Holographic correlation functions at strong coupling from integrability

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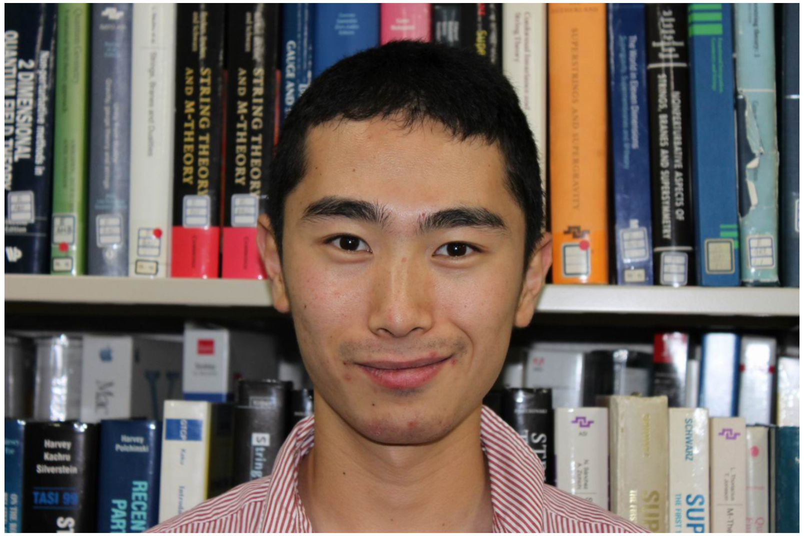
Based on

arXiv:1110.3949
arXiv:1205.6060

in collaboration with

Shota Komatsu



1 Introduction

AdS/CFT $1997 \sim$

Diverse aspects in diverse set-ups

The most basic aspect in the most basic set-up

Structure of **CFT** in $\mathcal{N}=4$ **SYM**/$AdS_5 \times S^5$ **string duality**

Basic ingredients for CFT

- ♦ **2-point functions** $\Leftrightarrow$ **spectrum**
- ♦ **3-point functions** $\Leftrightarrow$ **interaction**

$\Rightarrow$ **4-point functions : crossing symmetry, etc**

Correlation functions in the basic duality:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle$$

$$\mathcal{O}_i(x_i) = \begin{cases} \mathrm{Tr}\,(\phi_1(x_i)\phi_2(x_i)\cdots) & \textbf{SYM side} \\ \int d^2z_i V_i(z_i; x_i) \quad x_i \in \partial(AdS_5) & \textbf{string side} \end{cases}$$

Studies of the basic correlation functions have naturally evolved in the manner

BPS (kinematical)	$\Longrightarrow$	**Non-BPS** (dymanical)
2-point	$\Longrightarrow$	**3-point**

A large number of people contributed to this fascinating developments, using **integrability-based methods**: **integrable spin chains, Bethe ansatz, method of spectral curves**, etc. (See the review by Beisert et al (2010))

$\Downarrow$

Most recently, the focus has been on

Non-BPS 3-point functions using integrability

SYM side **Technology to compute the overlaps of Bethe eigenstates**

Okuyama, Tseng, Roiban, Volovich, Alday, Gava, Narain, . . . ,

2011 ∼ Escobedo, Gromov, Sever, Vieira, Caetano, Foda, Serban, Wheeler, Kostov, Matsuo, . . .

String side **Use of semi-classical integrability for "heavy" states**

- **Heavy-Heavy** : Tsuji, Janik-Surowka-Wereszczynski, Buchbinder-Tseytlin,. . .
- **Heavy-Heavy** $\oplus$ **Light(BPS)** or **near BPS**

  2010 ∼ Zarembo, Costa-Monteiro-Santos-Zoakos, Roiban-Tseytlin, . . . ,

  2011∼ Klose-McLoughlin, Buchbinder-Tseytlin, . . .
- Genuine **Heavy-Heavy-Heavy**: $\Longleftarrow$ focus of this talk

  2011 ∼ Janik-Wereszczynski, Kazama-Komatsu

Holographic 3-point function in the saddle-point approximation

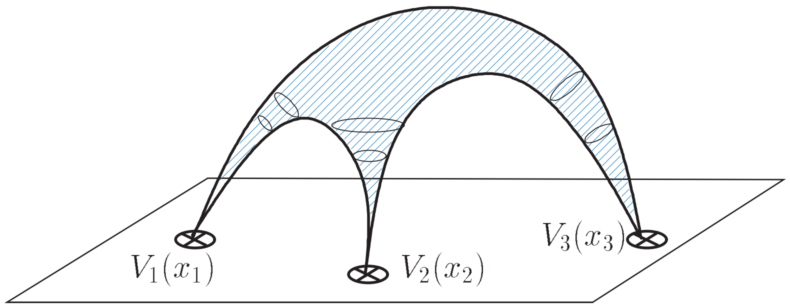
Structure

$$G(x_1, x_2, x_3) = e^{-S[X_*]} \prod_{i=1}^{3} V_i[X_*; z_i, x_i, Q_i]$$

$x_i =$ Points on the boundary of AdS

$$S \sim \log V_i[Q_i] \sim \mathcal{O}(\sqrt{\lambda})$$

$$\frac{\delta}{\delta X}\left(-S[X] + \sum_i \log V_i[X]\right)\Bigg|_{X_*} = 0$$



- $V_i = (1,1)$ primary $\Longrightarrow$ No z_i dependence.
- Near each x_i, the solution $X_* \sim$ the saddle point solution for $\langle V_i(x_1) V_i(x_2) \rangle$

Serious obstacles

- No systematic method to construct conformally invariant vertex operators of interest (even semi-classically) in curved spacetime.
- No three-pronged saddle solutions in curved spacetime are known.

Nontheless

It is possible to overcome these difficulties by exploiting the classical integrability of the string in $AdS_\star \times S^*$

Key: *The global information is connected to the local information through underlying integrability and analyticity*

- **R. Janik and A. Wereszczynski**, arXiv:1109.6262
  - **Strings in $AdS_2 \times S^k$**

  Computed the contribution of the AdS_2 part of the string $\sim$ evaluation of the action. (Contribution of the vertex operators $\sim$ trivial since string is structureless on the boundary)

  Contribution of the (spinning) S^k part (action $\oplus$ vertex) remains to be computed.

- **Y.K. and S. Komatsu**
  - arXiv:1110.3949: **Part I**
    - Large spin limit of **GKP spinning strings in AdS_3 (LSGKP)**

    Evaluated the finite part of the action $S[X_*]$
  - arXiv:1205.6060: **Part II**:

    ⋆ Developed a **general method** for evaluating **the contribution of the vertex operators** $\Rightarrow$ Applied to GKP strings

    ⋆ **Complete finite result for the LSGKP 3-point function** .

Part I

Computation of the finite part of the action

(∼ Calculation of the area of the Wilson loop for gluon-scattering)

- *Integrability for strings in* $\boldsymbol{AdS_3}$ *and GKP string I*
  - ⋆ ***Method of Pohlmeyer reduction***
- ***Action in terms of contour integrals***

  Generalized Riemann bilinear identity
- *Analysis of the* ***auxiliary linear problem*** *from two directions*
  - *Monodromy matrices and their eigenfunctions*
  - *WKB analysis of eigenfunctions*
- *Computation of the finite part of the action*

Part II

Contribution of the vertex operators

- ***state-operator correspondence***

  vertex operators $\Rightarrow$ ***wave functions***

  in terms of ***action-angle variables***

  – *Integrability for strings in* $\boldsymbol{AdS_3}$ *and GKP string II*

  ⋆ ***Framework of spectral curve and finite gap solution***

  – ***Sklyanin's method*** $\oplus$ ***global symmetry transformations***

  to construct and evaluate the ***action-angle variables****:*

  $\Rightarrow$ ***contributions of wave functions***

- *Computation of two point functions*

- *Computation of the* ***three point function*** *for LSGKP strings*

Part I

Computation of the finite part of the action

2 Integrability for strings in AdS_3 and GKP strings I

Method of Pohlmeyer reduction

2.1 String in Euclidean AdS_3

String in Euclidean AdS_3 (radius set to 1)

$$\vec{X} = (X_{-1}, X_0, X_1, X_2, X_3, X_4) \subset AdS_5$$

$$\vec{X} \cdot \vec{X} = -X_{-1}^2 + X_1^2 + X_2^2 + X_4^2 = -1$$

Poincaré coordinates: Boundary of AdS_3 at $z = 0$, described by $(x, \bar{x})$

$$X_+ \equiv X_{-1} + X_4 = \frac{1}{z}, \qquad X_- \equiv X_{-1} - X_4 = z + \frac{x\bar{x}}{z}$$

$$X \equiv X_1 + iX_2 = \frac{x}{z}, \qquad \bar{X} \equiv X_1 - iX_2 = \frac{\bar{x}}{z}$$

Convenient **matrix representation** and **global symmetry transformation**

$$\mathbb{X} \equiv \begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix}, \qquad \det \mathbb{X} = 1$$

$$\mathbb{X}' = V_L \mathbb{X} V_R$$

$$V_L \in SL(2,C)_L\,, \quad V_R \in SL(2,C)_R$$

Global symmetry: $G \equiv SO(4,C) = SL(2,C)_L \times SL(2,C)_R$,

Action

$$S = T \cdot \text{Area} = 2T \int d^2z \partial \vec{X} \cdot \bar{\partial} \vec{X}\,, \qquad \vec{X} \cdot \vec{X} = -1$$

Eq. of motion and Viraosoro conditions

$$\partial \bar{\partial} \vec{X} = (\partial \vec{X} \cdot \bar{\partial} \vec{X}) \vec{X}\,, \qquad \partial \vec{X} \cdot \partial \vec{X} = \bar{\partial} \vec{X} \cdot \bar{\partial} \vec{X} = 0$$

2.2 Pohlmeyer reduction

Describe the system with **G-invariant fields $\alpha, p, \bar{p}$** $\quad (\vec{N} \perp \vec{X}, \partial\vec{X}, \bar{\partial}\vec{X})$

$$e^{2\alpha} = \frac{1}{2}\partial\vec{X}\cdot\bar{\partial}\vec{X}\,, \quad p = \frac{1}{2}\vec{N}\cdot\partial^2\vec{X}\,, \quad \bar{p} = -\frac{1}{2}\vec{N}\cdot\bar{\partial}^2\vec{X}$$

Eq. of motion + Virasoro $\Leftrightarrow$ Flatness of certain left and right connections

$$[\partial + B_z^L, \bar{\partial} + B_{\bar{z}}^L] = 0\,, \qquad [\partial + B_z^R, \bar{\partial} + B_{\bar{z}}^R] = 0$$

$$\Downarrow$$

$$\partial\bar{\partial}\alpha - e^{2\alpha} + p\bar{p}e^{-2\alpha} = 0$$

$$p = p(z)\,, \qquad \bar{p} = \bar{p}(\bar{z})$$

Integrability $\Rightarrow$ Extend to **flat Lax connections** $B_z(\xi), B_{\bar z}(\xi)$
with $\xi =$ **complex spectral parameter**

$$B_z(\xi) = \frac{1}{\xi}\Phi_z + A_z\,, \qquad B_{\bar z}(\xi) = \xi\Phi_{\bar z} + A_{\bar z}$$

They are expressed in terms of α, p and $\bar p$ as

$$A_z \equiv \begin{pmatrix} \frac{1}{2}\partial\alpha & 0 \\ 0 & -\frac{1}{2}\partial\alpha \end{pmatrix}, \qquad A_{\bar z} \equiv \begin{pmatrix} -\frac{1}{2}\bar\partial\alpha & 0 \\ 0 & \frac{1}{2}\bar\partial\alpha \end{pmatrix}$$

$$\Phi_z \equiv \begin{pmatrix} 0 & -e^{\alpha} \\ -pe^{-\alpha} & 0 \end{pmatrix}, \qquad \Phi_{\bar z} \equiv \begin{pmatrix} 0 & -\bar p e^{-\alpha} \\ -e^{\alpha} & 0 \end{pmatrix}$$

B^L and B^R are identified as

- $B^L_z = B_z(\xi = 1)\,, \qquad B^L_{\bar z} = B_{\bar z}(\xi = 1)$
- $B^R_z = \mathcal{U}^\dagger B_z(\xi = i)\mathcal{U}\,, \qquad B^R_{\bar z} = \mathcal{U}^\dagger B_{\bar z}(\xi = i)\mathcal{U}$

$$\mathcal{U} = e^{i\pi/4}\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$$

□ **Auxiliary linear problem and reconstruction formula**:

Flatness condition $\Leftrightarrow$ compatibility of the set of linear equations:

Auxiliary linear problem

$$(\partial + B_z(\xi))\psi(\xi, z, \bar{z}) = 0\,, \qquad (\bar{\partial} + B_{\bar{z}}(\xi))\psi(\xi, z, \bar{z}) = 0$$

Two independent solutions for $\psi(\xi, z, \bar{z})$ contain all the important information

$\Rightarrow$ Two sets of independent solutions for the left and the right problems

$$\psi_a^L = \psi_a(\xi = 1)\,, \quad \psi_{\dot{a}}^R = U^\dagger \psi_{\dot{a}}(\xi = i)\,, \qquad a, \dot{a} = 1, 2$$

$SL(2)$-invariant product

$$\langle\psi,\chi\rangle \equiv \epsilon^{\alpha\beta}\psi_\alpha\chi_\beta\,, \qquad (\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}\,, \quad \epsilon^{12} \equiv 1)$$

$\psi^{L,R}$ are normalized as

$$\langle\psi^L_a,\psi^L_b\rangle = \epsilon_{ab}\,, \qquad \langle\psi^R_{\dot{a}},\psi^R_{\dot{b}}\rangle = \epsilon_{\dot{a}\dot{b}}$$

Reconstruction formula for the string coordinates

$$\mathbb{X}_{a\dot{a}} = \psi^L_{1,a}\psi^R_{\dot{1},\dot{a}} + \psi^L_{2,a}\psi^R_{\dot{2},\dot{a}}$$

2.3 GKP string spinning in X_1-X_2 plane

"Reference" (elliptic) GKP solution (Gubser-Klebanov-Polyakov, 2002)

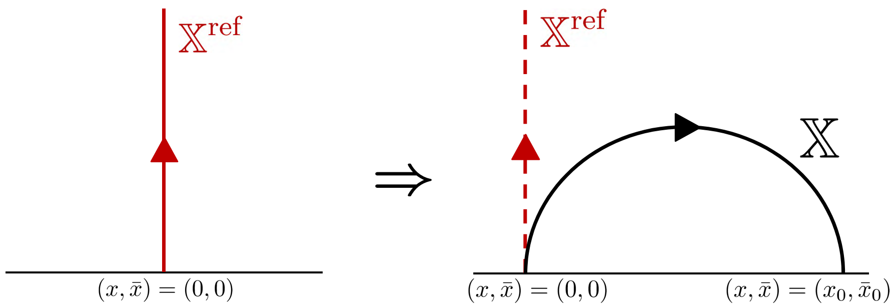
$$\mathbb{X}^{\mathrm{ref}}_{GKP} = \begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix} = \begin{pmatrix} e^{-\kappa\tau}\cosh\rho(\sigma) & e^{\omega\tau}\sinh\rho(\sigma) \\ e^{-\omega\tau}\sinh\rho(\sigma) & e^{\kappa\tau}\sinh\rho(\sigma) \end{pmatrix}, \quad \tau = it$$

It can be expressed in terms of the Jacobi elliptic functions[a] $\mathbf{dn}$ and $\mathbf{cn}$

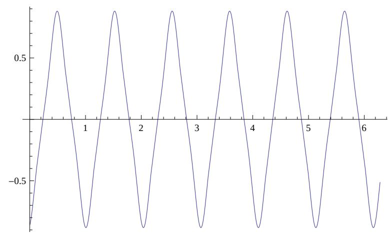
$$\kappa \equiv \omega k\,, \quad \omega \equiv \frac{2}{\pi}\mathcal{K}(k^2)\,, \quad k \leq 1$$

$$\cosh\rho(\sigma) \equiv \frac{\mathrm{dn}\,(\omega(\sigma+\pi/2))}{\sqrt{1-k^2}}$$

$$\sinh\rho(\sigma) \equiv \frac{k\,\mathrm{cn}\,(\omega(\sigma+\pi/2))}{\sqrt{1-k^2}}$$



$\rho(\sigma)$



[a] $\mathcal{K}(k^2)$ = complete elliptic integral of the first kind.

Large spin limit of GKP (LSGKP) : $k \longrightarrow 1 \Rightarrow \omega \longrightarrow \kappa$

$$\mathbb{X}^{\text{ref}}_{LSGKP} = \begin{pmatrix} e^{-\kappa\tau}\cosh\rho(\sigma) & e^{\kappa\tau}\sinh\rho(\sigma) \\ e^{-\kappa\tau}\sinh\rho(\sigma) & e^{\kappa\tau}\sinh\rho(\sigma) \end{pmatrix}$$

Dilatation charge and spin in terms of κ

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\kappa \int_0^{2\pi} d\sigma\ \cosh^2\rho = \frac{\sqrt{\lambda}}{2\pi}(\kappa\pi + \sinh\kappa\pi)$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\kappa \int_0^{2\pi} d\sigma\ \sinh^2\rho = \frac{\sqrt{\lambda}}{2\pi}(-\kappa\pi + \sinh\kappa\pi)$$

$SL(2)_L$ (left) charge $\quad \ell^+ \equiv \frac{1}{2}(\Delta + S) = \frac{\sqrt{\lambda}}{2\pi}\sinh\kappa\pi$

$SL(2)_R$ (right) charge $\quad \ell^- \equiv \frac{1}{2}(\Delta - S) = \frac{\sqrt{\lambda}}{2\pi}\kappa\pi \qquad \ll \ell^+$ for large κ

□ View from the Pohlmeyer reduction:

From the definitions of $\boldsymbol{p}, \bar{\boldsymbol{p}}$ and $\boldsymbol{\alpha}$,

$$p(z) = -\frac{\kappa^2}{4z^2}, \qquad \bar{p}(\bar{z}) = -\frac{\kappa^2}{4\bar{z}^2}$$

$$e^{2\alpha(z,\bar{z})} = \sqrt{p\bar{p}}$$

Auxiliary linear problem: $(\partial + B_z(\xi))\psi = 0$ and $(\bar{\partial} + B_{\bar{z}}(\xi))\psi = 0$

Solution

$$\psi = \mathcal{A}\tilde{\psi}, \qquad \mathcal{A} = \begin{pmatrix} p^{-1/4}e^{\alpha/2} & 0 \\ 0 & p^{1/4}e^{-\alpha/2} \end{pmatrix}$$

$$\tilde{\psi}_{\pm} = \exp\left(\pm\frac{\kappa i}{2}\left(\xi^{-1}\ln z - \xi \ln \bar{z}\right)\right)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

Monodromy around the origin

$$\begin{pmatrix} \tilde{\psi}'_+ \\ \tilde{\psi}'_- \end{pmatrix} = M \begin{pmatrix} \tilde{\psi}_+ \\ \tilde{\psi}_- \end{pmatrix}, \quad M = \begin{pmatrix} e^{i\hat{p}(\xi)} & 0 \\ 0 & e^{-i\hat{p}(\xi)} \end{pmatrix}$$

$$\hat{p}(\xi) = i\kappa\pi \left(\xi^{-1} + \xi\right)$$

This characterizes the behavior around each singularity (leg).

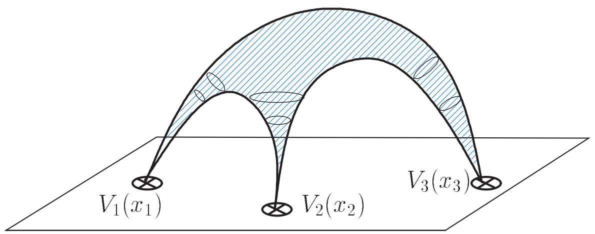
3 Action in terms of contour integrals

3.1 Finite part of the area

Definition of the "regularized area" (for N-point function)

$$A = 2\int d^2z\, \partial\vec{X}\cdot\bar{\partial}\vec{X} = 4\int d^2z\, e^{2\alpha} = A_{fin} + A_{div}$$

$$A_{div} = 4\int d^2z\,\sqrt{p\bar{p}} \quad \ni 4\int d^2z \frac{|\delta_i|^2}{|z-z_i|^2} \sim \text{log divergent}$$



$$A_{fin} = 4\int d^2z\, \left(e^{2\alpha} - \sqrt{p\bar{p}}\right) \overset{EoM}{=} 2A_{reg} + \pi(N-2)$$

$$A_{reg} \equiv \int d^2z\, \left(e^{2\alpha} + p\bar{p}\, e^{-2\alpha} - 2\sqrt{p\bar{p}}\right)$$

We can write A_{reg} as (cf. gluon scattering problem (Alday-Maldacena, ...))

$$A_{reg} = \frac{i}{4}\int_D \lambda dz \wedge \omega$$

$$\lambda = \sqrt{p}$$

$$\omega = ud\bar{z} + vdz = \text{ closed 1-form}$$

where

$$u = 2\sqrt{\bar{p}}(\cosh 2\hat{\alpha} - 1)\,, \quad v = \frac{1}{\sqrt{p}}(\partial\hat{\alpha})^2\,, \quad \hat{\alpha} = \alpha - \frac{1}{2}\ln p\bar{p}$$

Behavior of $p(z)$ near the insertion points

$$p(z) \overset{z \longrightarrow z_i}{\sim} \frac{-\kappa_i^2}{4(z - z_i)^2}$$

For **three point function**, $p(z)$ is actually **uniquely determined**

$$p(z) = -\frac{1}{4}\left(\frac{\kappa_1^2 z_{12} z_{13}}{z - z_1} + \frac{\kappa_2^2 z_{21} z_{23}}{z - z_2} + \frac{\kappa_3^2 z_{31} z_{32}}{z - z_3}\right)\frac{1}{(z - z_1)(z - z_2)(z - z_3)}$$

$$z_{ij} \equiv z_i - z_j$$

Define the function

$$\Lambda(z) \equiv \int_{z_0}^{z} \lambda(z')dz' = \int_{z_0}^{z} \sqrt{p(z')}dz'$$

$\Lambda(z)$ has

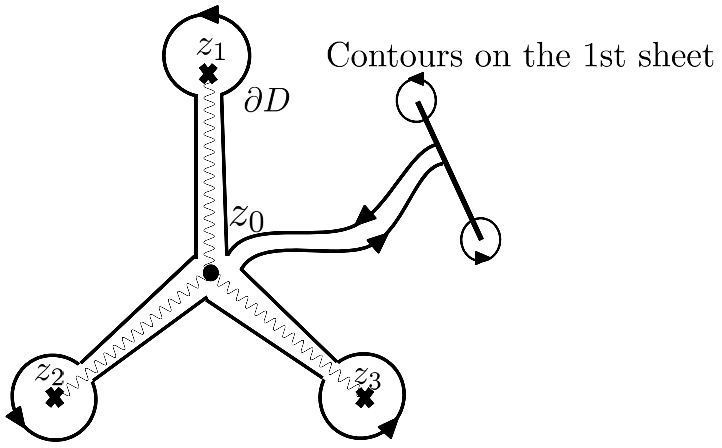
- **three log branch cuts** running from the singularities z_i
- **one square-root cut** connecting 2 zeros of $p(z)$

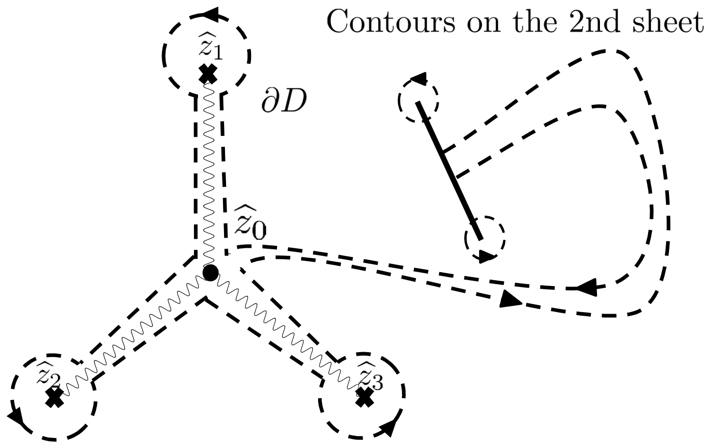
Λ is single-valued on the double cover D of the world-sheet.

Stokes theorem ⇒ **A_{reg} as a contour integral**

$$A_{reg} = \frac{i}{4}\int_D d\Lambda \wedge \omega = \frac{i}{4}\int_D d(\Lambda\omega) = -\frac{i}{4}\int_{\partial D} \Lambda\omega$$

The contour ∂D for the LSGKP three-point function



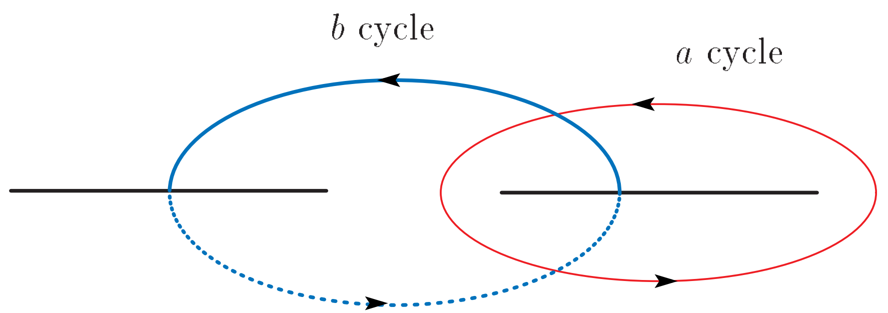


Further, we can **re-express $\int_{\partial D} \Lambda\omega$ more explicitly by using the generalization of the Riemann bilinear identities.**

3.2 Generalized Riemann bilinear identities

Usual Riemann bilinear identity for closed 1-forms $\boldsymbol{\lambda}$ and $\boldsymbol{\omega}$:

Example: Hyperelliptic Riemann surface with $\boldsymbol{g = 1}$



$$\int_{\partial D} \Lambda \omega = \oint_b \lambda \oint_a \omega - \oint_a \lambda \oint_b \omega$$

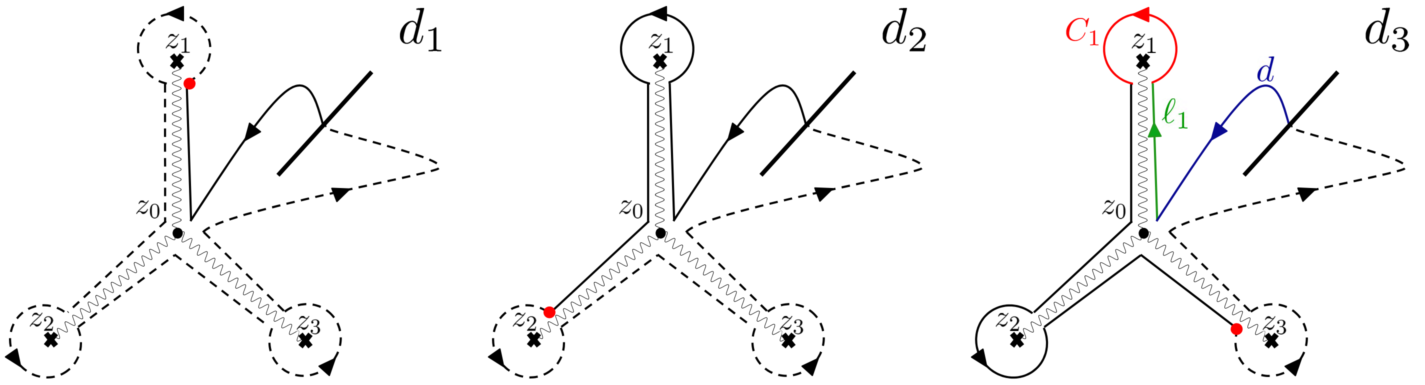
One can derive a generalization for the case with additional log branch cuts

The full identity is rather complicated.

• For LSGKP strings, substantial simplification occurs. The most convenient form is

$$\boldsymbol{A_{reg} = \frac{\pi}{12} + \frac{i}{4} \sum_{j=1}^{3} \oint_{C_i} \lambda dz \oint_{d_j} \omega}$$

The contours $\boldsymbol{d_j}$'s



- The major task will be the evaluation of **the integral $\oint_{d_j} \omega$**.

This information is contained in the behavior of the eigenfunctions of the auxiliary linear problem around z_i and along paths connecting $\{z_i, z_j\}$

4 Analysis of the auxililary linear problem

4.1 Monodromy matrices and their eigenfunctions

Globally we do not know the saddle point solution.

Locally around each z_i, the solution $\sim$ LSGKP solution

Characterized by the **local monodromy matrix** $M_i \in SL(2,C)$.

Each M_i, separately, can be diagonalized as

$$U_i M_i U_i^{-1} = \begin{pmatrix} e^{i\hat{p}_i(\xi)} & 0 \\ 0 & e^{-i\hat{p}_i(\xi)} \end{pmatrix}, \qquad \hat{p}_i(\xi) = i\kappa_i\pi\left(\xi^{-1}+\xi\right)$$

Eigenvectors $i_\pm$ of M_i

$$i_\pm \sim \exp\left[\pm\left(\frac{1}{\xi}\int\sqrt{p(z)}dz + \xi\int\sqrt{\bar{p}(\bar{z})}d\bar{z}\right)\right]$$

★ M_i's cannot be diagonalized simultaneously.

- $\det M_i = 1$
- Global consistency $M_1 M_2 M_3 = 1$

$\Rightarrow$ **M_i and the eigenvectors $i_\pm$ can be determined in terms of $\hat{p}_i(\xi)$ up to some unknown constants.**

- These constants cancel in some combinations of $\boxed{\langle i_\pm, j_\mp \rangle}$

Example

$$\log \langle 2_-, 1_+ \rangle + \log \langle 1_-, 2_+ \rangle = \log \left(\frac{\sin \frac{\hat{p}_1(\xi) - \hat{p}_2(\xi) + \hat{p}_3(\xi)}{2} \sin \frac{-\hat{p}_1(\xi) + \hat{p}_2(\xi) + \hat{p}_3(\xi)}{2}}{\sin \hat{p}_1(\xi) \sin \hat{p}_2(\xi)} \right)$$

To separate out the individual terms, we need to know the global analyticity property of $\langle i_\pm, j_\mp \rangle$ as a function of ξ.

4.2 WKB analysis of eigenfunctions

For this purpose, **solve the auxiliary linear problem in powers of ξ (and $1/\xi$)**

$$(\partial + B_z(\xi))\psi(\xi) = 0\,, \qquad (\bar{\partial} + B_{\bar{z}}(\xi))\psi(\xi) = 0$$

$$\psi = \mathcal{A}\tilde{\psi} = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}$$

$$\tilde{\psi}_1 = \exp\left[\frac{S_{-1}}{\xi} + S_0 + \xi S_1 + \xi^2 S_2 + \cdots\right]$$

We can solve for $S_{-1}, S_0, S_1, \ldots$.

In the vicinity of each z_i, classify the two independent solutions as

$s_i =$ small solution: exponentially decreasing, unambiguous

$b_i =$ big solution: exponentially increasing, ambiguous $\quad b_i' = b_i + a s_i$

5 Computation of the finite part of the action

Combine the analysis of monodromy eigenstates and the WKB eigenstates:

Relate s_i with $i_\pm$: This depends on the **sign of $\mathrm{Im}\,\xi$** (S_{-1} is imaginary)

Im $\xi > 0$ region (with $\kappa_2 > \kappa_1, \kappa_3, \quad \kappa_1 + \kappa_3 > \kappa_2$.)

$\Rightarrow$ Identification: $s_1 \sim 1_+ \,, s_2 \sim 2_- \,, s_3 \sim 3_+$

Contour integrals $\int_{d_i} \omega$ appear in ratios of $\langle s_i, s_j \rangle$

$$\frac{\langle s_2 \,, s_3 \rangle}{\langle s_2 \,, s_1 \rangle \langle s_1 \,, s_3 \rangle} = \frac{\langle 2_-, 3_+ \rangle}{\langle 2_-, 1_+ \rangle \langle 1_+, 3_+ \rangle}$$

$$= \exp \left[\frac{1}{\xi} \int_{d_1} \lambda dz + \xi \int_{d_1} \sqrt{\bar{p}} d\bar{z} + \frac{\xi}{2} \int_{d_1} \omega + \cdots \right]$$

Im $\xi < 0$ region Identification with $i_\pm$ are reversed.

Thus one finds

$$\langle s_1, s_2 \rangle = \begin{cases} \langle 1_+, 2_- \rangle & \mathrm{Im}\,\xi > 0 \\ \langle 1_-, 2_+ \rangle & \mathrm{Im}\,\xi < 0 \end{cases}, \qquad etc.$$

Apply **Wiener-Hopf decomposition formula**

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\xi' \frac{1}{\xi' - \xi}\left(F(\xi') + G(\xi')\right) = \begin{cases} F(\xi)\,, & (\mathrm{Im}\,\xi > 0) \\ -G(\xi)\,, & (\mathrm{Im}\,\xi < 0) \end{cases}$$

to the previously obtained relation

$$\log\langle 2_-, 1_+ \rangle + \log\langle 1_-, 2_+ \rangle = \log\left(\frac{\sin\frac{\hat{p}_1(\xi)-\hat{p}_2(\xi)+\hat{p}_3(\xi)}{2}\sin\frac{-\hat{p}_1(\xi)+\hat{p}_2(\xi)+\hat{p}_3(\xi)}{2}}{\sin\hat{p}_1(\xi)\sin\hat{p}_2(\xi)}\right)$$

$\Rightarrow$ We obtain $\log\langle 2_-, 1_+ \rangle$ **and** $\log\langle 1_-, 2_+ \rangle$ **separately in terms of** $\hat{p}_i(\xi)$**.**

So we can now evaluate A_{reg} in terms of κ_i in the manner

$$A_{reg} \Leftarrow \int_{d_j} \omega \Leftarrow \text{ratios of } \langle s_i, s_j \rangle \sim \langle i_\pm, j_\pm \rangle \Leftarrow \hat{p}_i(\xi) \ni \kappa_i$$

Result for A_{reg}

$$A_{reg} = \frac{\pi}{12} + \pi \Big[-\kappa_1 K(\kappa_1) - \kappa_2 K(\kappa_2) - \kappa_3 K(\kappa_3)$$
$$+ \frac{\kappa_1 + \kappa_2 + \kappa_3}{2} K(\frac{\kappa_1 + \kappa_2 + \kappa_3}{2})$$
$$+ \frac{|-\kappa_1 + \kappa_2 + \kappa_3|}{2} K(\frac{|-\kappa_1 + \kappa_2 + \kappa_3|}{2})$$
$$+ \frac{|\kappa_1 - \kappa_2 + \kappa_3|}{2} K(\frac{|\kappa_1 - \kappa_2 + \kappa_3|}{2})$$
$$+ \frac{|\kappa_1 + \kappa_2 - \kappa_3|}{2} K(\frac{|\kappa_1 + \kappa_2 - \kappa_3|}{2}) \Big]$$

where $K(x)$

$$K(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\theta \, e^{-\theta} \log \left(1 - e^{-4\pi x \cosh \theta}\right)$$

Part II

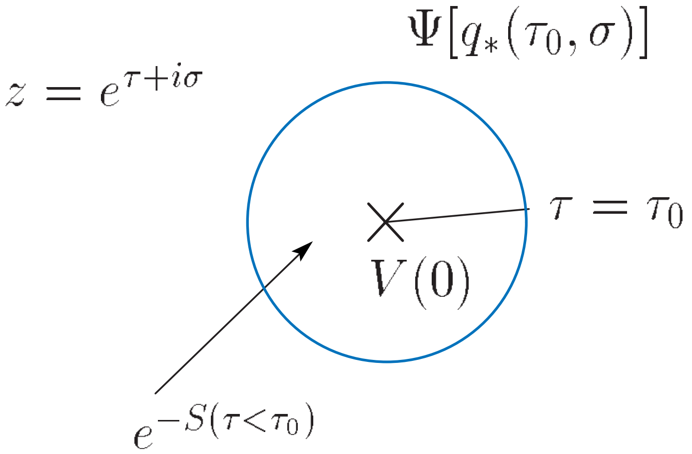
Contribution of the vertex operators

6 Evaluating the contribution of the vertex operators via state-operator correspondence

⋆ State-operator correspondence

In the saddle point approximation

$$V[q_*(z=0)]e^{-S_{q_*}(\tau<\tau_0)} = \Psi[q_*(\tau_0,\sigma)]$$



$q_*(\tau,\sigma)$ = saddle point configuration in some canonical variable $q(\tau,\sigma)$

If we can employ the **action-angle variables (J_n, θ_n)**, the **wave function** can be expressed simply as

$$\Psi[\theta] = \exp\left(i \sum_n J_n \theta_n - \mathcal{E}(\{J_n\})\tau \right)$$

♠ Extremely hard to construct action-angle variables for non-linear systems by solving Hamilton-Jacobi equation.

⋆ For integrable systems, we may use **Sklyanin's method** to construct action-angle variables

6.1 Integrability for strings in AdS_3 and GKP strings II

Framework of spectral curve and finite gap methods

To make use of the Sklyanin's method, we need to use the framework of spectral curve and finite gap methods.

□ **Right and left Lax connections**:

Basic object = **right flat current** ($SL(2)_R$-covariant, $SL(2)_L$-invariant)

$$j_z = \mathbb{X}^{-1}\partial\mathbb{X}\,, \qquad j_{\bar{z}} = \mathbb{X}^{-1}\bar{\partial}\mathbb{X}$$

Right Lax connection with spectral parameter x : $^{\exists}$ **singularities at** $x = \pm 1$

$$J^r_z(x) \equiv \frac{1}{1-x} j_z\,, \qquad J^r_{\bar{z}}(x) \equiv \frac{1}{1+x} j_{\bar{z}}$$

$$\left[\partial + J^r_z(x)\,, \bar{\partial} + J^r_{\bar{z}}(x)\right] = 0$$

Relation between x and the previous parameter ξ : $\quad x = \frac{1-\xi^2}{1+\xi^2}$

Similarly, we will need **left flat current** and left Lax connection

$$l_z = \partial \mathbb{X}\mathbb{X}^{-1}, \qquad l_{\bar{z}} = \bar{\partial}\mathbb{X}\mathbb{X}^{-1}$$

$$\left[\partial + J_z^l(x)\,,\bar{\partial} + J_{\bar{z}}^l(x)\right] = 0$$

$$J_z^l(x) \equiv -\frac{1}{1-(1/x)} l_z\,, \qquad J_{\bar{z}}^l(x) \equiv -\frac{1}{1+(1/x)} l_{\bar{z}}$$

Most important object: Monodromy matrix $\Omega(x, z_0)$

$$\Omega(x; z_0) = \mathcal{P} e^{-\oint \left(J_z^r(x)dz + J_{\bar{z}}^r(x)d\bar{z}\right)}$$

$$= u(x; z_0)^{-1} \begin{pmatrix} e^{i\hat{p}(x)} & 0 \\ 0 & e^{-i\hat{p}(x)} \end{pmatrix} u(x; z_0)$$

$\hat{p}(x) =$ quasi-momentum

Properties of Ω is encoded in

Spectral curve Γ : hyperelliptic Riemann surface with singularities

$$\Gamma \; : \quad \Gamma(x,y) \equiv \det\,(y\mathbf{1} - \Omega(x;z_0)) = 0$$
$$\Leftrightarrow \quad \left(y - e^{i\hat{p}(x)}\right)\left(y - e^{-i\hat{p}(x)}\right) = 0$$

Property of $\Gamma \Leftarrow$ behavior at $x = \infty, 0$ and at $x = \pm 1$.

- Conserved right and left global charges from the behaviors at $x = \infty, 0$

$$\hat{p}(x) = \frac{4\pi}{\sqrt{\lambda}x} S_\infty + O(\frac{1}{x^2}) \qquad (x \to \infty)$$
$$\hat{p}(x) = 2\pi m + \frac{4\pi x}{\sqrt{\lambda}} S_0 + O(x^2) \qquad (x \to 0)$$

- Leading singular behavior of $\hat{p}(x)$ around $x = \pm 1$ is dictated by the Virasoro condition

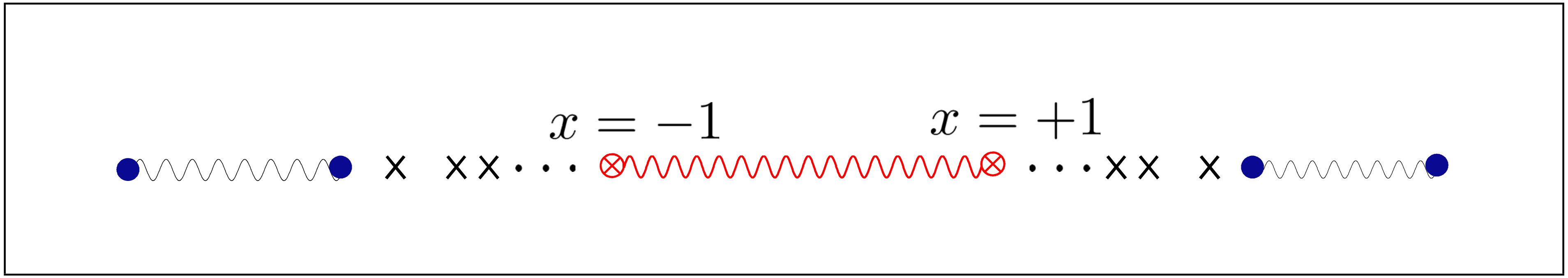
$$\mathrm{Tr}\,(j_z j_z) = 0 \quad \Rightarrow \quad j_z = u \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u^{-1} = \text{special Jordan block}$$

Diagonalizing $\Omega(x)$ carefully,

$$\hat{p}(x) = \pm \frac{\kappa_\pm}{\sqrt{1 \mp x}} + O((x \mp 1)) \qquad (x \to \pm 1)$$

"Half-poles" at $x = \pm 1$, as opposed to simple poles for $R \times S^3$ case.

Structure of the spectral curve for $g = 1$



(X's denote node-like singularities ($e^{i\hat{p}(x)} = e^{-i\hat{p}(x)}$) accumulating to ± 1.)

Spectral curve with finite g $\Rightarrow$ construct "finte gap" solution

6.2 Construction of the action-angle variables

Sklyanin's method

Normalized Baker-Akhiezer eigenvector $\vec{h}(x;\tau)$ of $\Omega(x;\tau,\sigma=0)$

$$(\star) \quad \Omega(x;\tau,\sigma=0)\vec{h}(x;\tau) = e^{i\hat{p}(x)}\vec{h}(x;\tau)$$

$$\boxed{\vec{n}\cdot\vec{h}=1}, \qquad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad \vec{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$\vec{h}(x;\tau)$ has $g+1$ poles, as a function of x.
Their positions on Γ : $(\gamma_1,\gamma_2,\ldots,\gamma_g,\gamma_\infty)(\tau)$
$\gamma_i(\tau)$ depends on $\vec{n}$

$\Omega(x)$ (hence $\hat{p}(\gamma_i)$)= dynamical variables $\Rightarrow \left\{\Omega(x), \Omega(x')\right\}_P$

Through $(\star)$, $\gamma_i(\tau)$'s become dynamical variables.

Sklyanin constructed canonical variables associated to these poles [1]

Canonical pairs "(q, p)"$\sim$ $(z(\gamma_i), \hat{p}(\gamma_i))$

$$\{z(\gamma_i)\,, \frac{\sqrt{\lambda}}{4\pi i}\hat{p}(\gamma_j)\}_P = \delta_{ij}$$
$$\{z(\gamma_i)\,, z(\gamma_j)\}_P = \{\hat{p}(\gamma_i)\,, \hat{p}(\gamma_j)\}_P = 0$$
$$z = x + \frac{1}{x} = \text{Zhukovski variable}$$

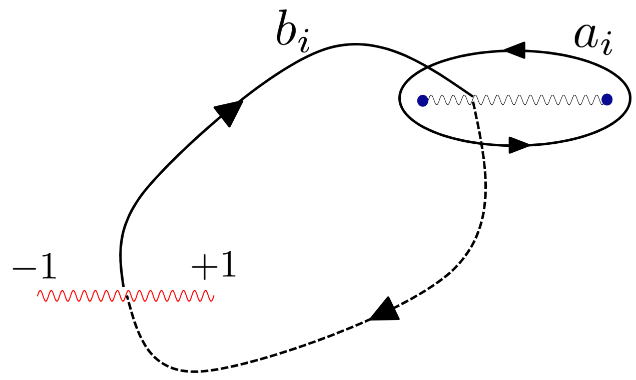
[1] Applied to string in $R \times S^3$ by Dorey and Vicedo. Applicable to Euclidean AdS_3 case as well.

Action variables S_i $(\sim \oint pdq)$

$$S_i \equiv \frac{i\sqrt{\lambda}}{8\pi^2}\int_{a_i} \hat{p}(x)dz$$

$$= \text{"filling fraction"}$$

$$(i = 1, 2, \ldots, g, \infty)$$



Angle variables ϕ_i conjugate to S_i:

Generating function $F(S_i\,, z(\gamma_i))$ for the canonical transformation

$$(*) \quad \frac{\partial F}{\partial z(\gamma_i)} = \frac{\sqrt{\lambda}}{4\pi i}\hat{p}(\gamma_i)\,, \qquad (**) \quad \frac{\partial F}{\partial S_i} = \phi_i$$

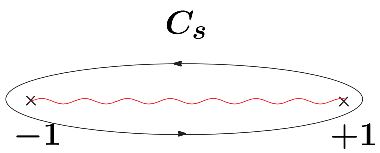
Integrating $(*)$

$$F(S_i\,, z(\gamma_i)) = \frac{\sqrt{\lambda}}{4\pi i}\sum_i \int_{z(x_0)}^{z(\gamma_i)} \hat{p}(x')dz'$$

To compute ϕ_i from $(**)$, vary S_i with all other S_j's fixed

$\Leftrightarrow$ Add to $\hat{p}dz$ a 1-form whose period integral along a_i is non-vanishing $\propto \omega_i$ with the properties

$$\oint_{a_j} \omega_i = \delta_{ij}\,, \qquad \oint_{C_s} \omega_i = -1$$



Using this we get

$$\phi_i(\tau) = \frac{\partial F}{\partial S_i} = 2\pi \sum_k \int_{x_0}^{\gamma_k(\tau)} \omega_i = \textbf{Abel map}$$

- $\phi_i(\tau)$ indeed evolves linearly in τ for classical solutions.

- Need **one more angle variable** $\tilde{\phi}_0$ conjugate to the **left global charge** S_0. This is obtained from the **left connection** J^l by the same procedure.

□ Illustration: The case of LSGKP string:

Explicit form of the right-current

$$j = \mathbb{X}^{-1} d\mathbb{X} = -\kappa d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \kappa \sigma \begin{pmatrix} 0 & e^{2\kappa\tau} \\ e^{-2\kappa\tau} & 0 \end{pmatrix}$$

j_τ and j_σ are independent of σ.

Monodromy matrix

$$\Omega(x,\tau) = \exp\left(\int_\sigma^{\sigma+2\pi} J_\sigma(x) d\sigma \right) = \frac{2\pi\kappa}{1-x^2} M(\tau, x)$$

where $$M(\tau,x) = \begin{pmatrix} -ix & e^{2\kappa\tau} \\ e^{-2\kappa\tau} & ix \end{pmatrix}$$

Eigenvalues of $M(\tau, x)$: $\lambda_\pm = \pm\sqrt{1-x^2}$ = time-independent (conserved)

Eigenfunctions

$$\psi_\pm = \begin{pmatrix} e^{2\kappa\tau} \\ \pm\sqrt{1-x^2} + ix \end{pmatrix}$$

Normalized Baker-Akhiezer vector (for λ_+)

$$h = \frac{1}{f}\psi_+ , \qquad 1 = n_1 h_1 + n_2 h_2$$

$$\Rightarrow \quad f = n_1 e^{2\kappa\tau} + n_2(\sqrt{1-x^2} + ix)$$

h has a moving pole at the zero of f.

$$x(t) = \frac{1 - \left(\frac{n_1}{n_2}\right) e^{4\kappa\tau}}{2i\frac{n_1}{n_2}e^{2\kappa\tau}} = \sin(2\kappa(t+t_0)) , \quad (\tau = it)$$

$$t_0 = -\frac{i}{2\kappa}\log\frac{n_1}{n_2}$$

Change of the normalization vector shifts the position of the pole.

The differential ω_∞ with the correct properties is given by

$$\omega_\infty = \frac{1}{2\pi}\frac{dx}{\sqrt{1-x^2}} \qquad \left(\oint_{a_\infty}\omega_\infty = 1 , \quad \oint_{C_s}\omega_\infty = -1\right)$$

Angle variable is given by the Abel map

$$\phi_\infty = 2\pi \int^{x(t)} \omega_\infty = \sin^{-1}(\sin(2\kappa\tilde{t})) + \text{const} = 2\kappa\tilde{t} + \text{const}$$

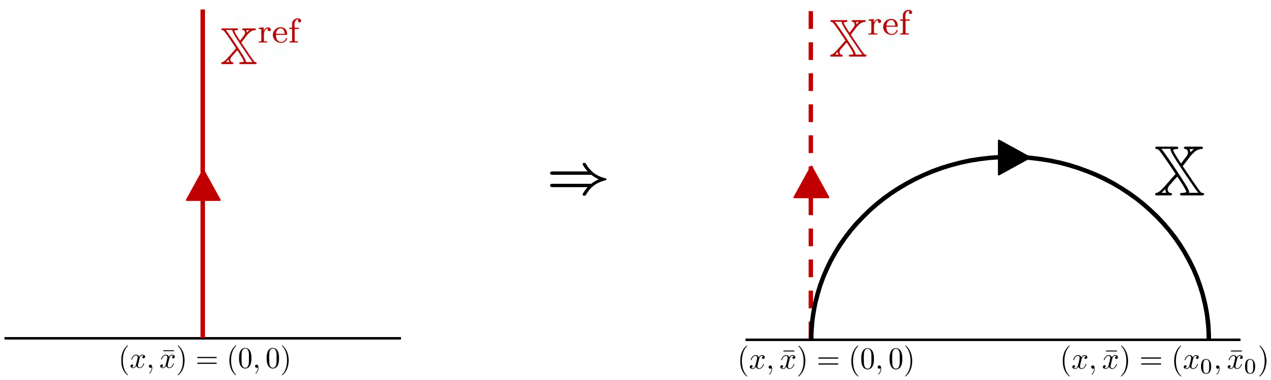
This is indeed linear in t.

6.3 Evaluation of the angle variables and the wave function

Need to evaluate the angle variables for a general "finite gap" solution $\mathbb{X}$

Main idea:

- Produce the solution of interest $\mathbb{X}$ from a suitable reference solution $\mathbb{X}^{\mathrm{ref}}$ by a **global transformation** $\mathbb{X} = V_L \mathbb{X}^{\mathrm{ref}} V_R$
- Compute the shift of angle variables $\Delta\phi_i$ under this transformation



Explicit formula:

- Case of the angle variables $\{\phi_1, \ldots, \phi_g, \phi_\infty\}$ describable by the **right-current**.

Angle variables $\Leftrightarrow$ Positions of the poles of BA vector

$\Rightarrow$ **How do the poles move under the global transformations ?**

Under a global right transformation V_R, the **normalized Baker-Akhiezer vector** gets transformed as

$$\vec{h}'(x;\tau) = \frac{1}{f(x;\tau)} V_R^{-1} \vec{h}^{\mathrm{ref}}(x;\tau)$$

$f(x;\tau)$ is needed to keep $\vec{h}'(x;\tau)$ normalized.

Under this transformations, **the positions of poles change** $\{\gamma_i\} \longrightarrow \{\gamma_i'\}$

$1/f(x;\tau)$ must remove the poles $\{\gamma_i\}$ and add the poles $\{\gamma_i'\}$

$\Leftrightarrow$ Divisor of f is $(f) = \sum_{i=1}^{g+1} (\gamma_i' - \gamma_i)$.

Meromorphic differential which encodes this is

$$\varpi = d(\log f) = \frac{df}{f} \ni \text{ poles at } \gamma_i' \text{ and } \gamma_i \text{ with residues } 1 \text{ and } -1$$

By studying the structure of ϖ, one can prove

- **ϕ_i with $i = 1 \sim g$ do not change under the global transformation**
  $\Rightarrow$ **Only ϕ_∞ can possibly change.**

- The change of ϕ_∞ can be expressed as

$$\int_{b_\infty} \varpi = \log\left(\frac{f(\infty^+)}{f(\infty^-)}\right) = 2\pi i \sum_{i=1}^{g+1} \int_{\gamma_i}^{\gamma_i'} \omega_\infty = i\Delta\phi_\infty$$

  One can explicitly evaluate this from the asymptotic behavior of $\vec{h}^{\rm ref}(x;\tau)$ at $x = \pm\infty$

- Similar analysis with the **left-current** $\Rightarrow$ **Similar formula for $\Delta\tilde{\phi}_0$**

Altogether we obtain

Master formula

$$\Delta\phi_\infty = -i\log\left(\frac{v_{22}-\frac{n_2}{n_1}v_{21}}{-\frac{n_1}{n_2}v_{12}+v_{11}}\right), \qquad \Delta\tilde{\phi}_0 = -i\log\left(\frac{\tilde{v}_{11}+\frac{\tilde{n}_2}{\tilde{n}_1}\tilde{v}_{21}}{\frac{\tilde{n}_1}{\tilde{n}_2}\tilde{v}_{12}+\tilde{v}_{22}}\right)$$

v_{ij} =**components of** V_R, $\tilde{v}_{ij}$ = **components of** V_L

- **Normalization vectors** $\vec{n}$ **and** $\vec{\tilde{n}}$ are fixed by the requirement that the wave function

$$\Psi[\tilde{\phi}_0[\vec{\tilde{n}}], \phi_i[\vec{n}], \phi_\infty[\vec{n}]] \equiv e^{iS_0\tilde{\phi}_0[\vec{\tilde{n}}]+iS_\infty\phi_\infty[\vec{n}]+i\sum_i S_i\phi_i[\vec{n}]}$$

carrying definite Δ and S $\Longleftrightarrow$ conformal primary $\mathcal{O}^{\Delta,S}(x=0)$ $\Leftrightarrow$ **Invariant under the special conformal transformation**

Practical master formula

$$\Delta\phi_\infty = -i \log\left(\frac{v_{22}}{v_{11}}\right), \qquad \Delta\tilde{\phi}_0 = -i \log\left(\frac{\tilde{v}_{11}}{\tilde{v}_{22}}\right)$$

They depend only on the **diagonal elements**

$\Leftrightarrow$ Effects of **dilatations** and **rotations**, as expected.

Dilatation

$$X_+ \to \lambda X_+\,, \quad X_- \to \frac{1}{\lambda} X_-\,, \quad X, \bar{X} : \text{ invariant}$$

$$V_L^d(\lambda) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}, \quad V_R^d(\lambda) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

Rotation

$$X \to \xi X\,, \quad \bar{X} \to \frac{1}{\xi}\bar{X}\,, \quad X_\pm : \text{ invariant}$$

$$V_L^r(\xi) = \begin{pmatrix} \sqrt{\xi} & 0 \\ 0 & \frac{1}{\sqrt{\xi}} \end{pmatrix}, \quad V_R^r(\xi) = \begin{pmatrix} \frac{1}{\sqrt{\xi}} & 0 \\ 0 & \sqrt{\xi} \end{pmatrix}$$

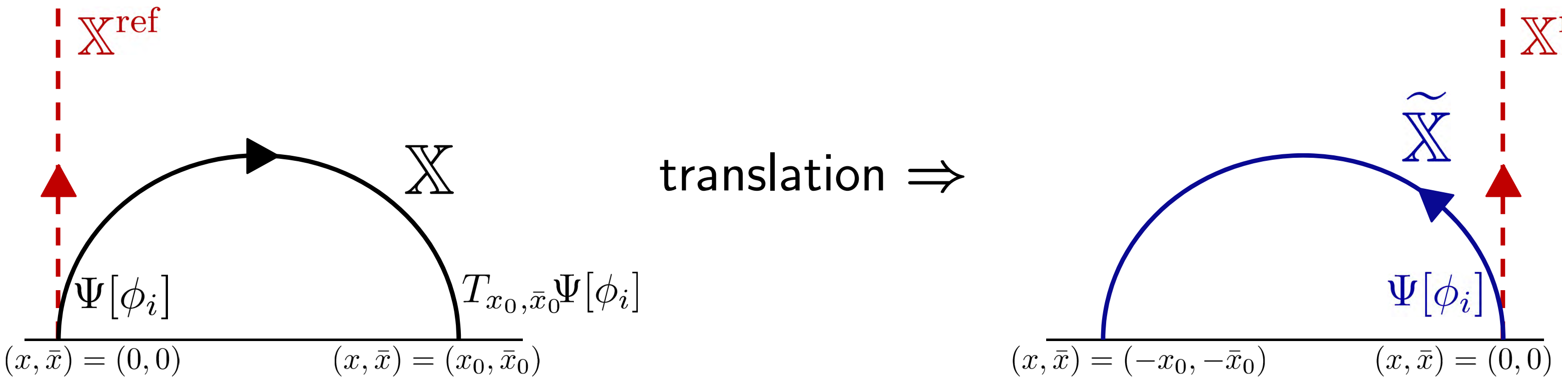
7 Computation of the two point functions

We now sketch how we can compute two-point functions:

Step 1. Wave function $\boldsymbol{\Psi_1}\big|_{\mathbb{X}}$ corresponding to $\boldsymbol{V(0,0)}\big|_{\mathbb{X}}$ can be computed relative to $\boldsymbol{\Psi_1}\big|_{\mathbb{X}^{\mathrm{ref}}}$ in terms of the relative shift of the angle variables $\sim$ $\boldsymbol{e^{iJ\Delta\theta^{\mathbb{X}}}}$ $(\boldsymbol{J} = \boldsymbol{S_\infty}, \boldsymbol{S_0}, \boldsymbol{\theta} = \boldsymbol{\phi_\infty}, \tilde{\boldsymbol{\phi}}_0)$

Step 2. For the evaluation of $\boldsymbol{\Psi_2}\big|_{\mathbb{X}}$ corresponding to $\boldsymbol{V(x_0, \bar{x}_0)}\big|_{\mathbb{X}}$, in order to compare with the angle variables corresponding to $\mathbb{X}^{\mathrm{ref}}$

- tranlate $\mathbb{X}$ so that the insertion point is brought to the origin.
- swith to the local cylinder coordinates $\Leftrightarrow$ effectively $(\boldsymbol{\tau}, \boldsymbol{\sigma}) \longrightarrow (-\boldsymbol{\tau}, -\boldsymbol{\sigma})$.



$\Rightarrow$ **"Translated reversed" solution** $\tilde{\mathbb{X}}$

Step 3. $\Psi_2\big|_{\mathbb{X}}$ can now be computed relative to $\Psi_1\big|_{\mathbb{X}^{\text{ref}}}$ by comparing $\tilde{\mathbb{X}}$ with $\mathbb{X}^{\text{ref}}$.

$\Rightarrow$ General formula for the contribution of the wave functions

$$\Psi_1\,\Psi_2\Big|_{\mathbb{X}} = (-1)^{\mathcal{P}} \frac{\left(\Psi_1\big|_{\mathbb{X}^{\text{ref}}}(0)\right)^2 e^{iJ(\Delta\theta^{\mathbb{X}}+\Delta\theta^{\tilde{\mathbb{X}}})}}{(z_1 - z_2)^{\mathcal{E}+\mathcal{P}}(\bar{z}_1 - \bar{z}_2)^{\mathcal{E}-\mathcal{P}}}\, e^{-(J\omega-\mathcal{E})(\tau_f-\tau_i)}$$

$$\overset{\text{Virasoro}}{\Longrightarrow} \left(\Psi_1\big|_{\mathbb{X}^{\text{ref}}}(0)\right)^2 e^{iJ(\Delta\theta^{\mathbb{X}}+\Delta\theta^{\tilde{\mathbb{X}}})} \times \underbrace{e^{+S\big|_{\tau_i}^{\tau_f}}}_{cancel\ with\ the\ action}$$

Step 4. Compute $\Delta\theta^{\mathbb{X}}+\Delta\theta^{\tilde{\mathbb{X}}}$ for the specific string states by using the master formula and add the contribution from the action $e^{-S\big|_{\tau_i}^{\tau_f}}$.

Example: Case of the elliptic GKP string

$$\Psi_1 e^{-S} \Psi_2 \Big|_{\mathbb{X}} = \frac{\left(\Psi_1 \Big|_{\mathbb{X}^{\mathrm{ref}}(0)} \right)^2}{x_0^{(\Delta - S)} \bar{x}_0^{(\Delta + S)}} \longrightarrow \frac{1}{x_0^{(\Delta - S)} \bar{x}_0^{(\Delta + S)}}$$

with the normalization $\Psi_1 \Big|_{\mathbb{X}^{\mathrm{ref}}(0)} = 1$

8 Computation of the three point function for LSGKP strings

Theme: Interlacing of local and global information

Around each vertex insertion point z_i

- we can compute the local eigensolutions $i^L_\pm$ and $i^R_\pm$ for the left and right auxiliary problems.
- We can expand the unknown global solutions ψ^L_a and $\psi^R_{\dot{a}}$ as

$$\psi^L_a = \langle \psi^L_a, i^L_- \rangle i^L_+ - \langle \psi^L_a, i^L_+ \rangle i^L_-$$
$$\psi^R_{\dot{a}} = \langle \psi^R_{\dot{a}}, i^R_- \rangle i^R_+ - \langle \psi^R_{\dot{a}}, i^R_+ \rangle i^R_-$$

Plug into the reconstruction formula

$$\begin{pmatrix} X_+ & X \\ \bar{X} & X_- \end{pmatrix}_{a,\dot{a}} = (\psi^L_a, \psi^R_{\dot{a}}) \equiv \psi^L_{1,a}\psi^R_{\dot{1},\dot{a}} + \psi^L_{2,a}\psi^R_{\dot{2},\dot{a}}$$

$$\Downarrow$$

Local string solutions around z_i

$$
\begin{aligned}
X_+ &\simeq e^{\hat\kappa_i \tau}\beta_i^-(\alpha_i^+ \sinh\hat\kappa_i\sigma - \alpha_i^- \cosh\hat\kappa_i\sigma) \\
&\quad + e^{-\hat\kappa_i \tau}\beta_i^+(\alpha_i^- \sinh\hat\kappa_i\sigma - \alpha_i^+ \cosh\hat\kappa_i\sigma) \\
X &\simeq e^{\hat\kappa_i \tau}\overline{\beta}_i^-(\alpha_i^+ \sinh\hat\kappa_i\sigma - \alpha_i^- \cosh\hat\kappa_i\sigma) \\
&\quad + e^{-\hat\kappa_i \tau}\overline{\beta}_i^+(\alpha_i^- \sinh\hat\kappa_i\sigma - \alpha_i^+ \cosh\hat\kappa_i\sigma) \\
\bar X &\simeq \cdots \\
X_- &\simeq \cdots
\end{aligned}
$$

Coefficients contain the **local** information about of the **global solution**

$$
\alpha_i^\pm \equiv \langle \psi_1^L, \hat i_\pm^L\rangle\,, \quad \beta_i^\pm \equiv \langle \psi_1^R, i_\pm^R\rangle\,, \qquad \hat i_\pm^L \equiv \frac{1}{\sqrt{2}}(\pm i_+^L + i_-^L)\,,
$$

$$
\overline{\alpha}_i^\pm \equiv \langle \psi_2^L, \hat i_\pm^L\rangle\,, \qquad \overline{\beta}_i^\pm \equiv \langle \psi_2^R, i_\pm^R\rangle
$$

$$
\hat\kappa_{1,3} = \kappa_{1,3}\,, \qquad \hat\kappa_2 = -\kappa_2
$$

Location of the vertex operators:

$$\boldsymbol{x}^{(i)} = \frac{\boldsymbol{X}}{\boldsymbol{X}_+}\bigg|_{\tau=-\infty,\sigma=0} = \begin{cases} \overline{\beta}_i^+/\beta_i^+ & \text{for } i = 1,3 \\ \overline{\beta}_i^-/\beta_i^- & \text{for } i = 2 \end{cases}$$

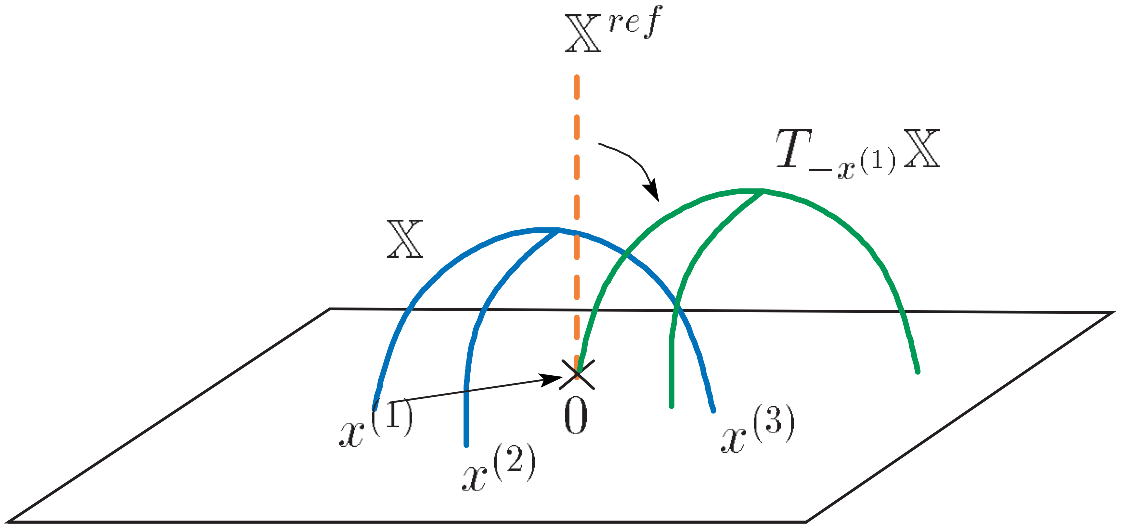
$$\bar{\boldsymbol{x}}^{(i)} = (\boldsymbol{\beta}, \bar{\boldsymbol{\beta}}) \longrightarrow (\boldsymbol{\alpha}, \bar{\boldsymbol{\alpha}})$$

□ Computation of the contribution of the wave functions:

(1) Translate each leg to the origin by
$\tilde{\mathbb{X}}_i = \boldsymbol{T}_{-\boldsymbol{x}^{(i)}}\mathbb{X}$

(2) Compare with $\mathbb{X}^{\mathrm{ref}}$:
Find $\boldsymbol{V_L}$ and $\boldsymbol{V_R}$ such that
$\tilde{\mathbb{X}}_i = \boldsymbol{V_L}\mathbb{X}^{\mathrm{ref}}\boldsymbol{V_R}$

(3) Use the master formula to find $\boldsymbol{\Delta\phi_0^{(i)}}$ and $\boldsymbol{\Delta\phi_\infty^{(i)}}$ from $\boldsymbol{V_L}$ and $\boldsymbol{V_R}$



$$\Downarrow$$

Contribution of the wave functions:

$$\Psi_1\Psi_2\Psi_3\big|_{\mathbb{X}} = \exp\left(i\sum_{i=1}^{3} S_0^{(i)}\Delta\phi_0^{(i)} + S_\infty^{(i)}\Delta\phi_\infty^{(i)} \right) \prod_{i=1}^{3} \Psi\big|_{\mathbb{X}^{\mathrm{ref}}(\log\epsilon_i)}$$

(⋆) $\Delta\phi_0^{(i)}$ and $\Delta\phi_\infty^{(i)}$: Expressed in terms of $\alpha_i^{\pm}$'s an $\beta_i^{\pm}$'s

(⋆⋆) They can be expressed in the extremely useful form, such as

$$(\beta_1^{+})^2 = -\frac{(x^{(2)} - x^{(3)})}{(x^{(1)} - x^{(2)})(x^{(3)} - x^{(1)})} \frac{\langle 1_+^R, 2_-^R\rangle\langle 3_+^R, 1_+^R\rangle}{\langle 2_-^R, 3_+^R\rangle}$$

Local information of the global solution ψ is written as

(info. about relative positions) × **(overlaps of local solutions)**

Moreover,

$$\frac{\langle 1^R_+, 2^R_- \rangle \langle 3^R_+, 1^R_+ \rangle}{\langle 2^R_-, 3^R_+ \rangle} \propto \frac{\langle s_1, s_2 \rangle \langle s_3, s_1 \rangle}{\langle s_2, s_3 \rangle} (\xi = i)$$

: computed in Part I

Substitution of the results for various parts gives

$$\Psi_1 \Psi_2 \Psi_3 \big|_{\mathbb{X}} = \frac{C_{\mathrm{w.f.}}}{(x^1 - x^2)^{\ell_1^- + \ell_2^- - \ell_3^-} (x^2 - x^3)^{\ell_2^- + \ell_3^- - \ell_1^-} (x^3 - x^1)^{\ell_3^- + \ell_1^- - \ell_2^-}} \times \frac{\left(\Psi \big|_{\mathbb{X}^{\mathrm{ref}}(0)} \right)^3}{(\bar{x}^1 - \bar{x}^2)^{\ell_1^+ + \ell_2^+ - \ell_3^+} (\bar{x}^2 - \bar{x}^3)^{\ell_2^+ + \ell_3^+ - \ell_1^+} (\bar{x}^3 - \bar{x}^1)^{\ell_3^+ + \ell_1^+ - \ell_2^+}}$$

where

$$\ell_i^- = \frac{1}{2} (\Delta^{(i)} - S^{(i)}) \,, \qquad \ell_i^+ \equiv \frac{1}{2} (\Delta^{(i)} + S^{(i)})$$

$$\log C_{\text{w.f.}} = H_{-}\left[h(x,\xi=i)\right] + H_{+}\left[h(x,\xi=1)\right] + \underbrace{\frac{i\sqrt{\lambda}}{2}\sum_{j=1}^{3}\hat{\kappa}_j\left(\int_{d_j}\sqrt{p}dz - \int_{d_j}\sqrt{\bar{p}}d\bar{z}\right)}_{\textit{cancel with } \log A_{div}} + \sum_j \ell_j^{+}\log\tilde{c}\,,$$

$$H_{\pm}\left[f(x)\right] \equiv 2\sum_{j=1}^{3}\ell_j^{\pm}f(\kappa_j) - \left(\ell_1^{\pm}+\ell_2^{\pm}+\ell_3^{\pm}\right)f\left(\frac{\kappa_1+\kappa_2+\kappa_3}{2}\right) - \sum_{(i,j,k)=(1,2,3)+\text{cyclic}}\left(-\ell_i^{\pm}+\ell_j^{\pm}+\ell_k^{\pm}\right)f\left(\frac{-\kappa_i+\kappa_j+\kappa_k}{2}\right)$$

$$h(x,\xi) \equiv -\frac{1}{\pi i}\int_0^{\infty}d\xi'\frac{1}{\xi'^2-\xi^2}\log\left(1-e^{-2\pi x({\xi'}^{-1}+\xi')}\right)$$

$$\tilde{c} = 1 - \sqrt{\frac{\prod_{(i,j,k)=(1,2,3)+\text{cyclic}}\sinh(\pi(-\kappa_i+\kappa_j+\kappa_k))}{\sinh(\pi(\kappa_1+\kappa_2+\kappa_3))}}$$

In this notation the contribution from the finite part of the action can be written as

$$\log C_{\mathrm{action}} = -\frac{\sqrt{\lambda}}{2\pi} A_{\mathrm{fin}} = -\frac{7\sqrt{\lambda}}{12} + H_{-}\left[K(x)\right]$$

$$K(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\theta e^{-\theta} \log\left(1 - e^{-4\pi x \cosh\theta}\right)$$

□ Final result for the 3-point function of LSGKP string

- ***Despite the lack of knowledge of V_i and X_*, one can obtain a completely explicit result.***
- ***Integrability is quite powerful, beyond the spectral problem.***

3pt function for LSGKP

$$= e^{-A}\Psi_1\Psi_2\Psi_3$$

$$= \frac{C^{LSGKP}(\{\kappa_i\})}{\prod_{i\neq j\neq k}(x^{(i)} - x^{(j)})^{\ell_i^- + \ell_j^- - \ell_k^-}(\bar{x}^{(i)} - \bar{x}^{(j)})^{\ell_i^+ + \ell_j^+ - \ell_k^+}}$$

3pt coupling

$$\log C^{\mathrm{LSGKP}}(\{\kappa_i\}) = -\frac{7\sqrt{\lambda}}{12} + \sum_j \ell_j^+ \log \tilde{c} + H_-[\widetilde{K}(x)] + H_+[h(x, \xi = 1)]$$

where

$$\begin{aligned}\widetilde{K}(x) &\equiv K(x) + h(x, \xi = i)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\, \frac{\cosh 2\theta}{\cosh\theta} \log\left(1 - e^{-4\pi x \cosh\theta}\right)\,,\\ h(x, \xi = 1) &= -\frac{1}{2}\log\left(1 - e^{-4\pi x}\right)\end{aligned}$$

- Corresponding result on the SYM side is not yet available.

- **Consistency check: In the limit $\kappa_3 \to 0, \kappa_2 \to \kappa_1$, the three point function above reduces to the properly normalized two point function.**

9 Discussions and perspectives

□ **What have been achieved** :

- We have developed a general method to compute semi-classical correlation functions at strong coupling for non-BPS string states with large quantum numbers, when they are describable by the "finite gap method" of integrable systems.

> **Our method is quite powerful in that it can be applied to cases where neither the vertex operators nor the saddle point configurations are explicitly known.**

- As an important example, we applied it to the three point function of the large spin limit of the GKP folded spinning strings and obtained completely finite answer with the expected dependence of the target space coordinates on $\boldsymbol{\Delta}$ and $\boldsymbol{S}$.

☐ Some future projects:

- Apply our method to correlation functions for other types of strings.

  In particular, it is important to study the case of the string in $AdS_2 \times S^3$, for which the computation on the SYM side, in the $SU(2)$ sector, should be easier. (Work in progress)

- Computation of the 4 point functions. Study how the crossing symmetry is realized.

Hope to report progress on this and related matters in the near future