# Short spinning strings, symmetries and the spectrum of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring


Radu Roiban
PSU


Based on 1203.5710
M. Beccaria, S. Giombi, G. Macorini and A. Tseytlin

## Integrability for $\mathcal{N}=4$ sYM and strings in $AdS_5 \times S^5$

- Allows in principle to determine the spectrum of dimensions/energies at any value of the coupling constant

Weak coupling integrability explicitly demonstrated though 4/5-loop order
-- large quantum numbers provide guidance

Strong coupling integrability explicitly demonstrated though 2-loop order

The current strategy:

- assume all-order integrability and construct Bethe ansatze of appropriate type that explain existing data
- check predictions for other dimensions

## Good understanding of operators with large quantum numbers

### Asymptotic Bethe ansatz (ABA)

(final form) Beisert, Eden, Staudacher

Firmly established: tested through 4/2 loops at weak/strong coupling

String theory tests: semi-classical expansion, states with large quantum numbers dual to operators with large classical dimension

- Most studied example of long operator: $\mathrm{Tr}[\Phi^{J-1} D_+^S \Phi]$

$$S \gg J: \quad E = J + 2 + f(\lambda) \ln S + \ldots$$

$$f(\lambda \ll 1) = \frac{\lambda}{2\pi^2}\Big[1 - \frac{\lambda}{48} + \frac{11\,\lambda}{2^8 \times 45} - \Big(\frac{73}{630} + \frac{4\,\zeta_3^2}{\pi^6}\Big)\frac{\lambda^3}{2^7} + \ldots\Big]$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}\Big[1 - \frac{3\,\ln 2}{\sqrt{\lambda}} - \frac{\mathrm{K}}{(\sqrt{\lambda})^2} + \ldots\Big]$$

Both regimes captured by BES equation; solution by Basso, Korchemsky, Kotanski

- Long and fast string generalization

$$\ell = \frac{J}{\sqrt{\lambda}\ln S} - \text{fixed}: \quad E = J + 2 + \left\{ \begin{array}{c} f_w(\lambda, \sqrt{\lambda}\ell) \\ f_s(\sqrt{\lambda}, \ell^2) \end{array} \right\} \ln S + \ldots$$

Captured by FRS equation

general “short” operators $\longleftrightarrow$ general string quantum states

- Integrability-based results; improvements of ABA
  - “Luscher corrections” — Ambjorn, Kristjansen, Janik; Bajnog, Janik, Lukowski; Bajnog, Hegedus, Janik, Lukowski
  - ABA $\longrightarrow$ TBA — Arutyunov, Frolov
  - TBA in Y variables — Gromov, Kazakov, Kozak, Vieira; Bombardelli, Fioravanti, Tateo

- Conjectured set of functional equations for the spectrum

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})} \qquad f^{\pm} = f(u \pm i/2) \quad f^{[a]} = f(u \pm ia/2)$$

Boundary conditions: $Y_{0s} = \infty$, $Y_{a>2,\pm 2} = 0$, $Y_{2,|s|>2} = \infty$, $Y_{2,\pm 2} = \text{finite}$

Energy:

$$E = \Delta - J = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a^*}{\partial u} \ln(1 + Y_{a0}^*(u))$$

$$\frac{u}{g} = x + \frac{1}{x} \qquad \epsilon_a(u) = a + \frac{2ig}{x^{[+a]}} - \frac{2ig}{x^{[-a]}}$$

Additional constraints needed for a physical solution

general “short” operators $\longleftrightarrow$ general string quantum states

- Integrability-based results; improvements of ABA
  - “Luscher corrections” — Ambjorn, Kristjansen, Janik; Bajnog, Janik, Lukowski; Bajnog, Hegedus, Janik, Lukowski
  - ABA $\longrightarrow$ TBA — Arutyunov, Frolov
  - TBA in Y variables — Gromov, Kazakov, Kozak, Vieira; Bombardelli, Fioravanti, Tateo

- Conjectured set of functional equations for the spectrum
- Additional constraints related to $\mathbb{Z}_4$ automorphism of $PSU(2,2|4)$

  FiNLIE — Gromov, Kazakov, Leurent, Volin

Main assumptions: -- quantum integrability in finite volume

-- string $\sigma-$ model/gt spin chain obeys Hirota eq.

$$T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$

general "short" operators $\longleftrightarrow$ general string quantum states

Simplest example: the Konishi multiplet, containing $\mathrm{Tr}[\Phi^i\bar{\Phi}_i]$

$$\Delta(\lambda \ll 1) = 2 + \frac{12\,\lambda}{(4\pi)^2}\Big[1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} - (208 - 48\zeta_3 + 120\zeta_5)\frac{\lambda^3}{(4\pi)^6} + 8(158 + 72\zeta_3 - 54\zeta_3^2 - 90\zeta_5 + 315\zeta_7)\frac{\lambda^4}{(4\pi)^8} + \dots\Big]$$

5-loop: Initially predicted by from Luscher corrections; confirmed by direct calculation

Bajnok, Hegedus, Janik, Lukowski; Eden, Korchemsky, Smirnov, Sokatchev

Strong coupling: Konishi multiplet $\longleftrightarrow$ 1st excited string level

Worldsheet analysis suggests that (more later)

$$\Delta(\lambda \gg 1) = 2\sqrt[4]{\lambda} + A_0 + \frac{A_1}{\sqrt[4]{\lambda}} + \frac{A_2}{(\sqrt[4]{\lambda})^3} + \dots$$

RR, Tseytlin

general "short" operators $\longleftrightarrow$ general string quantum states

Simplest example: the Konishi multiplet, containing $\mathrm{Tr}[\Phi^i \bar{\Phi}_i]$

- TBA/Y-system eqs. solved numerically; fit reproduces QFT calculation

Current strong coupling status: $\Delta(\lambda \gg 1) = 2\sqrt[4]{\lambda} + A_0 + \frac{A_1}{\sqrt[4]{\lambda}} + \frac{A_2}{(\sqrt[4]{\lambda})^3} + \frac{A_3}{(\sqrt[4]{\lambda})^5} + \ldots$

- TBA/Y-system eqs. solved numerically up $\lambda \sim 10^3$ and extrapolated to large $\lambda$

Gromov, Kazakov, Vieira
Frolov

$$A_0 = 0 \qquad A_1 = 1.998 \qquad A_2 = -3.1$$

- worldsheet calculation at "1-loop"

RR, Tseytlin
Gromov, Shenderovich, Serban, Volin

$$A_0 = 0 \qquad A_1 = 2$$

- using an all-order conjecture of Basso, the "2-loop" coefficient is

Gromov, Valatka

$$A_2 = \frac{1}{2} - 3\zeta_3 \simeq -3.10617$$

- higher-order coefficients: $A_k \sim \zeta_{2k-1} + \text{lower transcendentality}$

Beccaria, Giombi, Macorini, RR, Tseytlin

e.g. $A_3 = a_{3,3} + a_{3,2}\zeta_3 + a_{3,1}\zeta_5 \quad \text{with } a_{3,1} = \frac{15}{2}$

general “short” operators $\longleftrightarrow$ general string quantum states

Open questions:

- Identify general structures of the spectrum from the TBA/Y-system; what type of numbers can appear? $(\zeta_l, \beta_l, \mathrm{Li}_n(\mathrm{rational}), \ldots)$
- Analytic calculations for the TBA/Y-system approach?
- Other states?
- A systematic comparison with near-flat-space string spectrum?

general “short” operators $\longleftrightarrow$ general string quantum states

Open questions:

- Identify general structures of the spectrum from the TBA/Y-system; what type of numbers can appear? $(\zeta_l, \beta_l, \mathrm{Li}_n(\text{rational}), \dots)$
- Analytic calculations for the TBA/Y-system approach?
- Other states?
- A systematic comparison with near-flat-space string spectrum?

Here:

- understand the structure of the energy as a function of charges for some classes of string states
- evaluate $1/\sqrt{\lambda}$ corrections to energy of other states that should belong to the Konishi multiplet
- check symmetries (same *E* up to rational)/compare with TBA/Y-system

## Outline

- Organization of flat space string state in $AdS_5 \times S^5$ multiplets
- Structure of energy corrections
- Short strings vs. the Konishi multiplet:
  vertex operator-improved semiclassical approach
- Summary

## Organization of states

In large volume limit $AdS_5 \times S^5$ is nearly flat

$\longrightarrow$ Expect that some properties of the spectrum survive

Flat space spectrum: best constructed in light-cone gauge $n = \frac{1}{2}(N + \tilde{N})$

mass : $-p^2 = m^2 = 4(2\pi T)\,(n-1)\ (NSR \oplus GSO) \qquad -p^2 = m^2 = 4(2\pi T)\,n\ (GS)$

ground state : $|0\rangle = (8_v + 8_c)^{\otimes 2} \quad \mapsto \quad$ IIB supergravity multiplet $2^8$ states

1st excited : $(a^i_{-1} \oplus S^a_{-1})^{\otimes 2}|0\rangle \equiv ((8_v + 8_c) \otimes (8_v + 8_c))^{\otimes 2} \quad \mapsto \quad 2^{16}$ states

-- Organize in representations of $SO(9)$ $\quad (44_b + 84_b + 128_f)^{\otimes 2}$

-- A single supersymmetry multiplet

higher levels : More oscillators; different indices, etc; representations of $SO(9)$, all of the same mass, several supersymmetry multiplets

Some changes in the presence of a weakly curved $AdS_5 \times S^5$ background:

# Organization of states

- Length of multiplets of 10d super-Poincare group = Length of multiplets 4d superconformal group

$\longrightarrow$ Multiplet structure is not severely reorganized; extra quantum number; Lift of flat space mass degeneracy (no useful notion of mass)

- Worldsheet perturbation theory – series in $\frac{1}{\sqrt{\lambda}}$

$\longrightarrow$ Different flat space levels cannot belong to the same multiplet; level-by-level reorganization

- Assuming non-intersection principle — Polyakov

$\longrightarrow$ Same should be true for all values of $\sqrt{\lambda}$

- Flat space spectrum: reorganized in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$

Energy defined here — Bianchi, Morales, Samtleben

$$SO(9) \mapsto SO(4) \times SO(5) \to SO(4) \times SO(6) \subset SO(2,4) \times SO(6); \;\; [k,p,q]_{(s_L,s_R)}$$

Example: 1st excited level $\displaystyle\sum_{J=0}^{\infty} [0,J,0]_{(0,0)} \times (1 + Q + Q \wedge Q + \dots)[0,0,0]_{(0,0)}$

$J = 0:$ Same content as the Konishi multiplet

Konishi multiplet $\longleftrightarrow$ 1st excited level (lowest KK)

TBA/Y

| $\Delta_0$ | $[p_1,q,p_2]_{(s_L,s_R)}=[J_2-J_3,J_1-J_2,J_2+J_3]_{(\frac{S_1+S_2}{2},\frac{S_1-S_2}{2})}$ |
|---|---|
| $2$ | $[0,0,0]_{(0,0)}$ |
| $2+\frac{1}{2}$ | $[0,0,1]_{(0,\frac12)}+[1,0,0]_{(\frac12,0)}$ |
| $2+1$ | $[0,0,0]_{(\frac12,\frac12)}+[0,0,2]_{(0,0)}+[0,1,0]_{(0,1)+(1,0)}+[1,0,1]_{(\frac12,\frac12)}+[2,0,0]_{(0,0)}$ |
| $2+\frac{3}{2}$ | $[0,0,1]_{(\frac12,0)+(\frac12,1)+(\frac32,0)}+[0,1,1]_{(0,\frac12)+(1,\frac12)}+[1,0,0]_{(0,\frac12)+(0,\frac32)+(1,\frac12)}+[1,0,2]_{(\frac12,0)}$<br>$+[1,1,0]_{(\frac12,0)+(\frac12,1)}+[2,0,1]_{(0,\frac12)}$ |
| $2+2$ | $[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)}+[0,0,2]_{(\frac12,\frac12)+(\frac32,\frac12)}+[0,1,0]_{2(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)}+[2,0,2]_{(0,0)}+[2,1,0]_{(0,1)}$<br>$+[0,1,2]_{(1,0)}+[0,2,0]_{2(0,0)+(1,1)}+[1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[1,1,1]_{2(\frac12,\frac12)}+[2,0,0]_{(\frac12,\frac12)+(\frac12,\frac32)}$ |
| $2+\frac{5}{2}$ | $[0,0,1]_{(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)+(2,\frac12)}+[0,0,3]_{(\frac32,0)}+[0,1,1]_{3(\frac12,0)+2(\frac12,1)+(\frac32,0)+(\frac32,1)}+[0,2,1]_{(0,\frac12)+(1,\frac12)}$<br>$+[1,0,0]_{(\frac12,0)+2(\frac12,1)+(\frac12,2)+(\frac32,0)+(\frac32,1)}+[1,0,2]_{(0,\frac12)+2(1,\frac12)}+[1,1,0]_{3(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)}$<br>$+[1,1,2]_{(\frac12,0)}+[1,2,0]_{(\frac12,0)+(\frac12,1)}+[2,0,1]_{(\frac12,0)+2(\frac12,1)}+[2,1,1]_{(0,\frac12)}+[3,0,0]_{(0,\frac32)}$ |
| $2+3$ | $[0,0,0]_{(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)+(\frac32,\frac32)}+[0,0,2]_{2(0,0)+(1,0)+2(1,1)+(2,0)}+[0,1,0]_{3(0,1)+3(1,0)+2(1,1)+(1,2)+(2,1)}$<br>$+[0,1,2]_{2(\frac12,\frac12)+(\frac32,\frac12)}+[0,2,0]_{3(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)}+[0,2,2]_{(0,0)}+[0,3,0]_{(0,1)+(1,0)}$<br>$+[1,0,3]_{(1,0)}+[1,1,1]_{2(0,0)+2(0,1)+2(1,0)+2(1,1)}+[1,2,1]_{(\frac12,\frac12)}+[2,0,0]_{2(0,0)+(0,1)+(0,2)+2(1,1)}$<br>$+[2,0,2]_{(\frac12,\frac12)}+[2,1,0]_{2(\frac12,\frac12)+(\frac12,\frac32)}+[2,2,0]_{(0,0)}+[3,0,1]_{(0,1)}+[1,0,1]_{4(\frac12,\frac12)+2(\frac12,\frac32)+2(\frac32,\frac12)+(\frac32,\frac32)}$ |
| $2+\frac{7}{2}$ | $[0,0,1]_{2(\frac12,0)+3(\frac12,1)+(\frac32,0)+2(\frac32,1)+(\frac32,2)}+[0,0,3]_{(0,\frac12)+(1,\frac12)}+[0,1,1]_{3(0,\frac12)+(0,\frac32)+4(1,\frac12)+2(1,\frac32)+(2,\frac12)}$<br>$+[0,1,3]_{(\frac12,0)}+[0,2,1]_{2(\frac12,0)+2(\frac12,1)+(\frac32,0)}+[0,3,1]_{(0,\frac12)}+[1,0,0]_{2(0,\frac12)+(0,\frac32)+3(1,\frac12)+2(1,\frac32)+(2,\frac32)}$<br>$+[1,0,2]_{2(\frac12,0)+2(\frac12,1)+(\frac32,0)+(\frac32,1)}+[1,1,0]_{3(\frac12,0)+4(\frac12,1)+(\frac12,2)+(\frac32,0)+2(\frac32,1)}+[1,1,2]_{(0,\frac12)+(1,\frac12)}$<br>$+[1,2,0]_{2(0,\frac12)+(0,\frac32)+2(1,\frac12)}+[1,3,0]_{(\frac12,0)}+[2,0,1]_{2(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)}+[2,1,1]_{(\frac12,0)+(\frac12,1)}$<br>$+[3,0,0]_{(\frac12,0)+(\frac12,1)}+[3,1,0]_{(0,\frac12)}$ |
| $2+4$ | $[0,0,0]_{3(0,0)+3(1,1)+(2,2)}+[0,0,2]_{3(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)+(\frac32,\frac32)}+[0,1,0]_{4(\frac12,\frac12)+2(\frac12,\frac32)+2(\frac32,\frac12)+2(\frac32,\frac32)}$<br>$+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)}+[0,2,2]_{(\frac12,\frac12)}$<br>$+[0,3,0]_{2(\frac12,\frac12)}+[0,4,0]_{(0,0)}+[1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)}+[1,0,3]_{(\frac12,\frac12)}+[0,0,4]_{(0,0)}$<br>$+[1,1,1]_{4(\frac12,\frac12)+2(\frac12,\frac32)+2(\frac32,\frac12)}+[1,2,1]_{(0,0)+(0,1)+(1,0)}+[2,0,0]_{3(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)+(\frac32,\frac32)}$<br>$+[2,0,2]_{(0,0)+(1,1)}+[2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[2,2,0]_{(\frac12,\frac12)}+[3,0,1]_{(\frac12,\frac12)}+[4,0,0]_{(0,0)}$ |

## 1st excited level vs. higher levels

1st excited string level: minimal (2d RG) mixing of their vertex operators:

- energies of states in the multiplet may differ only by (half-)integers
- mild mixing may be possible between states in different KK multip's

Higher levels: lots of multiplets

- each can acquire a different energy (i.e. different $\frac{1}{\sqrt{\lambda}}$ corrections)
- nontrivial mixing problem of vertex operators (help from closed sectors)

► Is there an integrable model for it? (w/ Hamiltonian $L_0 + \bar{L}_0$)

implications for correlation functions, worldsheet form factors, etc.

► No level mixing; what is the gauge theory analog of string level?

# On vertex operators and the general structure of $E(Q)$

Bosonic Lagrangian with manifest $SO(2,4) \times SO(6)$ symmetry

$$\mathcal{L} \sim \partial Y \cdot \bar{\partial} Y + \Lambda(Y \cdot Y + 1) + \partial X \cdot \bar{\partial} X + \Lambda(X \cdot X - 1)$$

Vertex operators

$$\mathcal{V} = \underbrace{\phi_{m_1 \ldots m_s} Z^{m_1} \ldots Z^{m_s}}_{\text{charges unrelated to level}} \underbrace{\partial Z \ldots \partial Z \, \bar{\partial} Z \ldots \bar{\partial} Z}_{\text{determine the level}} \underbrace{(Y_5 + Y_{0e})^{-\Delta}}_{\text{bulk-boundary prop.}}$$

charges unrelated to level
e.g. KK charge $(n \cdot Z)^J$

determine the level
partly in terms of
charges

bulk-boundary prop.

$$K = \left( \frac{z}{z^2 + (x - \mathrm{x})^2} \right)^{\Delta}$$

analog of $e^{iEt}$

Physical state condition: $\gamma_{2\mathrm{d}}(Q) = 0 \quad \text{with } Q - \text{fixed as } \lambda \to \infty$

- $\gamma_{2\mathrm{d}}(Q)$ from renormalization of vertex operators

At L-loops: Feynman graphs with at most (L+1) fields attached to $\mathcal{V}$

$$\frac{1}{\sqrt{\lambda}^L} (Q^{L+1} + \text{lower powers}) \subset \gamma_{2\mathrm{d}}$$

Bosonic model: detailed expressions

Wegner

In general: $\gamma_{2\mathrm{d}}$ is a matrix – the matrix representation of $L_0 + \bar{L}_0$

$$\frac{d}{d\ln\Lambda_{2\mathrm{d}}}\mathcal{V}^i = (d_i\delta^i_j + \hat{\gamma}_{2\mathrm{d}}{}^i{}_j)\mathcal{V}^j$$

-- structure constrained by symmetries and charge conservation

-- eigenvalues are the anomalous dimensions of vertex operators

Worldsheet theory is integrable in conformal gauge; seems plausible that it diagonalizes $L_0 + \bar{L}_0$; eigenvectors related to physical vertex operators

Ground state and 1st excited level are special: no/minimal mixing

$\longrightarrow$ 2d dimension is a regular function of $\sqrt{\lambda}$

$$\begin{aligned} E(E-4) &= 2\sqrt{\lambda}\sum a_i Q_i \\ &+ \sum_{i,j} b_{ij}Q_iQ_j + \sum c_iQ_i \\ &+ \tfrac{1}{\sqrt{\lambda}}\Big(\sum_{ijk} d_{ijk}Q_iQ_jQ_k + \sum_{i,j} e_{ij}Q_iQ_j + \sum f_iQ_i\Big) + \dots \end{aligned}$$

There exist singlet vertex operators which do not mix

In general: $\gamma_{\rm 2d}$ is a matrix – the matrix representation of $L_0 + \bar{L}_0$

$$\frac{d}{d\ln\Lambda_{\rm 2d}}\mathcal{V}^i = (d_i\delta^i_j + \hat{\gamma}_{\rm 2d}{}^i{}_j)\mathcal{V}^j$$

-- structure constrained by symmetries and charge conservation

-- eigenvalues are the anomalous dimensions of vertex operators

Worldsheet theory is integrable in conformal gauge; seems plausible that it diagonalizes $L_0 + \bar{L}_0$; eigenvectors related to physical vertex operators

Ground state and 1st excited level are special: no/minimal mixing
$\longrightarrow$ 2d dimension is a regular function of $\sqrt{\lambda}$

$$E(E-4) = 2\sqrt{\lambda}\sum a_i Q_i$$

← Tree level

$$+ \sum_{i,j} b_{ij} Q_i Q_j + \sum c_i Q_i$$

1 loop

2 loops

$$+ \frac{1}{\sqrt{\lambda}}\Big(\sum_{ijk} d_{ijk} Q_i Q_j Q_k + \sum_{i,j} e_{ij} Q_i Q_j + \sum f_i Q_i\Big) + \dots$$

There exist singlet vertex operators which do not mix

## Simple examples:

- strings in flat space: $E^2 \propto M^2 \propto J$ (1-loop corr. is exact)
- supergravity states: $E(E-4) = J(J+4)$
- $\mathcal{V} = Y_+^{-\Delta}\Big[(\partial X^k \bar{\partial}\bar{X}_k)^r \; + \; \dots\Big] \implies$ (use NSR formulation)

$$0 = -2(r-1) + \frac{1}{2\sqrt{\lambda}}\left[\Delta(\Delta-4) + 2r(r-1)\right] + \frac{1}{\sqrt{\lambda}^2}\left[\frac{2}{3}r(r-1)(r-\frac{7}{2}) + \dots\right]$$

- Solve for $\Delta$

Krastov, Lerner, Yudson; Castilla, Chakravarti

- Inclusion of fermions may change some of the details

- An excited string state: choose effective level $N$ (+ constant shifts)

$$0 = N + \frac{1}{2\sqrt{\lambda}}\Big(-E^2 + J^2 + n_{02}N^2 + n_{11}N\Big)$$
$$+ \frac{1}{2(\sqrt{\lambda})^2}\Big(n_{01}NJ^2 + n_{03}N^3 + n_{12}N^2 + n_{21}N\Big) + O(\tfrac{1}{(\sqrt{\lambda})^3})$$

Target space energy, in the presence of two charges; more subleading terms

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N$$
$$+ \frac{1}{\sqrt{\lambda}}\left(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\right)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\left(\tilde{n}_{11}J^2N + \tilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N\right)$$
$$+ \frac{1}{(\sqrt{\lambda})^3}\left(\tilde{n}_{01}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + ...\right)$$
$$+ \frac{1}{(\sqrt{\lambda})^4}\left(\bar{n}_{11}J^4N + ...\right) + O\left(\frac{1}{(\sqrt{\lambda})^5}\right)$$

- Other possible terms:

$E^k, E^l J^m, J^p$: spurious sol's to marginality condition in BMN limit; similar to higher powers of Laplacian in eom $\longrightarrow$ scheme dependent

$E^k N^p$ : treated perturbatively; slight redefinitions of $n-$coef's

- Direct flat space interpretation; $N-\mathrm{level}$

- Valid for all states that have a vertex operator description, regardless of values of charges

Target space energy, in the presence of two charges; more subleading terms

$$
\begin{aligned}
E^2 = {} & 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N \\
& + \frac{1}{\sqrt{\lambda}}\left(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\right) \\
& + \frac{1}{(\sqrt{\lambda})^2}\left(\tilde{n}_{11}J^2N + \tilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N\right) \\
& + \frac{1}{(\sqrt{\lambda})^3}\left(\tilde{n}_{01}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + \ldots\right) \\
& + \frac{1}{(\sqrt{\lambda})^4}\left(\bar{n}_{11}J^4N + \ldots\right) + O\left(\frac{1}{(\sqrt{\lambda})^5}\right)
\end{aligned}
$$

- the plan -- use universality of this structure to:
  - find the values for the coefficients $n_{ij}$ from semiclasscial short string expansion
  - if several suitable expansions exist, check that they give same values
  - justify semiclassical approach to quantum string states
  - test realization of symmetries: classical solutions in the same multiplet have same energy corrections (up to integer shifts)

## Expansions:

1) Connection to quantum string states: large $\sqrt{\lambda}$ , fixed charges $N$ and $J$

$$E = \sqrt{2\sqrt{\lambda}N}\Big[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + \frac{A_3}{(\sqrt{\lambda})^3} + O(\tfrac{1}{(\sqrt{\lambda})^4})\Big]$$

$$A_1 = \frac{1}{4N}J^2 + \frac{1}{4}(n_{02}N + n_{11}) \ ,$$

$$A_2 = -\frac{1}{2}A_1^2 + \frac{1}{4}(n_{01}J^2 + n_{03}N^2 + n_{12}N + n_{21})$$

$$A_3 = \frac{1}{128}\Big[(n_{11}^3 - 8n_{11}n_{21} + 32n_{31}) + (3n_{02}n_{11}^2 - 8n_{11}n_{12} - 8n_{02}n_{21} + 32n_{22})N$$
$$+ (3n_{02}^2n_{11} - 8n_{03}n_{11} - 8n_{02}n_{12} + 32n_{13})N^2 + ...\Big]$$

- Konishi multiplet states: $N = 2, J = 2$
- Higher KK levels: tensor with $[0, J, 0]$

## Expansions:

1) Connection to quantum string states: large $\sqrt{\lambda}$ , fixed charges $N$ and $J$

- Konishi multiplet states: $N=2, J=2$
- Higher KK levels: tensor with $[0,J,0]$

$$E_{N=2}=\sqrt{2\sqrt{\lambda}N}\Big[1+\frac{(A_1)_{N=2}}{\sqrt{\lambda}}+\frac{(A_2)_{N=2}}{(\sqrt{\lambda})^2}+\frac{(A_3)_{N=2}}{(\sqrt{\lambda})^3}+O(\tfrac{1}{(\sqrt{\lambda})^4})\Big]$$

$$(A_1)_{N=2}=\frac{1}{8}J^2+\frac{1}{4}(2n_{02}+n_{11})$$

$$(A_2)_{N=2}=-\frac{1}{2}(A_1)^2_{N=2}+\frac{1}{4}n_{01}J^2+\frac{1}{4}(4n_{03}+2n_{12}+n_{21})$$

$$(A_3)_{N=2}=-(A_1A_2)_{N=2}+\frac{1}{4}(2\tilde{n}_{02}+\tilde{n}_{11})J^2+\frac{1}{4}(8n_{04}+4n_{13}+2n_{22}+n_{31})\Big]$$

- Symmetry consequences (uniqueness of multiplet)

  $n_{ij}$ are not universal; however --

$$2n_{02}+n_{11}=\text{fixed} \qquad n_{01}=\text{fixed} \qquad 4n_{03}+2n_{12}+n_{21}=\text{fixed}$$

$$2\tilde{n}_{02}+\tilde{n}_{11}=\text{fixed} \qquad 8n_{04}+4n_{13}+2n_{22}+n_{31}=\text{fixed}$$

## Expansions:

2) Semiclassical expansion: fixed $\mathcal{Q} = Q/\sqrt{\lambda}$ large $\sqrt{\lambda}$ , then small $\mathcal{N}$ and $\mathcal{J}$

$$
\begin{aligned}
\frac{E}{\sqrt{\lambda}} = \mathcal{J} + \Big[&\frac{\mathcal{N}}{\mathcal{J}}(1 + \tfrac{1}{2}n_{01}\mathcal{J}^2 + \tfrac{1}{2}\tilde{n}_{01}\mathcal{J}^4 + ...) \\
&- \frac{\mathcal{N}^2}{2\mathcal{J}^3}\Big(1 + (n_{01} - n_{02})\mathcal{J}^2 + (\tilde{n}_{01} - \tilde{n}_{02} + \tfrac{1}{4}n_{01}^2)\mathcal{J}^4 + ...\Big) + ...\Big] \\
+ \frac{1}{\sqrt{\lambda}}\Big[&\frac{\mathcal{N}}{2\mathcal{J}}(n_{11} + \tilde{n}_{11}\mathcal{J}^2 + \bar{n}_{11}\mathcal{J}^4 + ...) \\
&+ \frac{\mathcal{N}^2}{2\mathcal{J}^3}\Big(-n_{11} + (n_{12} - \tfrac{1}{2}n_{01}n_{11} - \tilde{n}_{11})\mathcal{J}^2 + (\tilde{n}_{12} - \bar{n}_{11} - \tfrac{1}{2}n_{01}\tilde{n}_{11} - \tfrac{1}{2}\tilde{n}_{01}n_{11})\mathcal{J}^4 + ...\Big) \\
&+ \frac{\mathcal{N}^3}{4\mathcal{J}^5}\Big(3n_{11} + [3\tilde{n}_{11} - 2n_{12} + (3n_{01} - n_{02})n_{11}]\mathcal{J}^2 \\
&+ \big[2(n_{13} - \tilde{n}_{12}) - n_{01}n_{12} + 3\bar{n}_{11} + (3\tilde{n}_{01} - \tilde{n}_{02} + \tfrac{3}{4}n_{01}^2)n_{11} + (3n_{01} - n_{02})\tilde{n}_{11}\big]\mathcal{J}^4 + ...\Big) + ...\Big] \\
+ \frac{1}{(\sqrt{\lambda})^2}\Big[&\frac{\mathcal{N}}{2\mathcal{J}}\big(n_{21} + \tilde{n}_{21}\mathcal{J}^2 + ...\big) + ...\Big] + \mathcal{O}\big(\tfrac{1}{(\sqrt{\lambda})^3}\big)
\end{aligned}
$$

- Each loop order is a nontrivial function of charges

⟶ $E^2$ is a much more economical presentation of the spectrum

## Expansions:

Resummation of small $\mathcal{J}$ expansion: contact with the slope function(s)

$$E^2 = J^2 + h_1(\lambda, J)N + h_2(\lambda, J)N^2 + h_3(\lambda, J)N^3 + \ldots$$

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + \ldots + J^2\left(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\tilde{n}_{11}}{(\sqrt{\lambda})^2} + \frac{\tilde{n}_{21}}{(\sqrt{\lambda})^3} + \ldots\right) + \ldots$$

$$h_2 = n_{02} + \frac{n_{12}}{\sqrt{\lambda}} + \frac{n_{22}}{(\sqrt{\lambda})^2} + \ldots + J^2\left(\frac{\tilde{n}_{02}}{(\sqrt{\lambda})^2} + \frac{\tilde{n}_{12}}{(\sqrt{\lambda})^3} + \ldots\right) + \ldots$$

$$h_3 = \frac{n_{03}}{\sqrt{\lambda}} + \frac{n_{13}}{(\sqrt{\lambda})^2} + \ldots, \qquad h_4 = \frac{n_{03}}{(\sqrt{\lambda})^2} + \ldots$$

$$E \text{ at small } N \text{ and fixed } J: \; E = J + \frac{1}{2J} h_1(\lambda, J) N + \ldots$$

- Conjectured by Basso in $sl(2)$ sector based on ABA
- Proven by Basso and Gromov

$su(2)$ sector is more intricate

- Conjectured by Gromov and Beccaria and Tseytlin

• information on the 2-loop coefficient $n_{21}$ -- why rational?

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + \ldots + J^2\Big(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\tilde{n}_{11}}{(\sqrt{\lambda})^2} + \frac{\tilde{n}_{21}}{(\sqrt{\lambda})^3} + \ldots\Big) + \ldots$$

Basso

$$h_1^{sl(2)} = 2\sqrt{\lambda}\frac{d\ln I_J(\sqrt{\lambda})}{d\sqrt{\lambda}}$$

$$= 2\sqrt{\lambda} - 1 - \frac{\frac{1}{4} - J^2}{\sqrt{\lambda}} - \frac{\frac{1}{4} - J^2}{(\sqrt{\lambda})^2} - \frac{\frac{25}{64} - \frac{13}{8}J^2 + \frac{1}{4}J^4}{(\sqrt{\lambda})^3} + \ldots$$

$$= 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} - \frac{1}{1+\mathcal{J}^2} - \frac{\frac{1}{4} - \mathcal{J}^2}{\sqrt{\lambda}(1+\mathcal{J}^2)^{5/2}} - \frac{\frac{1}{4} - \frac{5}{2}\mathcal{J}^2 + \mathcal{J}^4}{(\sqrt{\lambda})^2(1+\mathcal{J}^2)^4} + \ldots$$

$$h_1^{su(2)}(\sqrt{\lambda}, \mathcal{J}) = -h_1^{su(2)}(-\sqrt{\lambda}, \mathcal{J}) \qquad h_1^{su(2)}(\sqrt{\lambda}, J) = -h_1^{su(2)}(-\sqrt{\lambda}, -J)$$

Gromov

Becaria, Tseytlin

$$h_1^{su(2)} = -2\sqrt{\lambda}\frac{d\ln K_J(\sqrt{\lambda})}{d\sqrt{\lambda}}$$

$$= 2\sqrt{\lambda} + 1 - \frac{\frac{1}{4} - J^2}{\sqrt{\lambda}} + \frac{\frac{1}{4} - J^2}{(\sqrt{\lambda})^2} - \frac{\frac{25}{64} - \frac{13}{8}J^2 + \frac{1}{4}J^4}{(\sqrt{\lambda})^3} + \ldots$$

$$= 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{1}{1+\mathcal{J}^2} - \frac{\frac{1}{4} - \mathcal{J}^2}{\sqrt{\lambda}(1+\mathcal{J}^2)^{5/2}} + \frac{\frac{1}{4} - \frac{5}{2}\mathcal{J}^2 + \mathcal{J}^4}{(\sqrt{\lambda})^2(1+\mathcal{J}^2)^4} + \ldots$$

- information on the 2-loop coefficient $n_{21}$ -- why rational?

# Classical solutions in the same multiplet: flat space

- Circular strings:

$$t = \kappa\tau \ , \quad \mathrm{x}_1 = x_1 + ix_2 = a e^{i(\tau+\sigma)} \ , \quad \mathrm{x}_2 = x_3 + ix_4 = a e^{i(\tau-\sigma)}$$

$$E^2_{\rm flat} = \frac{2}{\alpha'} N \ , \quad N = J_1 + J_2 \ , \quad J_1 = J_2 = \frac{a^2}{\alpha'}$$

-- Vertex operator of the corresponding quantum state:

$$\mathcal{V} = e^{-iEt}\left[(\partial \mathrm{x}_1 \bar\partial \mathrm{x}_1)^{J_1/2} (\partial \mathrm{x}_2 \bar\partial \mathrm{x}_2)^{J_2/2} + \ldots\right]$$

- Other orientations related by Lorentz transformations
- Add translational momentum

- Folded strings:

$$t = \kappa\tau \ , \quad x_1 + ix_2 = a \sin\sigma e^{i\tau}$$

$$E^2_{\rm flat} = \frac{2}{\alpha'} N \ , \quad N = S \ , \quad S = \frac{2a^2}{\alpha'}$$

-- Vertex operator of the corresponding quantum state:

$$\mathcal{V} = e^{-iEt}\left[(\partial \mathrm{x}_1 \bar\partial \mathrm{x}_1)^{S/2} + \ldots\right]$$

## Classical solutions in the same multiplet: $AdS_5 \times S^5$

-- 1st excited level: $N = 2$
-- member of Konishi multiplet: add orbital angular momentum: $J = 2$

- Circular strings: three inequivalent embeddings $\Delta_0 = 6$

  -- two planes in $S^5$: $J_1 = J_2 = 1\,,\ J = 2 \ \rightarrow\ [0,1,2]_{(0,0)}$
  -- two planes in $AdS_5$ : $S_1 = S_2 = 1\,,\ J = 2 \ \rightarrow\ [0,2,0]_{(1,0)}$
  -- one plane in each: $S_1 = J_1 = 1\,,\ J = 2 \ \rightarrow\ [1,1,1]_{(\frac{1}{2},\frac{1}{2})}$

- Folded strings: two inequivalent embeddings $\Delta_0 = 4$

  -- one planes in $S^5$: $J_1 = J = 2 \ \rightarrow\ [2,0,2]_{(1,1)}$
  -- one planes in $AdS_5$ : $S_1 = J = 2 \ \rightarrow\ [0,2,0]_{(1,1)}$

- Evaluate loop corrections to their energies in the semiclassical regime
  (no longer in the same multiplet)
- Extract $n-$coefficients
- Find (same) correction in short string regime; predict other $n-$coefficients

## Regularization and symmetries

- Lorentz-invariant field theories in infinite volume

  -- limited choices of regulator: dimensional, higher-derivative, etc

- QFT in finite volume:

  -- more choices as space/time different: e.g. mode number cutoff

  * for diagonal propagators argued to preserve supersymmetry

Rebhan, van Nieuwenhuizen, et al

► Off-diagonal propagators, finite volume:

  -- more choices due to absence of natural $i\epsilon$ prescription

  * expand each field in modes, each mode may have its own independent $i\epsilon$ prescription

$$\int \frac{dp_0}{2\pi} \ln(p_0 - \omega(n) - i s_{i,n}\epsilon) = \frac{s_{i,n}}{2}\omega(n) \rightarrow E_{2\mathrm{d}} = \frac{1}{4}\sum_i \sum_{=\infty}^{\infty} \left( s^b_{i,n}\omega^b_i(n) - s^f_{i,n}\omega^f_i(n) \right)$$

- Choice of signs ⟷ realization of target space symmetry algebra

- transcendental part of $E_{2\mathrm{d}}$ controlled by $n \gg 1$; little sensitivity to summation prescription

$$
\begin{aligned}
E^2 &= 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + \underline{n_{11}}N \\
&+ \frac{1}{\sqrt{\lambda}}(\underline{n_{01}}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N) \\
&+ \frac{1}{(\sqrt{\lambda})^2}(\underline{\tilde{n}_{11}}J^2N + \underline{\tilde{n}_{02}}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N) \\
&+ \frac{1}{(\sqrt{\lambda})^3}(\underline{\tilde{n}_{01}}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + \dots) \\
&+ \frac{1}{(\sqrt{\lambda})^4}(\underline{\bar{n}_{11}}J^4N + \dots) + O(\tfrac{1}{(\sqrt{\lambda})^5})
\end{aligned}
$$

# Properties of the coefficients:

- Near-BMN expansion is universal

$$E^2 = J^2 + 2N\sqrt{\lambda + J^2} + \dots = J^2 + N\Big(2\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}J^2 - \frac{1}{4(\sqrt{\lambda})^3}J^4 + \dots\Big)$$

$\Rightarrow \quad n_{01} = 1 \quad \tilde{n}_{01} = -\frac{1}{4}$ Other classical coefficients: rational + state-dependent

- From analysis of folded strings and circular strings:

$$2n_{02} + n_{11} = 2$$

RR, Tseytlin

Gromov, Serban, Shenderovich, Volin

$$\tilde{n}_{11} = -n_{11} \ , \quad \bar{n}_{11} = n_{11} \ , \quad 2\tilde{n}_{02} + \tilde{n}_{11} = 0$$

Beccaria, Giombi, Macorini, RR, Tseytlin

⟶ Universality of $\mathcal{J}$-dependence in terms in slope function(s)

$$h_1 = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{n_{11}}{1+\mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}(n_{12} + \tilde{n}_{12}\mathcal{J}^2 + \mathcal{O}(\mathcal{J})^4)) + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^2})$$

$$h_2 = \frac{n_{02} + \mathcal{J}^2}{1+\mathcal{J}^2} + \mathcal{O}(\tfrac{1}{\sqrt{\lambda}})$$

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N$$
$$+ \frac{1}{\sqrt{\lambda}}\left(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\right)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\left(\tilde{n}_{11}J^2N + \tilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N\right)$$
$$+ \frac{1}{(\sqrt{\lambda})^3}\left(\tilde{n}_{01}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + ...\right)$$
$$+ \frac{1}{(\sqrt{\lambda})^4}\left(\bar{n}_{11}J^4N + ...\right) + O\left(\frac{1}{(\sqrt{\lambda})^5}\right)$$

Tirziu, Tseytlin;
RR, Tseytlin; Gromov, Valatka

## More properties and predictions:

$n_{12} = n'_{12} - 3\zeta_3$ universal

- Folded string slope function: $n_{21} = -\frac{1}{4}$ Basso

$\longrightarrow$ $4n_{03} + 2n'_{12} + n_{21} = -1$ $\longleftarrow$ should be universal

$n'_{12} = -\frac{3}{8} - 2n_{03}$

$\longrightarrow$ Remarkably val's of $n_{03}$ and $n'_{12}$ imply that $n_{21} = -\frac{1}{4}$ should be universal

- from a 2-loop sigma model calculation; why rational?

- More universal behavior: $n_{1k} \supset \zeta_{2k-1}$

e.g. $\tilde{n}_{12} = \tilde{n}'_{12} + 3\zeta_3 + \frac{15}{4}\zeta_5$ $\quad n_{13} = n'_{13} + n''_{13}\zeta_3 + \frac{15}{4}\zeta_5$

Top transcendental component: high modes, local ws structure

| $n_{ij}$ | $(S,J)$ | $(J',J)$ | $(J_1=J_2,J)$ | $(S_1=S_2,J)$ | $(S=J',J)$ |
|---|---|---|---|---|---|
| $n_{01}$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\widetilde{n}_{01}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ |
| $n_{02}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $0$ | $2$ | $1$ |
| $\widetilde{n}_{02}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $-1$ | $0$ |
| $n_{03}$ | $-\frac{3}{8}$ | $\frac{1}{8}$ | $0$ | $-1$ | $-\frac{1}{2}$ |
| $n_{04}$ | $\frac{31}{64}$ | $\frac{1}{64}$ | $0$ | $2$ | $\frac{3}{4}$ |
| $n_{11}$ | $-1$ | $1$ | $2$ | $-2$ | $0$ |
| $\widetilde{n}_{11}$ | $1$ | $-1$ | $-2$ | $2$ | $0$ |
| $\overline{n}_{11}$ | $-1$ | $1$ | $2$ | $-2$ | $0$ |
| $n'_{12}$ | $\frac{3}{8}$ | $-\frac{5}{8}$ | $-\frac{3}{8}$ | $\frac{13}{8}$ | $\frac{5}{8}(?)$ |
| $\widetilde{n}'_{12}$ | $-\frac{27}{16}$ | $-\frac{3}{16}$ | $-\frac{57}{16}$ | $-\frac{105}{16}$ | $-$ |
| $n'_{13}$ | $-\frac{9}{16}$ | $-\frac{7}{16}$ | $-\frac{3}{16}$ | $-\frac{85}{16}$ | $-$ |
| $n''_{13}$ | $\frac{15}{4}$ | $-\frac{3}{4}$ | $-\frac{3}{4}$ | $\frac{15}{4}$ | $\frac{3}{2}$ |
| $n_{21}$ | $-\frac{1}{4}$ | $-\frac{1}{4}(?)$ | $-\frac{1}{4}(?)$ | $-\frac{1}{4}(?)$ | $-\frac{1}{4}(?)$ |

An example: $J' \equiv J_1 = J_2 \neq J_3 \equiv J \quad \text{in} \quad S^5$

The solution in embedding coordinates: $X_1^2 + \cdots + X_6^2 = 1$

$$X_1 = \; a \; e^{i(w\tau+\sigma)} \;, \qquad X_2 = \; a \; e^{i(w\tau-\sigma)} \;, \qquad X_3 = \sqrt{1-2a^2} \; e^{i\nu\tau}$$

$$\mathcal{E}_0^2 = \kappa^2 = 4a^2 + \nu^2 = \nu^2 + \frac{4\mathcal{J}'}{\sqrt{1+\nu^2}} \;, \qquad w^2 = 1 + \nu^2$$

$$\mathcal{J}' \equiv \mathcal{J}_1 = \mathcal{J}_2 = a^2 w \;, \quad \mathcal{J} \equiv \mathcal{J}_3 = \sqrt{1-2a^2}\,\nu \;, \quad \nu = \frac{\mathcal{J}}{\sqrt{1 - \frac{2\mathcal{J}'}{\sqrt{1+\nu^2}}}}$$

Expansions:

$$E_0 = 2\sqrt{\sqrt{\lambda} J'} \left[ 1 + \frac{1}{\sqrt{\lambda}} \frac{J^2}{8J'} - \frac{1}{(\sqrt{\lambda})^2} \frac{J^4}{128 J'^2} + \ldots \right]$$

$$\mathcal{E}_0 = \mathcal{J} + \frac{2}{\mathcal{J}} \sqrt{1+\mathcal{J}^2}\,\mathcal{J}' - \frac{2(1+2\mathcal{J}^2)}{\mathcal{J}^3(1+\mathcal{J}^2)} \mathcal{J}'^2 + \mathcal{O}(\mathcal{J}'^3)$$

$$E_0 = 2\sqrt{1+\rho^2}\sqrt{\sqrt{\lambda} J'} \left[ 1 + \frac{1}{\sqrt{\lambda}} \frac{\rho^2 J'}{(1+\rho^2)} + \frac{1}{(\sqrt{\lambda})^2} \frac{(4\rho^2 + \rho^4 - 2\rho^6) J'^2}{2(1+\rho^2)^2} + \mathcal{O}((\sqrt{\lambda})^{-3}) \right]$$

1-loop correction to energy: from 2d partition function

- find frequencies of small oscillations about the classical solution
- sum them up (signs for fermions)

$$
\begin{aligned}
B_8(\Omega) &= \Omega^4 + \Omega^3\left(-8 - 4n^2 + 20q - 8\kappa^2\right) \\
&\quad + \Omega^2(16 + 8n^2 + 6n^4 - 80q - 36n^2q + 96q^2 + 32\kappa^2 + 16n^2\kappa^2 - 80q\kappa^2 + 16\kappa^4) \\
&\quad + \Omega(-32n^2 + 8n^4 - 4n^6 + 96n^2q + 12n^4q - 96n^2q^2 - 32n^2\kappa^2 - 8n^4\kappa^2 + 48n^2q\kappa^2) \\
&\quad + 16n^4 - 8n^6 + n^8 - 16n^4q + 4n^6q\ , \\
F_8(\Omega) &= 2\Omega^4 + \Omega^3(-8 - 12\kappa^2 - 8n^2 + 20q) \\
&\quad + \Omega^2(12 + 28\kappa^2 + 18\kappa^4 + 8n^2 + 28\kappa^2n^2 + 12n^4 - 52q - 64\kappa^2q - 36n^2q + 59q^2) \\
&\quad + \Omega(-8 - 20\kappa^2 - 20\kappa^4 - 8\kappa^6 + 8n^2 + 8n^2\kappa^2 - 20\kappa^4n^2 + 8n^4 - 20\kappa^2n^4 \\
&\qquad - 8n^6 + 44q + 80\kappa^2q + 44\kappa^4q - 24n^2q + 32\kappa^2n^2q + 12n^4q - 78q^2 \\
&\qquad - 79\kappa^2q^2 + 2n^2q^2 + 45q^3) \\
&\quad + 2 + 4\kappa^2 + 2\kappa^4 - 8n^2 - 4\kappa^2n^2 - 4\kappa^4n^2 + 12n^4 - 4\kappa^2n^4 + 2\kappa^4n^4 \\
&\quad - 8n^6 + 4\kappa^2n^6 + 2n^8 - 12q - 16\kappa^2q - 4\kappa^4q + 28qn^2 + 16\kappa^2n^2q \\
&\quad + 4\kappa^4n^2q - 20n^4q + 4n^6q + 27q^2 + 21\kappa^2q^2 + 2\kappa^4q^2 - 30n^2q^2 \\
&\quad - 11\kappa^2n^2q^2 + 11n^4q^2 - 27q^3 - 9\kappa^2q^3 + 9n^2q^3 + \tfrac{81}{8}q^4\ .
\end{aligned}
$$

- $\Omega = \omega^2 \qquad q = 2\mathcal{J}'/\sqrt{1+\nu^2}$
- solve perturbatively in $\nu$

1-loop energy in terms of the parameters of the solution:

$$E_1 = \frac{1}{\kappa}\left(f_0(\nu) + f_1(\nu)\,a^2 + f_2(\nu)\,a^4 + \dots\right)$$

$$f_0(\nu) = 0 \qquad f_1(\nu) = 2 - \nu^2 + \mathcal{O}(\nu^4) \qquad f_2(\nu) = -\frac{3}{4} - 6\zeta_3 + \mathcal{O}(\nu^2)$$

- semiclassical expansion

$$E_1 = \left(\frac{2}{\mathcal{J}} - 2\mathcal{J} + 2\mathcal{J}^3 + \mathcal{O}(\mathcal{J}^5)\right)\mathcal{J}_1 + \left(-\frac{4}{\mathcal{J}^3} + \frac{2}{\mathcal{J}}\left(\frac{5}{8} - 3\zeta_3\right) + \mathcal{O}(\mathcal{J})\right)\mathcal{J}_1^2 + \dots$$

- fixed $-\rho$ expansion ( $\rho = \mathcal{J}/(2\sqrt{\mathcal{J}'})$ )

$$E_1 = 2\frac{\sqrt{\sqrt{\lambda}J'}}{\sqrt{1+\rho^2}}\left[\frac{1}{2\sqrt{\lambda}} + \frac{J'}{2\lambda}\left(-\frac{3 - 43\rho^2 - 32\rho^4}{8(1+\rho^2)} - 3\zeta_3\right) + \mathcal{O}(\sqrt{\lambda}^{-3})\right]$$

$$E_1 = \left(\frac{2}{\mathcal{J}} - 2\mathcal{J} + 2\mathcal{J}^3 + \mathcal{O}(\mathcal{J}^5)\right)\mathcal{J}_1 + \left(-\frac{4}{\mathcal{J}^3} + \frac{2}{\mathcal{J}}\left(\frac{5}{8} - 3\zeta_3\right) + \mathcal{O}(\mathcal{J})\right)\mathcal{J}_1^2 + \dots$$

Vs. $(\mathcal{N} = 2\mathcal{J}_1)$

$$E_1 = \frac{\mathcal{N}}{2\mathcal{J}}(n_{11} + \tilde{n}_{11}\mathcal{J}^2 + \bar{n}_{11}\mathcal{J}^4 + \dots)$$
$$+ \frac{\mathcal{N}^2}{2\mathcal{J}^3}\Big( - n_{11} + (n_{12} - \tfrac{1}{2}n_{01}n_{11} - \tilde{n}_{11})\mathcal{J}^2 + (\tilde{n}_{12} - \bar{n}_{11} - \tfrac{1}{2}n_{01}\tilde{n}_{11} - \tfrac{1}{2}\tilde{n}_{01}n_{11})\mathcal{J}^4 + \dots\Big)$$
$$+ \mathcal{O}\left(\frac{\mathcal{N}^3}{\mathcal{J}^5}\right)$$

Read off $n_{ij}$ coeficients:

RR, Tseytlin

Beccaria, Giombi, RR, Tseytlin

$$n_{01} = 1 \quad n_{02} = 0 \quad n_{03} = 0$$

$$n_{11} = 2\ , \quad \tilde{n}_{11} = -2\ , \quad n_{12} = n'_{12} - 3\zeta_3 = -\frac{3}{8} - 3\zeta_3$$

- coefficients and relations as mentioned
- $n_{1j\geq 3}$ contain higher zeta numbers in expected pattern

Konishi dimension: (assume susy restriction for $n_{21}$)

$$E = 2\sqrt[4]{\lambda}\left[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + \frac{b_3}{2(\sqrt{\lambda})^3} + \dots\right]$$

$$b_1 = 2$$
$$b_2 = \tfrac{1}{2} - 3\zeta_3$$
$$b_3 = a_{3,3} + a_{3,2}\zeta_3 + \tfrac{15}{2}\zeta_5$$

$$S = S_1 = S_2 \;\; \text{in} \;\; AdS_5; \quad J \;\; \text{in} \;\; S^5$$

- solution in embedding coordinates: $-Y_5^2 - Y_0^2 + Y_1^2 + \cdots + Y_4^2 = -1$

$$Y_0 + iY_5 = \sqrt{1+2r^2}\; e^{i\kappa\tau} \;,\; Y_1 + iY_2 = r\; e^{i(w\tau+\sigma)} \;,\; Y_3 + iY_4 = r\; e^{i(w\tau-\sigma)} \;,\; X_1 + iX_2 = e^{i\nu\tau}$$

$$\mathcal{E}_0 = (1+2r^2)\kappa = 2\sqrt{\mathcal{S}}\,\Big(1 + \mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} + \ldots\Big)$$

- Slightly more involved technical details

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda}S}\;\Big[1 + \frac{1}{\sqrt{\lambda}}\underline{\Big(S + \frac{J^2}{8S} - \frac{1}{2}\Big)} + \mathcal{O}\big(\frac{1}{(\sqrt{\lambda})^2}\big)\Big] + b_0$$

- level 1 state for $S = 1$
- minimal nontrivial $J = 2 \quad \longrightarrow \quad b_1 = 2$
- unique state in representation $[0,2,0]_{(1,0)}$
- $n_{12} = n'_{12} - 3\zeta_3 = \frac{13}{8} - 3\zeta_3$ ; $n_{21}$ still needed at 2 loops

$S = J_1 \quad \text{with} \quad J_2 \neq 0$

- solution in embedding coordinates:

$$Y_0 + iY_5 = \sqrt{1+r^2}\; e^{i\kappa\tau} \;, \quad Y_1 + iY_2 = r\; e^{i(w\tau+\sigma)}$$

$$X_1 + iX_2 = a\; e^{i(w'\tau-\sigma)} \;, \quad X_3 + iX_4 = \sqrt{1-a^2} e^{i\nu\tau}$$

$$\mathcal{E}_0 = 2\sqrt{\mathcal{S}}\left(1 + \frac{1}{2}\mathcal{S} + \frac{\mathcal{J}_2^2}{8\mathcal{S}} + \ldots\right)$$

- Slightly more involved technical details; $n_{11}$ vanishes

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda}S}\left[1 + \frac{1}{\sqrt{\lambda}}\left(\frac{1}{2}S + \frac{J_2^2}{8S}\right) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)\right] + b_0$$

- level 1 state for $S = 1$
- minimal nontrivial $J_2 = 2 \longrightarrow b_1 = 2$
- state in representation $[1,1,1]_{(1/2,1/2)}$ not unique; choose dim=6
- $n_{12} = n'_{12} - 3\zeta_3 = \frac{5}{8}(?) - 3\zeta_3$; $n_{21}$ still needed at 2 loops

## Summary

- general structure of energy as function of conserved charges

  - based on the structure of vertex operators for general states

  - only some states in the multiplet should be visible semiclasically

- discussed a number of examples of members of Konishi multiplet

$$E = 2\sqrt[4]{\lambda} + b_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + \dots \qquad b_1 = 2 \qquad b_2 = \frac{1}{2} - 3\zeta_3$$

  - direct calculation of $b_1$ and of transcendental part of $b_2$

  - $b_1$ agrees with TBA/Y-system conjecture

    Gromov, Kazakov, Vieira

    Frolov

    Test of finite-size integrability

  - consistent with supersymmetry

  - transcendental part of $b_2$ is universal

  - prediction for transcendental structure of higher coefficients

  - susy: 2-loop contrib. to $b_2$ is rational (surprising) and universal

    Special simplifications in the 2-loop sigma model calculation?

- Most open question are still open

# Extra slides