has been observed at both weak [Drummond Henn Smirnov Sokatchev 2006] and strong couplings [Alday Maldacena 2007]. The symmetry has been generalized to a dual superconformal symmetry [Drummond Henn Korchemsky Sokatchev 2008]. The tree-level S-matrix is invariant under the dual $\mathfrak{psu}(2,2|4)$ symmetry.

- The four-gluon amplitude has an all-loop, exponentiated form [Anastasiou Bern Dixon Kosower 2003],

$$A_4 = \exp[-\Gamma_{\text{cusp}} \log \frac{-s-i\epsilon}{\mu^2} \log \frac{-t}{\mu^2} + d(\log \frac{-s-i\epsilon}{\mu^2} + \log \frac{-t}{\mu^2}) + \text{const}].$$

  A general ansatz to remove all infrared and collinear divergences [Bern Dixon Smirnov 2005]:

$$A_n^{\text{BDS}} = 1 + \sum_{\ell=1}^{\infty} g^{2\ell} A_n^{(\ell)}(\epsilon) = \exp\left[\sum_{\ell=1}^{\infty} g^{2\ell}\left(\Gamma_{\text{cusp}}^{(\ell)}(\epsilon) A_{n,0}^{(1)}(\ell\epsilon) + C^{(\ell)} + E_n^{(\ell)}(\epsilon)\right)\right].$$

- Loop amplitudes are not invariant under the dual conformal symmetry, but they satisfy an anomalous Ward identity [Drummond Henn Korchemsky Sokatchev 2007]. BDS ansatz is exact for $n = 4, 5$, since it is the only solution. In general, a finite remainder function is allowed, which depends on $3(n-5)$ cross-ratios, e.g. $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$ etc. for $n = 6$.

The symmetry of the S-matrix in planar $\mathcal{N}=4$ SYM

- There is strong evidence for a duality between MHV amplitude and a null polygonal Wilson loop in dual spacetime [Alday Maldacena 2007] [Drummond Korchemsky Sokatchev 2007] [Brandhuber Heslop Travaglini 2007], tested up to two-loop six-point [Drummond Henn Korchemsky Sokatchev 2007] [Bern Dixon Kosower Roiban Spradlin Vergu Volovich 2008].

- The original superconformal symmetry of the amplitude are mapped to the dual symmetry of the Wilson loop by T-dualities [Berkovits Maldacena 2008] [Beisert Ricci Tseytlin Wolf 2008]. Their closure is an infinite-dimensional Yangian symmetry, $Y[\mathfrak{psu}(2,2|4)]$ [Drummond Henn Plefka 2009].

- A generalized duality between the superamplitude and a supersymmetric Wilson loop has been derived at the integrand level [Mason Skinner 2010][Caron-Huot 2010], although a rigorous UV regularization for the super-loop has not been carried out [Belitzky Korchemsky Sokatchev 2011],

$$A_n(\lambda_i,\bar{\lambda}_i,\eta_i)=W_n(x_i,\theta_i)(1+\mathcal{O}(\epsilon)),\quad W_n=\frac{1}{N_c}\langle \mathrm{Tr}\mathcal{P}e^{-\oint \mathbf{A}(x_i,\theta_i)}\rangle.$$

- The chiral super Wilson loop obscures one chiral half of superconformal symmetries. As a natural generalization, Wilson loops in non-chiral $\mathcal{N}=4$ superspace generally manifest the full symmetry [Caron-Huot 2011] [Beisert Vergu 2012] [Beisert SH Schwab Vergu 2012].

The symmetry of the S-matrix in planar $\mathcal{N}=4$ SYM

- We define *BDS-subtracted* S-matrix: $A_{n,k} = A_n^{\rm BDS} \times R_{n,k}$, which is a finite object depending on dual conformal cross-ratios and the so-called R-invariants. It has simple collinear limits, and by definition, $R_{4,0} = R_{5,0} = R_{5,1}/R_{5,1}^{\rm tree} = 1$.

- Such invariants can be constructed using twistors of the dual (super)space [Hodges 2009],

$$\text{momentum twistor}: \quad \mathcal{Z}_i = (Z_i^a, \chi_i^A) = (\lambda_i^\alpha, x_i^{\alpha\dot\alpha}\lambda_{i\alpha}, \theta_i^{\alpha A}\lambda_{i\alpha});$$

$$\text{four-bracket}: \quad \langle ijkl\rangle = \varepsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d, \quad \text{e.g.} \quad u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle};$$

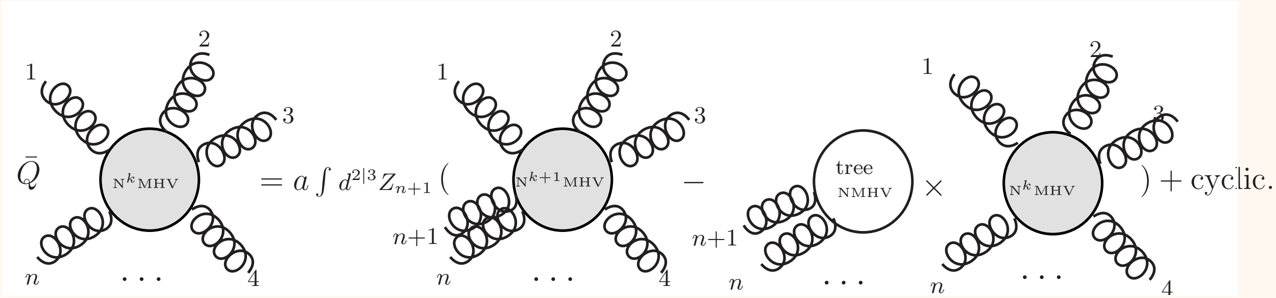
$$\text{R-invariant}: \quad [i\,j\,k\,l\,m] = \frac{\delta^{0|4}(\chi_i^A\langle jklm\rangle + \text{cyclic})}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}.$$

They form the fundamental representation of the dual superconformal algebra,

$$Q_A^a = (\mathfrak{Q}_A^\alpha, \bar{\mathfrak{S}}_A^{\dot\alpha}) = \sum_{i=1}^n Z_i^a \frac{\partial}{\partial \chi_i^A}, \qquad \bar{Q}_a^A = (\mathfrak{S}_\alpha^A, \bar{\mathfrak{Q}}_{\dot\alpha}^A = \bar{s}_{\dot\alpha}^A) = \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial Z_i^a},$$

$$K_b^a = (\mathfrak{P}_{\alpha\dot\alpha}, \mathfrak{K}_{\alpha\dot\alpha}, \mathfrak{M}_{\alpha\beta}, \bar{\mathfrak{M}}_{\dot\alpha\dot\beta}, \mathfrak{D}) = \sum_{i=1}^n Z_i^a \frac{\partial}{\partial Z_i^b}, \qquad R_B^A = \mathfrak{R}_B^A = \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial \chi_i^B}.$$

The S-matrix from the symmetry: a new proposal



- The BDS-subtracted S-matrix is not invariant under the naive $\bar{Q}_a^A$. We propose an all-loop equation for the "anomaly" as collinear integral (see also [Bullimore Skinner 2011]),

$$\bar{Q}_a^A R_{n,k} = \Gamma_{\text{cusp}}\, \text{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3} \mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R^{\text{tree}}_{n+1,1}\right] + \text{cyclic},$$

where the cusp anomalous dimension is known $\Gamma_{\text{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \ldots$.

- The RHS is an 1d integral over τ; one then computes the residue at $\epsilon \to 0$,

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon(\mathcal{Z}_{n-1} - \frac{\langle n{-}1 n 2 3\rangle}{\langle n 1 2 3\rangle} \tau \mathcal{Z}_1) + \mathcal{O}(\epsilon^2)\,,$$

$$\text{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} (d^{2|3} \mathcal{Z}_{n+1})_a^A = \frac{\langle n{-}1 n 2 3\rangle}{\langle n 1 2 3\rangle} (n{-}1\, n\, 1)_a \oint_{\epsilon=0} \epsilon d\epsilon \int_0^\infty d\tau (d^{0|3} \chi_{n+1})^A\,.$$

The S-matrix from the symmetry: a new proposal

- Using the discrete parity symmetry, we derive an equivalent equation for level-one generator, $Q_A^{(1)a} = (s_A^\alpha, \ldots) = \frac{1}{2}\sum_{i,j} \text{sgn}(j-i)\left(Z_i^a \frac{\partial}{\partial Z_i^b} Z_j^b \frac{\partial}{\partial \chi_j^A} - Z_i^a \frac{\partial}{\partial \chi_i^B} \chi_j^B \frac{\partial}{\partial \chi_j^A}\right)$,

$$Q_A^{(1)a} R_{n,k} = \Gamma_{\text{cusp}} Z_n^a \lim_{\epsilon\to 0}\int_0^\infty \frac{d\tau}{\tau}(d\eta_{n+1})_A \left(R_{n+1,k} - \sum_{i,j} C_{i,j}\frac{\partial R_{n,k}}{\partial \chi_j}\right) + \text{cyclic}.$$

- The equations essentially amount to Yangian invariance of the S-matrix. RHS are not anomalies: they should be interpreted as *quantum corrections* of (naive) symmetry generators acting on the S-matrix [Bargheer Beisert Galleas Loebbert McLoughlin 2009] [Sever Vieira 2009] [Beisert Henn McLoughlin Plefka 2010].

- We claim that the equations are valid for any value of the coupling. When expanded in powers of Γ_{cusp}, they recursively give derivatives of all-loop amplitudes.

- The differential equations are nice: both sides are finite, regulator independent, and manifest the transcendentality of loop amplitudes. They are powerful: together with collinear limits, the solutions uniquely determine the full S-matrix.

The S-matrix from the symmetry: outline of a derivation

- The way $\bar{Q}$ acts on a Wilson loop is by inserting a fermion operator on the edges, which was calculated in explicit examples using Feynman diagrams [Caron-Huot 2011]

$$\bar{Q}^A_{\dot{\alpha}}\langle W_n\rangle \propto g^2 \oint dx_{\dot{\alpha}\alpha}\langle(\psi^A + F\theta^A + \ldots)^\alpha W_n\rangle.$$

- The key new ingredient: the fermion insertion is the unique excitation with given quantum numbers. The Operator Product Expansion [Alday Gaiotto Maldacena Sever Vieira 2010] allows us to extract the excited n-gon Wilson loop from an $(n+1)$-gon in collinear limit,

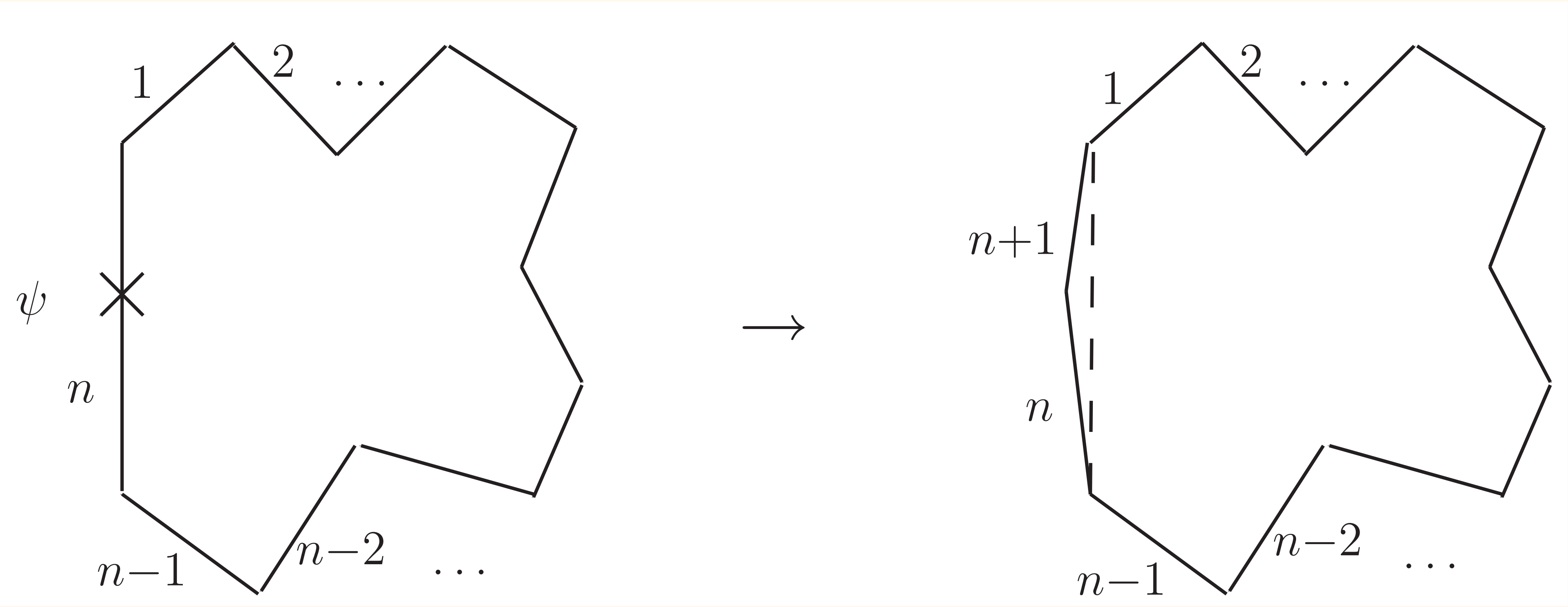
$$\frac{1}{A_n^{\text{BDS}}}\bar{Q}\langle W_{n,k}\rangle = \frac{g^2}{F(g^2)}\,\text{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}R_{n+1,k+1}(\tau,\epsilon) + \text{cyclic}.$$

  Given that BDS ansatz is one-loop exact, we obtain the $\bar{Q}$ of BDS,

$$\langle W_{n,k}\rangle\bar{Q}\frac{1}{A_n^{\text{BDS}}} = -\Gamma_{\text{cusp}}R_{n,k}\,\text{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}R^{\text{tree}}_{n+1,1}(\tau,\epsilon) + \text{cyclic}.$$

- Both τ integrals diverge, but the sum must be finite, so we have $g^2/F(g^2) = \Gamma_{\text{cusp}}$. A crucial test of our derivation is to check the dispersion relation of the insertion.

The S-matrix from the symmetry: outline of a derivation



The S-matrix from the symmetry: outline of a derivation

- The fermion operators are labeled by a momentum, p, conjugate to its position along the edge. We want to understand the $\log\epsilon$ term in momentum space,

$$\lim_{\epsilon\to 0}\log\left(\int_0^\infty d\tau\,\tau^{i\frac{p}{2}}\,d^{0|3}\chi_{n+1}\,R_{n+1,1}\right)\to\log\epsilon\times\gamma(p)+C(p),$$

where the dispersion relation $\gamma(p)$ has to match that of a fermion excitation of the null edge, known for any values of the coupling thanks to integrability [Basso 2010].

- We have derived $R_{6,1}$ up to two loops, which can be used to give $\gamma(p)$ to order $\Gamma_{\rm cusp}^2$,

$$\gamma(p)=\Gamma_{\rm cusp}\left(\psi_+-\psi(1)\right)-\frac{\Gamma_{\rm cusp}^2}{8}\left(\psi_+''+4\psi_-'(\psi_--\frac{1}{p})+6\zeta(3)\right).$$

This agrees precisely with [Basso 2010], and it also confirms the prefactor must be $\Gamma_{\rm cusp}$.

- For RHS of the equations, we only need the total-τ integral (zero-momentum). The cancelation of $\log\epsilon$ divergences in that case is guaranteed by the Goldstone theorem: the fermion with $p=0$ is a Goldstone fermion, thus $\gamma(0)=0$.

The S-matrix from the symmetry: jumpstarting amplitudes I

- The simplest case, MHV remainder function, $R_{n,0}$, is independent of Grassmann variables. We can obtain all the derivatives from its $\bar{Q}$,

$$\frac{\partial}{\partial \chi_i^1} \bar{Q}_a^1 R_{n,0} = \frac{\partial}{\partial Z_i^a} R_{n,0},$$

which uniquely determine $R_{n,0}$, up to a constant (fixed by collinear limit). From the RHS, we can already deduce its total derivative must be of the form

$$dR_{n,0} = \sum_{i,j} F_{i,j} d \log \langle i-1\, i\, i+1\, j \rangle,$$

which holds to all loops. This proves the conjecture of [Caron-Huot 2011].

- Remarkably, the solution to $\bar{Q}$ equation is also unique for NMHV amplitude, up to a linear combination of R-invariants, which can be fixed by collinear limits.

- We need both equations beyond NMHV. For all-loop N^{k}MHV, the solutions are unique, up to invariants under naive Q, $\bar{Q}$ and $Q^{(1)}$. It is known [Korchemsky Sokatchev 2010][Drummond Ferro 2010] that all such invariants are given by the Grassmannian formula [Arkani-Hamed Cachazo Cheung Kaplan 2009].

The S-matrix from the symmetry: jumpstarting amplitudes I

- From the collinear integral of $R^{\text{1-loop}}_{7,1}$, one can easily compute the derivative of two-loop MHV hexagon, reproducing the formula in [Goncharov Spradlin Vergu Volovich 2010] [Del Duca Duhr Smirnov 2010]

$$R^{\text{2-loop}}_{6,0} = 4\sum_{i=1}^{3}\left(L_4^+(u_i) - \frac{1}{2}\mathrm{Li}_4(1-\frac{1}{u_i})\right) - \frac{1}{2}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-\frac{1}{u_i})\right)^2 + \frac{1}{6}J^4 + \frac{\pi^2}{3}J^2 + \frac{\pi^4}{18}.$$

  Higher-point amplitudes are similar; we found the symbol agrees with [Caron-Huot 2011].

- We derived the two-loop NMHV hexagon, and found agreement with results in [Kosower Roiban Vergu 2011] and [Dixon Drummond Henn 2011]. Similarly we computed the symbol for the heptagon.

- An ansatz was proposed for $S[R^{\text{3-loop}}_{6,0}]$ [Dixon Drummond Henn 2011], based on physical considerations, e.g. OPE constraints, and assumptions on possible forms of the symbol. We confirmed their assumptions, and fixed the two undetermined parameters,

$$S[R^{\text{3-loop}}_{6,0}] = \left(S[X] - \frac{3}{8}S[f_1] + \frac{7}{32}S[f_2]\right)(u_1, u_2, u_3).$$

Jumpstarting amplitudes II: restricted kinematics

- Amplitudes/Wilson loops simplify significantly for the restricted kinematics when the $2n$ external momenta/edges are embedded in a two-dimensional subspace [Alday Maldacena 2009][Del Duca Duhr Smirnov 2009] [Heslop Khoze 2010]. It is natural to do the reduction supersymmetrically, and the symmetry factorizes $PSU(2,2|4) \to SL(2|2)_{\text{even}} \times SL(2|2)_{\text{odd}}$:

$$\mathcal{Z}_{2i-1} = (\lambda^1_{2i-1}, 0, \lambda^3_{2i-1}, 0, \chi^1_{2i-1}, 0, \chi^3_{2i-1}, 0), \quad \mathcal{Z}_{2i} = (0, \lambda^2_{2i}, 0, \lambda^4_{2i}, 0, \chi^2_{2i}, 0, \chi^4_{2i}).$$

  Four-brackets factorize, $\langle 2i{-}1\, 2j{-}1\, 2k\, 2l\rangle = \langle 2i{-}1\, 2j{-}1\rangle\, [2k\, 2l]$; even and odd cross-ratios are built from 1d distances, $u_{a,b,c,d} = \frac{\langle a\, b\rangle\, \langle c\, d\rangle}{\langle a\, c\rangle\, \langle b\, d\rangle}$.

- Superamplitudes will be built from “mini” R-invariants in even and odd sector,

$$(a\, b\, c) = \frac{\delta^{0|2}(\langle a\, b\rangle \chi_c + \langle b\, c\rangle \chi_a + \langle c\, a\rangle \chi_b)}{\langle a\, b\rangle\, \langle b\, c\rangle\, \langle c\, a\rangle},$$

  Tree amplitudes are trivial combinations of R-invariants, which, e.g. for N^{2}MHV, are products of $(a\, b\, c\, d) := -(a\, b\, c)(a\, c\, d)$. Loop amplitudes are combinations with coefficients being pure, transcendental functions of conformal cross-ratios.

Jumpstarting amplitudes II: restricted kinematics

- The $\bar{Q}$ equation in restricted kinematics is derived by considering the overlap of a $2n$-gon with the collinear limit of $(2n+2)$-gon. In the even sector, we have,

$$\bar{Q}_a^A R_{2n,k} = \Gamma_{\rm cusp} \int d^{1|2}\lambda_{2n+1} \int d^{0|1}\lambda_{2n+2}(R_{2n+2,k+1} - R^{\rm tree} R_{2n,k}) + \text{cyclic},$$

where we take $\lambda_{2n+2} = \lambda_{2n} + \epsilon\lambda_2$ supersymmetrically, and explicitly the measure is

$$\int d^{1|2}\lambda_{2n+1} \int d^{0|1}\lambda_{2n+2} = \lambda_{2n,a} \lim_{\epsilon\to 0} \int_{\lambda_{2n-1}}^{\lambda_1} \langle\lambda_{2n+1} d\lambda_{2n+1}\rangle \int d^2\chi_{2n+1}(d\chi_{2n+2})^A.$$

- From a reasonably nice form of N^{2}MHV tree, we applied the equation twice and derived the $2n$-point two-loop MHV, which agrees with [Heslop Khoze 2010] [Gaiotto Maldacena Sever Vieira 2010].

- A nice byproduct from the computation is the one-loop NMHV, now written in a basis of R-invariants, in terms of functions of cross-ratios, e.g. the octagon

$$R_{8,1} = \big((3\,5\,7)[2\,4\,6] f^1_{8,1}(u_1,u_2) + 7\text{ cyclic}\big) + R^{\rm tree}_{8,1}\, f^2_{8,1}(u_1,u_2);$$
$$f^{1,\text{1-loop}}_{8,1} = \log(1-u_1)\log(1-u_2), \quad f^{2,\text{1-loop}}_{8,1} = \log u_1(1-u_1)\log u_2(1-u_2).$$

Jumpstarting amplitudes II: one-loop N^{2}MHV

- For $k{+}\ell{=}3$, i.e. one-loop N^{2}MHV, two-loop NMHV and three-loop MHV, new structures, such as combinations $x{-}y, 1{-}x{-}y$, appear. We computed the amplitudes explicitly using the equations. The result is highly non-trivial and interesting.

- The one-loop N^{2}MHV octagon can be put into a nice form ($u_i := u_{i,i+2,i+4,i+6}$)

$$R_{8,2} = R_{8,2}^{\text{tree}} \frac{u_1 u_2}{1 - u_1 - u_2} \big(f_{8,2}(u_1, u_2) + f_{8,2}(u_2, u_1) \big) + \text{(3 cyclic)},$$

where $R_{8,2}^{\text{tree}} = (1\,3\,5\,7)[2\,4\,6\,8]$, $\quad f_{8,2}(x,y) = \mathrm{Li}_2(x) + \frac{1}{2} \log x \log\left(\frac{1-x}{y}\right) - \frac{\pi^2}{8}$.

- The same pattern also appears in higher-point N^{2}MHV, e.g. the decagon reads,

$$\begin{aligned} R_{10,2} = \; & (1357)[26810] f^1_{10,2}(u_1, u_6) + \text{(4 cyclic)} + [(1357)[46810] f^1_{10,2}(u_1, u_4) \\ & + (1357)[2468] f^2_{10,2}(u_1, u_2) + 2(1357)[2410][468] f_{8,2}(1{-}u_1, u_{10}) + \text{(9 cyclic)}] + \ldots\,, \end{aligned}$$

where $\ldots$ denotes remaining $\log\log$ terms with pure R invariants as coefficients;

$$f^1_{10,2}(x,y) = 2\frac{xy}{1-x-y}\big(f_{8,2}(1-x, 1-y) - f_{8,2}(y,x)\big),$$

$$f^2_{10,2}(x,y) = 2\frac{y(1-x)}{x-y} f_{8,2}(y, 1-x) - 2\frac{x(1-y)}{x-y} f_{8,2}(x, 1-y).$$

Jumpstarting amplitudes II: two-loop NMHV

- We determined the two-loop NMHV octagon, up to one parameter corresponding to adding a multiple of the one-loop amplitude, in terms of the two functions:

$$f_{8,1}^{1,\text{2-loop}} = \mathrm{Li}_{2,2}(x, \frac{1-y}{x}) + \mathrm{Li}_{2,2}(1-x, \frac{y}{1-x}) - \mathrm{Li}_{2,2}(x, \frac{1}{x}) - \mathrm{Li}_{2,2}(1, y) + C(x,y) + (x \leftrightarrow y),$$

where the "classical part" $C(x,y)$ involves only polylogarithms of degree 3 or less:

$$\begin{aligned} C(x,y) = &-\left(\mathrm{Li}_3(\frac{xy}{(1-x)(1-y)}) - \mathrm{Li}_3(\frac{x}{1-y}) - \mathrm{Li}_3(\frac{y}{1-x}) + \mathrm{Li}_3(x) + \mathrm{Li}_3(y) + (\mathrm{Li}_2(\frac{y}{1-x}) - \mathrm{Li}_2(y)) \log\frac{1-y}{x}\right) \log y(1-x) \\ &+ \left(4\mathrm{Li}_3(y) + 2\mathrm{Li}_3(1-y) + \mathrm{Li}_2(y) \log\frac{x^2(1-y)}{y} - 2\zeta(3)\right) \log(1-x) \\ &+ \left(\frac{1}{2}\log xy \log(1-x)(1-y) - \frac{1}{2}\log x \log y\right) \log(1-x) \log(1-y) \\ &+ \frac{1}{2}\mathrm{Li}_2(y) \log^2(1-x) + \frac{3}{2}\mathrm{Li}_2(x)\mathrm{Li}_2(y) + \frac{5}{8}\log^2(1-x)\log^2(1-y), \end{aligned}$$

and a simpler function $f_{8,1}^{2,\text{2-loop}} = g(x,y) + (x \leftrightarrow 1-x) + (y \leftrightarrow 1-y) + (x \leftrightarrow y)$:

$$\begin{aligned} g(x,y) = &\left(6\mathrm{Li}_3(1-x) - \mathrm{Li}_2(1-x) \log\frac{1-x}{x} + \log^2 x \log 1-x\right) \log y + \left(\frac{1}{8}\log x + \frac{3}{4}\log 1-x\right) \log x \log^2 y \\ &- \frac{1}{8}\log x \log 1-x \log y \log 1-y - 3\zeta(3)\log x + \frac{\pi^2}{6}\left(\frac{1}{4}\log x \log\frac{x}{(1-x)} - \log x \log y\right) + \frac{\pi^4}{160}. \end{aligned}$$

Jumpstarting amplitudes II: two-loop NMHV

- The function $f^1_{8,1}$ is basically a component amplitude, $f^1_{8,1} = \langle 13\rangle[68]R_{8,1}|_{\chi^1\chi^3\chi^6\chi^8}$. We consider small x expansion, $f^1_{8,1}(v=\frac{x}{1-x}, w=\frac{y}{1-y}) = \sum_{n=1}^\infty f_n(w)v^n$:

$$
\begin{aligned}
f_n^{\text{2-loop}} = & \quad \log v\, f_n^{\text{2-loop}}|_{\log v} + [\frac{w^n}{n^2}(2\,\mathrm{Li}_2(-w)+\log w\log(1+w))]_{\text{reg}} + [\frac{2w^n}{n^3}\log\frac{1+w}{w}]_{\text{reg}} \\
& + \frac{4(-)^n}{n^3}\log(1+w) + \frac{(-)^n}{n}(\frac{1}{n}-2S_1(n))\log w\log(1+w) - \frac{(-)^n}{n^2}\mathrm{Li}_2(-w) \\
& + \frac{4(-)^n}{n}(S_1(n)-\frac{1}{n})\log(1+w)^2 - \frac{(-)^n}{n}(6\,\mathrm{Li}_3(-w) - \log w\,\mathrm{Li}_2(-w)+\pi^2\log(1+w)), \\
f_n^{\text{2-loop}}|_{\log v} = & \; [\frac{w^n}{n^2}\log\frac{w}{1+w}]_{\text{reg}} + \frac{(-1)^n}{n}\log^2(1+w) - \frac{(-1)^n}{n^2}\log(1+w),
\end{aligned}
$$

where the $\log v$ part agrees with OPE leading-order predictions. The most interesting part is in terms which mix v with w, while the remaining terms are factorized.

- The result becomes remarkably simple after doing a Fourier (Mellin) transform,

$$
f(p,q) = \int_0^1 \frac{dv}{v}\int_0^1 \frac{dw}{w} f(v,w) v^{i\frac{p}{2}} w^{i\frac{q}{2}};
$$

$$
f_{8,1}^{1,\text{2-loop}}(p,q) = \frac{\pi}{p\sinh(\frac{\pi p}{2})}\frac{\pi}{q\sinh(\frac{\pi q}{2})}\frac{\coth(\frac{\pi p}{2})-\coth(\frac{\pi q}{2})}{p-q} + \text{factorized}.
$$

Jumpstarting amplitudes II: three-loop MHV

- We also derived the two-loop NMHV decagon, whose non-trivial, mixed part is essentially a sum of octagons. Based on this, we obtained the complete *function* for the three-loop MHV octagon, up to two constants multiplying two-loop MHV and NMHV octagons. All other beyond-the-symbol ambiguities were fixed.

- The result, in terms of functions like $\mathrm{Li}_{3,3}$, is relatively involved, but the small x expansion is compact; in particular the mixed part is similar to two-loop NMHV,

$$\begin{aligned} f_n^{\text{3-loop}} = & \sum_{i=1}^{n} \Big[c_i w^i (\log v \log \frac{w}{1+w} + 2\,\mathrm{Li}_2(-w) + \log w \log(1+w)) + c_i' w^i \log \frac{w}{1+w}\Big]_{\text{reg}} \\ & + \sum_{i=1}^{n} \Big[\frac{c_i}{w^i} (\log v \log(1+w) + 2\,\mathrm{Li}_2(-w) + \log w \log(1+w)) + \frac{c_i'}{w^i} \log(1+w)\Big]_{\text{reg}} \\ & + \text{factorized}\,. \end{aligned}$$

- We expect it to have a nice Mellin representation, and possibly also for higher points. We have a rich set of data: non-trivial but simple, suggesting some underlying picture. How to understand such nice structures from integrability?

Summary and outlook

- The all-loop S-matrix in planar $\mathcal{N} = 4$ SYM is invariant under a suitably deformed Yangian symmetry at the quantum level, and is fully determined by it.

- We derived new, elegant equations based on the quantum-corrected symmetry, and tested them extensively against e.g. results of multi-loop amplitudes and OPE.

- The equations have provided new data for the S-matrix of planar $\mathcal{N} = 4$ SYM; we hope that they will provide more insights into its integrability.

- Open questions
  - OPE interpretations of the result, especially how to understand multi-particle states? Relations to the spin chain picture in [Sever Wang Vieira 2012]?
  - Understanding the equations at strong coupling? Relations to TBA, Y-system?
  - Beyond amplitudes in $\mathcal{N} = 4$ SYM: non-chiral Wilson loops/correlation functions in the light-cone limit? the S-matrix of super Chern-Simons from symmetries?