Alleviating the non-ultralocality of the $AdS_5 \times S^5$ superstring

Marc MAGRO

ENS DE LYON

1204.0766, 1204.2531, 1206.6050

With
F. DELDUC (ENS Lyon) and B. VICEDO (York / Hertfordshire)

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Plan

Part I:
- Motivation
- Non-ultralocality
- Difficulties

Part II: Review of first steps of Faddeev-Reshetikhin procedure

Part III: Generalization to the superstring
- Alleviation of non-ultralocality
- Link with Pohlmeyer reduction

Part IV: Remarks and conclusion

Motivation

Goal = Quantization of the $AdS_5 \times S^5$ superstring from first principles

→ Construct corresponding **Quantum Integrable Lattice** Model

- Long-term goal...
- Would provide a proof of quantum integrability

Motivation & Difficulties

Goal = Quantization of the $AdS_5 \times S^5$ superstring from first principles

→ Construct corresponding **Quantum Integrable Lattice** Model

- Necessary Step = Classical integrable discretization

Already very difficult !

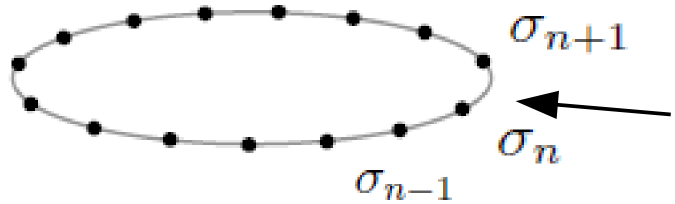
Old problem for integrable Sigma Models !

Difficulty comes from their **Non-ultralocality**

→ Review: - What is Non-ultralocality
- Why it is the origin of difficulties

Classical integrable discretization

- Lattice



Dynamical variables encoded in matrices $T^n(\lambda)$
λ: Spectral parameter

N sites with periodic conditions

What kind of **Poisson bracket** for $T^n(\lambda)$ in order to have a classical integrable lattice model ?

Freidel-Maillet Quadratic Algebra

[Freidel-Maillet '91]

$$\{T_1^n, T_2^m\} = a_{12} T_1^n T_2^m \delta^{m,n} - T_1^n T_2^m d_{12} \delta^{m,n} + T_1^n b_{12} T_2^m \delta^{m+1,n} - T_2^m c_{12} T_1^n \delta^{m,n+1}$$

With $T_1^n = T^n(\lambda) \otimes \mathbb{I}$

$T_2^m = \mathbb{I} \otimes T^m(\mu)$

$a_{12} = a_{12}(\lambda, \mu)$ and similarly for b, c and d

Freidel-Maillet Quadratic Algebra

$$\{T_1^n, T_2^m\} = a_{12} T_1^n T_2^m \delta^{m,n} - T_1^n T_2^m d_{12} \delta^{m,n} + T_1^n b_{12} T_2^m \delta^{m+1,n} - T_2^m c_{12} T_1^n \delta^{m,n+1}$$

$$a_{12} = -a_{21}, \quad d_{12} = -d_{21}, \quad b_{12} = c_{21}$$

→ **Antisymmetry**

Freidel-Maillet Quadratic Algebra

$$\{T_1^n, T_2^m\} = a_{12} T_1^n T_2^m \delta^{m,n} - T_1^n T_2^m d_{12} \delta^{m,n} + T_1^n b_{12} T_2^m \delta^{m+1,n} - T_2^m c_{12} T_1^n \delta^{m,n+1}$$

$$[a_{12}, a_{13}] + [a_{13}, a_{23}] + [a_{13}, a_{23}] = -[C_{12}, C_{13}]$$
$$[d_{12}, d_{13}] + [d_{13}, d_{23}] + [d_{13}, d_{23}] = -[C_{12}, C_{13}]$$
$$[a_{12}, c_{13}] + [a_{12}, c_{23}] + [c_{13}, c_{23}] = 0,$$
$$[d_{12}, b_{13}] + [d_{12}, b_{23}] + [b_{13}, b_{23}] = 0$$

C_{12} : Quadratic Casimir

→ **Jacobi identity**

Freidel-Maillet Quadratic Algebra

$$\{T_1^n, T_2^m\} = a_{12} T_1^n T_2^m \delta^{m,n} - T_1^n T_2^m d_{12} \delta^{m,n} + T_1^n b_{12} T_2^m \delta^{m+1,n} - T_2^m c_{12} T_1^n \delta^{m,n+1}$$

$a - d + b - c = 0$ ⟶ **Integrability**

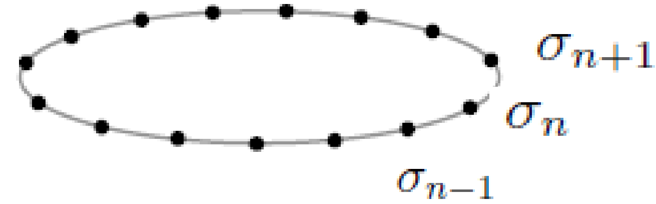
Monodromy $M = T^N T^{N-1} \ldots T^2 T^1$ has Poisson bracket:

$$\{M_1, M_2\} = a_{12} M_1 M_2 - M_1 M_2 d_{12} + M_1 b_{12} M_2 - M_2 c_{12} M_1$$

⟶ $\mathrm{Tr} M^k$ in involution

Non-ultralocality

Comes from b and c



$$\{T_1^n, T_2^m\} = a_{12} T_1^n T_2^m \delta^{m,n} - T_1^n T_2^m d_{12} \delta^{m,n} + T_1^n b_{12} T_2^m \delta^{m+1,n} - T_2^m c_{12} T_1^n \delta^{m,n+1}$$

Consider ***b=0***:

- Previous conditions imply ***c=b=0*** and ***a=d*** with ***a*** solution of modified classical Yang-Baxter equation

- Corresponds to ultralocal model and in particular

$$\{M_1, M_2\} = [a_{12}, M_1 M_2]$$

Continuum Limit

$$T^n(\lambda) = P\overleftarrow{\exp} \int_{\sigma_n}^{\sigma_{n+1}} \mathscr{L}(\sigma,\lambda) d\sigma$$

$\mathscr{L}(\sigma,\lambda)$:

Spatial Lax Matrix

$$\{T_1^n, T_2^m\} = a_{12} T_1^n T_2^m \delta^{m,n} - T_1^n T_2^m d_{12} \delta^{m,n} + T_1^n b_{12} T_2^m \delta^{m+1,n} - T_2^m c_{12} T_1^n \delta^{m,n+1}$$

↓

$$\{\mathscr{L}_1, \mathscr{L}_2\} = [r_{12}, \mathscr{L}_1 + \mathscr{L}_2] \delta_{\sigma\sigma'} + [s_{12}, \mathscr{L}_1 - \mathscr{L}_2] \delta_{\sigma\sigma'} + 2 s_{12} \delta'_{\sigma\sigma'}$$

$$r = a + \frac{1}{2}(b - c) \qquad s = \frac{1}{2}(b + c)$$

- PB of Lax matrix are of the r/s form with r and s stemming from (a,b,c,d)
- **Non-ultralocality** carried by the matrix **s**

[Maillet '85 '86]

First difficulty for the superstring

Good news

1. Start from continuum
2. Hamiltonian Lax matrix known
3. Its PB are of the r/s form

[M.M. '08, Vicedo '09]

$$\mathfrak{f} = \mathfrak{psu}(2,2|4) = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)} \oplus \mathfrak{f}^{(2)} \oplus \mathfrak{f}^{(3)}$$

$$\text{with} \quad [\mathfrak{f}^{(i)}, \mathfrak{f}^{(j)}] \subset \mathfrak{f}^{(i+j)} \quad \text{mod} \quad 4$$

$$\text{and} \quad \mathfrak{f}^{(0)} = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5)$$

$$s_{12}(\lambda,\mu) = \frac{1}{4}(2-\lambda^4-\mu^4)C_{12}^{(00)} + \frac{1}{4}(\lambda^{-2}\mu^{-2}-\lambda^2\mu^2)C_{12}^{(22)} + \frac{1}{4}(\lambda^{-3}\mu^{-1}-\lambda\mu^3)C_{12}^{(13)} + \frac{1}{4}(\mu^{-3}\lambda^{-1}-\mu\lambda^3)C_{12}^{(31)}$$

First difficulty for the superstring

Bad news

r and s do not stem from any (a,b,c,d)

Same situation for Principal Chiral Model, Coset Models !

→ r/s algebra of the continuum
cannot be discretized
as a lattice Freidel-Maillet algebra

More difficulties

Could say:

1. Start from known PB algebra of the Lax matrix on continuum
2. Compute then PB of $T^n(\lambda) = P\overleftarrow{\exp}\int_{\sigma_n}^{\sigma_{n+1}} \mathscr{L}(\sigma,\lambda)d\sigma$

Problem: These PB are not well defined

Schematically, comes from:

$$\int_{\sigma_n}^{\sigma_{n+1}} d\sigma \int_{\sigma_m}^{\sigma_{m+1}} d\sigma' \partial_\sigma \delta_{\sigma\sigma'} = \chi(\sigma_{n+1};[\sigma_m,\sigma_{m+1}]) - \chi(\sigma_n;[\sigma_m,\sigma_{m+1}])$$

Characteristic function of the interval:
Undefined when two points coincide !

→ Old problem ! What to do ?

Faddeev-Reshetikhin approach

[FR '86]

Concerns SU(2) Principal Chiral Model

Described by:

- Hamiltonian $H = \int d\sigma \, \mathrm{Tr}\big((j^0)^2 + (j^1)^2\big)$
- Canonical Poisson bracket

$$
\begin{aligned}
\{j_1^0(\sigma), j_2^0(\sigma')\} &= [C_{12}, j_2^0(\sigma)]\delta_{\sigma\sigma'} \\
\{j_1^0(\sigma), j_2^1(\sigma')\} &= [C_{12}, j_2^1(\sigma)]\delta_{\sigma\sigma'} - C_{12}\delta'_{\sigma\sigma'} \\
\{j_1^1(\sigma), j_2^1(\sigma')\} &= 0
\end{aligned}
$$

- Lax matrix $\mathscr{L}(\lambda) = \frac{1}{1-\lambda^2}(j^1 + \lambda j^0)$

⟶ Satisifies a non-ultralocal r/s algebra

FR Strategy = To **get rid** of Non-ultralocality

First steps of FR approach

1. Keep the **same** Lax matrix

2. Replace **canonical non-ultralocal** PB by the **ultralocal** PB

$$\{j_1^0(\sigma), j_2^0(\sigma')\}' = [C_{12}, j_2^0(\sigma)]\,\delta_{\sigma\sigma'}$$
$$\{j_1^0(\sigma), j_2^1(\sigma')\}' = [C_{12}, j_2^1(\sigma)]\,\delta_{\sigma\sigma'}$$
$$\{j_1^1(\sigma), j_1^1(\sigma')\}' = [C_{12}, j_2^0(\sigma)]\,\delta_{\sigma\sigma'}$$

3. Find Hamiltonian H' such that $(H', \{\cdot,\cdot\}')$ has **same classical dynamics** as $(H, \{\cdot,\cdot\})$

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3. Find Hamiltonian H' such that $(H', \{\cdot,\cdot\}')$ has **same classical dynamics** as $(H, \{\cdot,\cdot\})$

Degeneracy of ultralocal bracket

- A priori, look for H' s.t. $\forall f, \quad \{H', f\}' = \{H, f\}$
- But ultralocal PB is degenerate !

$$T_{\pm\pm} = \text{Tr}\left[(j^0 \pm j^1)^2\right] \text{ are Casimirs } i.e. \ \{h, T_{\pm\pm}\}' = 0 \quad \forall h$$

↓

1. Only possible to reproduce **Reduction** of PCM dynamics defined by setting Casimirs to constants

2. Can be done in a **consistent way** because these quantities are chiral/antichiral

Reduction of conformal symmetry

⟶ Hamiltonian H' for reduced dynamics

How to generalize FR approach to the superstring ?

- Keep Lax matrix

$$\mathscr{L}(\lambda) = A^{(0)} + \tfrac{1}{4}(\lambda^{-3} + 3\lambda)A^{(1)} + \tfrac{1}{2}(\lambda^{-2} + \lambda^{2})A^{(2)} + \tfrac{1}{4}(3\lambda^{-1} + \lambda^{3})A^{(3)}$$
$$+ \tfrac{1}{2}(1 - \lambda^{4})\Pi^{(0)} + \tfrac{1}{2}(\lambda^{-3} - \lambda)\Pi^{(1)} + \tfrac{1}{2}(\lambda^{-2} - \lambda^{2})\Pi^{(2)} + \tfrac{1}{2}(\lambda^{-1} - \lambda^{3})\Pi^{(3)}$$

- Replace canonical PB

$$\{A_1^{(i)}(\sigma), A_2^{(j)}(\sigma')\} = 0$$
$$\{A_1^{(i)}(\sigma), \Pi_2^{(j)}(\sigma')\} = [C_{12}^{(i4-i)}, A_2^{(i+j)}(\sigma)]\,\delta_{\sigma\sigma'} - \delta_{i+j} C_{12}^{(i4-i)}\,\delta'_{\sigma\sigma'}$$
$$\{\Pi_1^{(i)}(\sigma), \Pi_2^{(j)}(\sigma')\} = [C_{12}^{(i4-i)}, \Pi_2^{(i+j)}(\sigma)]\,\delta_{\sigma\sigma'}$$

by ultra-local one:

$$\{\cdot,\cdot\}' = ?$$

How to generalize FR approach to the superstring ?

Try to mimick FR:

$$\{\Pi_1^{(i)}(\sigma),\Pi_2^{(j)}(\sigma')\}' = [C_{12}^{(ii)},\Pi_2^{(i+j)}(\sigma)]\delta_{\sigma\sigma'}$$
$$\{A_1^{(i)}(\sigma),\Pi_2^{(j)}(\sigma')\}' = [C_{12}^{(ii)},A_2^{(i+j)}(\sigma)]\delta_{\sigma\sigma'}$$
$$\{A_1^{(i)}(\sigma),A_2^{(j)}(\sigma')\}' = [C_{12}^{(ii)},\Pi_2^{(i+j)}(\sigma)]\delta_{\sigma\sigma'}$$

But

- No sign of **Coset**

- Its Casimirs are **inconsistent** with the dynamics of the superstring !

→ **Impossible to guess !**

The way out of the tunnel

Needs to understand better the **algebraic setting** behind **Hamiltonian** integrability of the superstring and more precisely, the **deep origin** of Non-ultralocality.

→ Rephrase integrability in the right framework = **R-matrix approach**

[Semenov-Tian-Shansky '83]

Already done by B. Vicedo in 1003.1192

Quartet behind integrability

* **Loop algebra:** Twisted loop algebra

$$\widehat{\mathfrak{f}}^{\sigma} = \{X(\lambda) \in \widehat{\mathfrak{f}} \quad | \quad \sigma[X(-i\lambda)] = X(\lambda)\}$$

σ: Usual $\mathbb{Z}_4$ automorphism

* **Lax matrix** $\mathscr{L}$

* **R-matrix**

$$R = \pi_{\geq 0} - \pi_{<0}$$

Projection on $\mathfrak{f} \otimes \mathbb{C}[[\lambda]]$

Projection on $\mathfrak{f} \otimes \lambda^{-1}\mathbb{C}[[\lambda^{-1}]]$

Solution of modified classical Yang-Baxter equation:

$$[RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y] \qquad \forall X, Y \in \widehat{\mathfrak{f}}$$

Quartet behind integrability

$*$ **Twist** function

$$\phi(\lambda) = \frac{16\lambda^4}{(1-\lambda^4)^2}$$

Defines **twisted inner product** on $\widehat{\mathfrak{f}}^{\sigma}$

$$(X,Y)_\phi = \mathrm{Residue}_{\lambda=0} \frac{d\lambda}{\lambda} \phi(\lambda)\, \mathrm{Str}[X(\lambda)Y(\lambda)]$$

Note that $\frac{d\lambda}{\lambda}\phi = du$ with $u(\lambda)$ = Zhukovsky map

Origin of non-ultralocality

1. At abstract level: Quartet enables to define Kirillov-Kostant PB (associated with R-matrix) on central extension of $C^\infty(S^1, \widehat{\mathfrak{f}}^\sigma)$

2. In practice:

$$\{\mathscr{L}_1, \mathscr{L}_2\} = [r_{12}, \mathscr{L}_1 + \mathscr{L}_2]\delta_{\sigma\sigma'} + [s_{12}, \mathscr{L}_1 - \mathscr{L}_2]\delta_{\sigma\sigma'} + 2s_{12}\delta'_{\sigma\sigma'}$$

PB corresponds to r/s form with

$$r = \tfrac{1}{2}(R - R^*) \qquad \text{and} \qquad s = \tfrac{1}{2}(R + R^*)$$

R^* defined w.r.t. twisted inner product: $(R^* X, Y)_\phi = (X, RY)_\phi$

3. Algebraic origin of non-ultralocality

$$\text{Non-ultralocality} \Longleftrightarrow R^* \neq -R$$

4. **Now possible to quantify Non-ultralocality !**

Origin of non-ultralocality

For twist ϕ:

$$R^* = -\left[(\lambda^{-1}\phi)^{-1}\right] \circ R \circ \left[\lambda^{-1}\phi\right]$$

First try

$$R^* = -\left[(\lambda^{-1}\phi)^{-1}\right] \circ R \circ \left[\lambda^{-1}\phi\right]$$

Suggests procedure to get rid of non-ultralocality

$$(\widehat{\mathfrak{f}}^{\sigma}, \mathscr{L}, R, \lambda^{-1}\phi) \longrightarrow (\widehat{\mathfrak{f}}^{\sigma}, \mathscr{L}, R, 1)$$

Problem: Leads to **completely degenerate** PB !

Reason: Loop algebra is twisted

$$\mathrm{Str}(\mathfrak{f}^{(i)}\mathfrak{f}^{(j)}) = 0 \text{ if } i + j \neq 0 \bmod 4$$

↓

$$(X, Y)_1 = \mathrm{Residue}_{\lambda=0} d\lambda\, \mathrm{Str}[X(\lambda)Y(\lambda)] = 0$$

→ **Impossible** to **completely get rid** of non-ultralocality !

→ Good lead: Can recast original FR procedure in this language

Second try

$$(\widehat{\mathfrak{f}}^{\sigma}, \mathscr{L}, R, \lambda^{-1}\phi) \longrightarrow (\widehat{\mathfrak{f}}^{\sigma}, \mathscr{L}, R, \lambda^{-1})$$

$$R = \pi_{\geq 0} - \pi_{<0} \longrightarrow \quad r = \pi_{>0} - \pi_{<0}$$

$\mathbf{s} = \pi_0$ = Projector on constant part $\mathfrak{f}^{(0)}$ of $\widehat{\mathfrak{f}}^{\sigma}$

$$s_{12}(\lambda, \mu) = C_{12}^{(00)}$$

[Semenov-Tian-Shansky and Sevostyanov '95]

r and s stem from (a,b,c,d) satisfying Freidel-Maillet conditions !

$$a = r + \alpha, \quad b = -s - \alpha, \quad c = -s + \alpha, \quad d = r - \alpha$$

α = skew-symmetric solution of modified classical Yang-Baxter equation on $\mathfrak{f}^{(0)}$

Alleviation of non-ultralocality

$$s_{12}(\lambda, \mu) = C_{12}^{(00)}$$

[Semenov-Tian-Shansky and Sevostyanov '95]

r and s stem from (a,b,c,d) satisfying Freidel-Maillet conditions !

$$a = r + \alpha, \quad b = -s - \alpha, \quad c = -s + \alpha, \quad d = r - \alpha$$

α = skew-symmetric solution of modified classical Yang-Baxter equation on $\mathfrak{f}^{(0)}$

→ **Alleviation** of non-ultralocality
Resulting non-ultralocality is **mild**

Modified PB for phase space variables

$$\{A_1^{(0)}(\sigma),A_2^{(0)}(\sigma')\}' = -[C_{12}^{(00)},2A_2^{(0)}+\mathscr{C}_2^{(0)}]\delta_{\sigma\sigma'}+2C_{12}^{(00)}\delta'_{\sigma\sigma'}.$$

$$\{A_1^{(0)}(\sigma),A_2^{(1)}(\sigma')\}' = -[C_{12}^{(00)},\mathscr{C}_2^{(1)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(0)}(\sigma),A_2^{(3)}(\sigma')\}' = -[C_{12}^{(00)},2A_2^{(3)}+\mathscr{C}_2^{(3)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(1)}(\sigma),A_2^{(2)}(\sigma')\}' = -[C_{12}^{(13)},\mathscr{C}_2^{(3)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(2)}(\sigma),A_2^{(2)}(\sigma')\}' = -[C_{12}^{(22)},\mathscr{C}_2^{(0)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(3)}(\sigma),A_2^{(3)}(\sigma')\}' = -2[C_{12}^{(31)},A_{-2}^{(2)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(0)}(\sigma),\Pi_2^{(2)}(\sigma')\}' = -2[C_{12}^{(00)},A_{-2}^{(2)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(1)}(\sigma),\Pi_2^{(1)}(\sigma')\}' = [C_{12}^{(13)},A_{+2}^{(2)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(1)}(\sigma),\Pi_2^{(3)}(\sigma')\}' = \tfrac{3}{2}[C_{12}^{(13)},\mathscr{C}_2^{(0)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(2)}(\sigma),\Pi_2^{(2)}(\sigma')\}' = [C_{12}^{(22)},\mathscr{C}_2^{(0)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(3)}(\sigma),\Pi_2^{(1)}(\sigma')\}' = \tfrac{1}{2}[C_{12}^{(31)},\mathscr{C}_2^{(0)}]\delta_{\sigma\sigma'},$$
$$\{A_1^{(3)}(\sigma),\Pi_2^{(3)}(\sigma')\}' = -[C_{12}^{(31)},A_{-2}^{(2)}]\delta_{\sigma\sigma'},$$
$$\{\Pi_1^{(1)}(\sigma),\Pi_2^{(2)}(\sigma')\}' = -\tfrac{1}{2}[C_{12}^{(13)},\mathscr{C}_2^{(3)}]\delta_{\sigma\sigma'},$$
$$\{\Pi_1^{(2)}(\sigma),\Pi_2^{(2)}(\sigma')\}' = -[C_{12}^{(22)},\mathscr{C}_2^{(0)}]\delta_{\sigma\sigma'},$$
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$$\{A_1^{(0)}(\sigma),A_2^{(2)}(\sigma')\}' = -2[C_{12}^{(00)},A_{-2}^{(2)}]\delta_{\sigma\sigma'},$$
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$$\{A_1^{(0)}(\sigma),\Pi_2^{(3)}(\sigma')\}' = -[C_{12}^{(00)},A_2^{(3)}+\tfrac{1}{2}\mathscr{C}_2^{(3)}]\delta_{\sigma\sigma'},$$
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$$\{\Pi_1^{(1)}(\sigma),\Pi_2^{(3)}(\sigma')\}' = -\tfrac{3}{4}[C_{12}^{(13)},\mathscr{C}_2^{(0)}]\delta_{\sigma\sigma'},$$
$$\{\Pi_1^{(2)}(\sigma),\Pi_2^{(3)}(\sigma')\}' = -\tfrac{1}{2}[C_{12}^{(22)},\mathscr{C}_2^{(1)}]\delta_{\sigma\sigma'},$$

$$\mathscr{C}^{(0)}\equiv\Pi^{(0)},\ \mathscr{C}^{(1)}\equiv\tfrac{1}{2}A^{(1)}+\Pi^{(1)},\ A_{\pm}^{(2)}\equiv\tfrac{1}{2}(\Pi^{(2)}\mp A^{(2)},\ \mathscr{C}^{(3)}\equiv-\tfrac{1}{2}A^{(3)}+\Pi^{(3)}$$

Casimirs of the modified Poisson bracket

Case of Sigma Model on symmetric space F/G

$$\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)} \quad \text{with} \quad \mathfrak{f}^{(0)} = \mathfrak{g}$$

Phase space variables: $\left(A^{(0)}, \Pi^{(0)}, A_{\pm}^{(1)} = A^{(1)} \pm \Pi^{(1)}\right)$

$$\mathscr{L} = A^{(0)} + \tfrac{1}{2}(\lambda^{-1} + \lambda)A^{(1)} + \tfrac{1}{2}(1 - \lambda^2)\Pi^{(0)} + \tfrac{1}{2}(\lambda^{-1} - \lambda)\Pi^{(1)}$$

Non-vanishing PB:

$$\{A_1^{(0)}(\sigma), A_2^{(0)}(\sigma')\}' = -[C_{12}^{(00)}, 2A_2^{(0)}(\sigma) + \Pi_2^{(0)}(\sigma)]\delta_{\sigma\sigma'} + 2C_{12}^{(00)}\delta'_{\sigma\sigma'}$$

$$\{A_1^{(0)}(\sigma), A_2^{(1)}(\sigma')\}' = -[C_{12}^{(00)}, A_2^{(1)}(\sigma) + \Pi_2^{(1)}(\sigma)]\delta_{\sigma\sigma'}$$

$$\{A_1^{(0)}(\sigma), \Pi_2^{(1)}(\sigma')\}' = -[C_{12}^{(00)}, A_2^{(1)}(\sigma) + \Pi_2^{(1)}(\sigma)]\delta_{\sigma\sigma'}$$

$$\{A_1^{(1)}(\sigma), A_2^{(1)}(\sigma')\}' = -[C_{12}^{(11)}, \Pi_2^{(0)}(\sigma)]\delta_{\sigma\sigma'}$$

$$\{A_1^{(1)}(\sigma), \Pi_2^{(1)}(\sigma')\}' = [C_{12}^{(11)}, \Pi_2^{(0)}(\sigma)]\delta_{\sigma\sigma'}$$

$$\{\Pi_1^{(1)}(\sigma), \Pi_2^{(1)}(\sigma')\}' = -[C_{12}^{(11)}, \Pi_2^{(0)}(\sigma)]\delta_{\sigma\sigma'}$$

Casimirs of the modified Poisson bracket

Case of Sigma Model on symmetric space F/G

$$\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)} \quad \text{with} \quad \mathfrak{f}^{(0)} = \mathfrak{g}$$

Casimirs	
$\Pi^{(0)}$	Hamiltonian constraint $\rightarrow \Pi^{(0)} = 0$
$A_+^{(1)}$	$A_+^{(1)}(\sigma,\tau) = \mu_+ T$ with $\mu_+ \in \mathbb{R}$ and $T \in \mathfrak{f}^{(1)}$
$\mathrm{Tr}\left[(A_-^{(1)})^n\right]$	Polar decomposition theorem $A_-^{(1)}(\sigma,\tau) = \mu_- g^{-1}(\sigma,\tau) T g(\sigma,\tau)$ with $\mu_- \in \mathbb{R}$ and $g(\sigma,\tau) \in G$

Casimirs and Pohlmeyer reduction

[Pohlmeyer '76]

Casimirs are consistent with dynamics of Sigma Model !

Casimirs	Interpretation
$\Pi^{(0)}$	$\Pi^{(0)} = 0$ Hamiltonian constraint
$A_+^{(1)}$	$A_+^{(1)} = \mu_+ T$ Partial gauge fixing condition and Partial conformal symmetry reduction
$\mathrm{Tr}[(A_-^{(1)})^n]$	$A_-^{(1)} = \mu_- g^{-1} T g$ Partial conformal symmetry reduction

Fixing Casimirs $\Longleftrightarrow$ Pohlmeyer reduction

Result of the procedure

- Left with phase space variables $(g \in G, A^{(0)} \in \mathfrak{g})$
- Satisfy the PB

$$
\begin{aligned}
\{g_1(\sigma), g_2(\sigma')\}' &= 0 \\
\{g_1(\sigma), A_2^{(0)}(\sigma')\}' &= -2 g_1(\sigma) C_{12}^{(00)} \delta_{\sigma\sigma'} \\
\{A_1^{(0)}(\sigma), A_2^{(0)}(\sigma')\}' &= -2[C_{12}^{(00)}, A_2^{(0)}(\sigma)]\delta_{\sigma\sigma'} + 2C_{12}^{(00)}\delta'_{\sigma\sigma'}
\end{aligned}
$$

- Lax matrix

$$\mathscr{L}(\lambda) = A^{(0)} + \tfrac{1}{2}\lambda^{-1}\mu_- g^{-1} T g - \tfrac{1}{2}\lambda \mu_+ T$$

- Hamiltonian H' determined

Consequence for symmetric space sine-Gordon (SSSG) models

Pohlmeyer reduction of Sigma model on symmetric space F/G = SSSG model

[Eichenherr, Pohlmeyer '79]

Lagrangian formulation: G/H gauged WZW model + Potential term

Lie algebra $\mathfrak{h}$ = Elements of $\mathfrak{g}$ commuting with T

[Bakas et al. '96, Grigoriev and Tseytlin '07]

Results: 1. $\{\cdot,\cdot\}'$ is **the canonical structure** of SSSG models !

2. H' and $\mathscr{L}$ are the corresponding Hamiltonian and Lax matrix

↓

Non-ultralocality of symmetric space sine-Gordon models viewed as gauged WZW models + Potential is mild

Case of the superstring

Same results for $AdS_5 \times S^5$ superstring:

- Alleviation procedure leads to Pohlmeyer reduction

$\exists$ Casimirs of $\{\cdot,\cdot\}'$ that correspond to gauge fixing conditions of κ-symmetry

- Non-ultralocality of $AdS_5 \times S^5$ semi-symmetric space sine-Gordon model is mild

[Grigoriev and Tseytlin '07, Mikhailov and Schäfer-Nameki '07]

Remarks

1. **No freedom** in the procedure !

Mild non-ultralocality $\rightarrow$ Modified PB $\rightarrow$ Pohlmeyer reduction

$*$

2. Canonical structure of (semi) symmetric space σ models

[Mikhailov '05, '06, Schmiddt '10, '11] $\downarrow$ Pohlmeyer reduction

Poisson bracket on reduced phase space is non-local !

Remarks

3. **Complex sine-Gordon**

- As a gauged $SU(2)/U(1)$ WZW model + Potential:

Non-ultralocality mild

- However, if $U(1)$ invariance is gauge fixed:

  $\rightarrow$ Gauge fixed action:

$$\int d\sigma d\tau \frac{1}{2}\left(\frac{|\partial_\alpha \psi|^2}{1-|\psi|^2} - m^2|\psi|^2\right)$$

  $\rightarrow$ Compute associated r/s structure: Dynamical ! [Maillet '86]

$\Longrightarrow$ **Try to discretize at the level of gauged WZW**

Reminiscent of other results within factorized scattering theory

[Dorey and Hollowood '95, Hoare and Tseytlin '10]

Conclusion

Non-ultralocality of generalized sine-Gordon models is mild

Generalization of first steps of **FR procedure**

=

Pohlmeyer reduction

Challenge: Reach same situation as for ultralocal models. One knows from [Freidel-Maillet '91 '92] that one has to search for representation of **Quantum Algebra**

$$A_{12}T_1^nT_2^n = T_2^nT_1^nD_{12}$$
$$T_1^{n+1}B_{12}T_2^n = T_2^nT_1^{n+1}$$
$$T_1^nT_2^{n+1} = T_2^{n+1}C_{12}T_1^n$$

$$A_{12}A_{13}A_{23} = A_{23}A_{13}A_{12}$$
$$D_{12}D_{13}D_{23} = D_{23}D_{13}D_{12}$$
$$A_{12}C_{13}C_{23} = C_{23}C_{13}A_{12}$$
$$D_{12}B_{13}B_{23} = B_{23}B_{13}D_{12}$$

Conclusion

Appealing structure which brings **hope** that one may be able to quantize from first principles **at least** the **Pohlmeyer reduction** of the superstring !