Hidden beauty of correlation functions in $\mathcal{N} = 4$ SYM

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Outline

- ✔ Properties of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM
- ✔ Structure of the four-point correlation function
- ✔ Hidden permutation symmetry of the integrand
- ✔ Three, four, five, six, . . . loops
- ✔ The Konishi anomalous dimension
- ✔ Conclusions

$\mathcal{N}=4$ SYM stress-tensor multiplet in analytic superspace

- ✔ $\mathcal{N}=4$ SYM stress-tensor multiplet in ordinary superspace
  - ✗ Half-BPS operator made of 6 scalars Φ^I, $I=1,\ldots,6$:

$$\mathcal{O}^{IJ}_{\mathbf{20'}} = \mathrm{tr}(\Phi^I\Phi^J) - 1/6\,\delta^{IJ}\,\mathrm{tr}(\Phi^K\Phi^K)$$

  - ✗ Lowest-weight state of the $\mathcal{N}=4$ stress-tensor supermultiplet:

$$\mathcal{T}(x,\theta^A,\bar\theta_A) = \mathcal{O} + \ldots + (\theta)^4\mathcal{L}_{\mathcal{N}=4} + \ldots + (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)T_{\mu\nu} + \ldots$$

  - ✗ $\mathcal{T}$ is not chiral, but depends on $\theta^A,\bar\theta_A$ $(A=1,2,3,4)$ in a restricted half-BPS way
- ✔ $\mathcal{N}=4$ analytic (harmonic) superspace and half-BPS shortening (Hartwell&Heslop&Howe; Eden&Ferrara&ES):
  - ✗ Break $SU(4)\to SU(2)\times SU(2)'\times U(1)$ with the help of auxiliary harmonic coordinates $y^a_{a'}$,

$$\theta^A_\alpha \;\to\; (\rho^a_\alpha\,,\theta^{a'}_\alpha)\,, \quad \text{with}\;\; \rho^a_\alpha = \theta^a_\alpha + \theta^{a'}_\alpha y^a_{a'}$$

  - ✗ half-BPS = Grassmann analyticity:

$$\mathcal{T} = \mathcal{T}(x^{\dot\alpha\alpha},\,\rho^a_\alpha,\,\bar\rho^{\dot\alpha}_{a'},\,y^a_{a'}) = \mathcal{O}(x,y) + \ldots + (\rho)^4\mathcal{L}_{\mathcal{N}=4}(x) + \ldots + (\rho\sigma^\mu\bar\rho)(\rho\sigma^\nu\bar\rho)T_{\mu\nu}(x) + \ldots$$

$\mathcal{N}=4$ SYM stress-tensor multiplet in analytic superspace II

- Lowest weight state has harmonic dependence

$$\mathcal{O}(x,y) = Y_I\, Y_J\, \mathcal{O}^{IJ}_{\mathbf{20'}}(x) = Y_I\, Y_J\, \mathrm{tr}\left(\Phi^I \Phi^J\right)\,,$$

where $Y^I(y)$, $Y^2 = 0$ are null vectors of $SO(6)$.

- Restrict the odd expansion to the chiral sector

$$\mathcal{T}(x,\rho,\bar\rho = 0, y) = \mathcal{O}(x,y) + \ldots + (\rho)^4 \mathcal{L}_{\mathcal{N}=4}(x)$$

- $\mathcal{N}=4$ SYM action as an integral over $1/4$ superspace (Howe et al):

$$S_{\mathcal{N}=4} = \int d^4x\, \mathcal{L}_{\mathcal{N}=4}(x) = \int d^4x \int d^4\rho\; \mathcal{T}(x,\rho,0,y)$$

  - Supersymmetric due to the special properties of the (on-shell) stress-tensor multiplet

Correlation functions of the $\mathcal{N} = 4$ stress-tensor multiplet

- $n-$point correlation function of analytic supermultiplets $\mathcal{T}(x, \rho, 0, y)$

$$G_n = \langle \mathcal{T}(1) \ldots \mathcal{T}(n) \rangle = \sum_{k=0}^{n-4} \sum_{\ell=0}^{\infty} a^{\ell+k}\, G_{n;k}^{(\ell)}(1, \ldots, n)\,, \qquad a = g^2 N_c/(4\pi^2)$$

The $\ell-$loop correction $G_{n;k}^{(\ell)} \sim (\rho)^{4k}$ is a homogeneous polynomial in the odd variables

- Consider the four-point case $n = 4 \;\Rightarrow\; k = 0$: no ρ dependence in the chiral sector. So, we can replace $\mathcal{T}(x, \rho, 0, y)$ by just the bosonic 1/2-BPS operator $\mathcal{O}(x, y)$:

$$G_4 = \langle \mathcal{O}(x_1, y_1) \ldots \mathcal{O}(x_4, y_4) \rangle = \sum_{\ell=0}^{\infty} a^{\ell}\, G_4^{(\ell)}(1, 2, 3, 4)$$

- Born level (with $x_{ij}^2 = (x_i - x_j)^2$, $y_{ij}^2 = (y_i - y_j)^2$)

$$G_4^{(0)}(1, 2, 3, 4) = \frac{N_c^2 - 1}{(4\pi^2)^4} \left(\frac{y_{12}^2}{x_{12}^2} \frac{y_{23}^2}{x_{23}^2} \frac{y_{34}^2}{x_{34}^2} \frac{y_{41}^2}{x_{41}^2} + \frac{y_{12}^2}{x_{12}^2} \frac{y_{24}^2}{x_{24}^2} \frac{y_{34}^2}{x_{34}^2} \frac{y_{13}^2}{x_{13}^2} + \frac{y_{13}^2}{x_{13}^2} \frac{y_{23}^2}{x_{23}^2} \frac{y_{24}^2}{x_{24}^2} \frac{y_{41}^2}{x_{41}^2} \right) + \text{disconnected}$$

- Duality with super-amplitudes/Wilson loops (Alday&Eden&Korchemsky&Maldacena&ES):

$$\lim_{x_{i,i+1}^2 \to 0} (G_{n;k}/G_n^{(0)}) = [\mathcal{A}_n^{N^k MHV} / \mathcal{A}_n^{MHV \text{ tree}}]^2$$

Correlation functions II

✔ Loop correction via Lagrangian insertions

$$a\frac{d}{da}G_4 = \int d^4x_5 \, \langle \mathcal{O}(x_1,y_1)\ldots\mathcal{O}(x_4,y_4)\mathcal{L}_{\mathcal{N}=4}(x_5)\rangle$$

✗ Repeat ℓ times: the $\ell-$loop 4-point function is given by the Born-level $(4+\ell)-$point function

$$G^{(0)}_{4+\ell;\ell}|_{\rho_1=\ldots=\rho_4=0} = \langle \mathcal{O}(x_1,y_1)\ldots\mathcal{O}(x_4,y_4)\mathcal{L}(x_5)\ldots\mathcal{L}(x_{4+\ell})\rangle^{(0)}(\rho_5)^4\ldots(\rho_{4+\ell})^4 \propto a^\ell$$

This is a particular component of the super-correlator of $4+\ell$ stress-tensor multiplets:

$$\langle \mathcal{T}(\rho_1=0)\ldots\mathcal{T}(\rho_4=0)\mathcal{T}(5)\ldots\mathcal{T}(4+\ell)\rangle$$

✔ Integrand of the 4-point function as a Born-level correlator of stress-tensor multiplets

$$G^{(\ell)}_4(1,2,3,4) = \int d^4x_5\ldots d^4x_{4+\ell}\left(\frac{1}{\ell!}\int d^4\rho_5\ldots d^4\rho_{4+\ell}\,G^{(0)}_{4+\ell;\ell}(1,\ldots,4+\ell)\right)$$

What do we know about this tree-level correlator?

Correlation functions III

- ✔ Examples at one and two loops

$$G^{(0)}_{5;1}(1,2,3,4,5) = \frac{2\,(N_c^2-1)}{(4\pi^2)^5} \times \mathcal{I}_5 \times \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2}$$

$$G^{(0)}_{6;2}(1,2,3,4,5,6) = \frac{2\,(N_c^2-1)}{(4\,\pi^2)^6} \times \mathcal{I}_6 \times \frac{\frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2} x^2_{\sigma_3\sigma_4} x^2_{\sigma_5\sigma_6}}{\prod_{1\le i<j\le 6} x_{ij}^2}$$

- ✔ Essential ingredient: nilpotent n-point superconformal invariant of Grassmann degree $4(n-4)$

$$\mathcal{I}_n|_{\rho_1=\ldots=\rho_4=0} = (x^2_{12}x^2_{13}x^2_{14}x^2_{23}x^2_{24}x^2_{34}) \times R(1,2,3,4) \times (\rho_5)^4 \ldots (\rho_n)^4$$

$$R(1,2,3,4) = \frac{y^2_{12}y^2_{23}y^2_{34}y^2_{14}}{x^2_{12}x^2_{23}x^2_{34}x^2_{14}}(x^2_{13}x^2_{24} - x^2_{12}x^2_{34} - x^2_{14}x^2_{23}) + \text{similar terms}$$

  - ✗ $\mathcal{I}_n$ can be constructed by using the odd part of $PSU(2,2|4)$ to restore $\rho_1,\ldots,\rho_4$.
  - ✗ $\mathcal{I}_n$ has $SU(4)$ and conformal weights matching those of $\mathcal{O}(x,y)$
  - ✗ Crucial property: $\mathcal{I}_n(1,\ldots,n)$ is fully permutation invariant.
- ✔ In summary: the $(4+\ell)-$point tree-level correlator has the general form

$$G^{(0)}_{4+\ell;\ell}(1,\ldots,4+\ell) = \frac{2\,(N_c^2-1)}{(4\,\pi^2)^{4+\ell}} \times \mathcal{I}_{4+\ell} \times f^{(\ell)}(x_1,\ldots,x_{4+\ell})$$

$f^{(\ell)}$ is a permutation invariant function of $x_1,\ldots,x_{4+\ell}$ with conformal weight $(+4)$ at each point.

Hidden permutation symmetry of the integrand

✔ We predict the form of the four-point correlator at ℓ loops:

$$G_4^{(\ell)}(1,2,3,4) = \frac{2\,(N_c^2-1)}{(4\pi^2)^4} \times R(1,2,3,4) \times F^{(\ell)} \qquad \text{for } \ell \geq 1$$

$$F^{(\ell)}(x_1,x_2,x_3,x_4) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{\ell!\,(4\pi^2)^\ell} \int d^4x_5 \ldots d^4x_{4+\ell}\, f^{(\ell)}(x_1,\ldots,x_{4+\ell})$$

$$f^{(\ell)}(x_1,\ldots,x_{4+\ell}) = \frac{P^{(\ell)}(x_1,\ldots,x_{4+\ell})}{\prod_{1\leq i<j\leq 4+\ell} x_{ij}^2}$$

✗ The form of the denominator is dictated by the tree-level OPE of

$$\langle \mathcal{O}(1)\ldots\mathcal{O}(4)\mathcal{L}(5)\ldots\mathcal{L}(4+\ell)\rangle^{(0)} \sim R(1,2,3,4)\,(x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2)\, f^{(\ell)}(x)$$

✗ The numerator $P^{(\ell)}$ is a homogeneous polynomial in x_{ij}^2 of conformal weight $-(\ell-1)$ at each point, invariant under $S_{4+\ell}$ permutations of x_i.

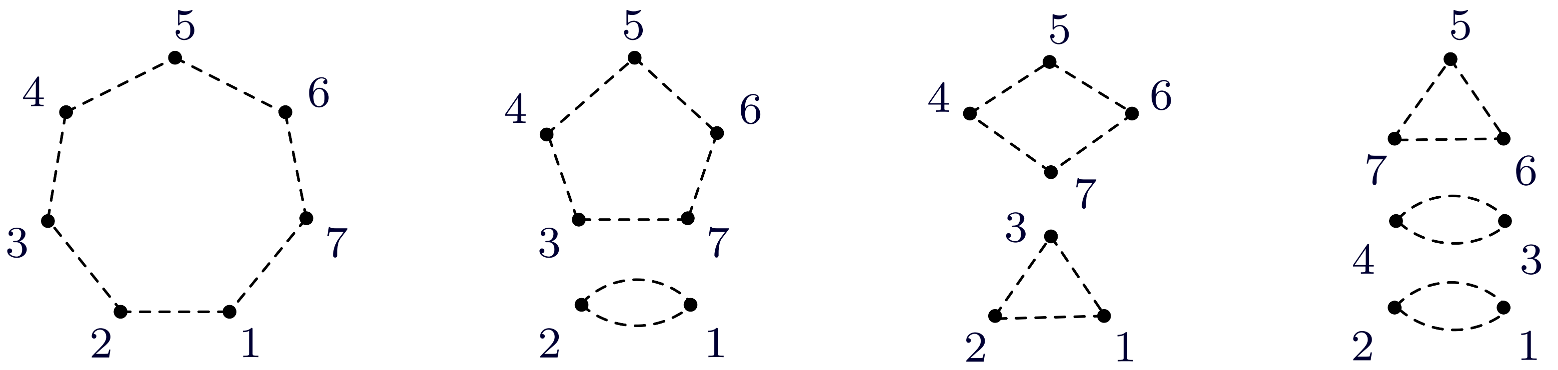
✗ Examples at 1 and 2 loops:

$$P^{(1)}(x_1,\ldots,x_5) = 1\,, \qquad P^{(2)}(x_1,\ldots,x_6) = \frac{1}{48}\sum_{\sigma\in S_6} x_{\sigma(1)\sigma(2)}^2 x_{\sigma(3)\sigma(4)}^2 x_{\sigma(5)\sigma(6)}^2$$

✔ Loop corrections in all $SU(4)$ channels given by single function $F^{(\ell)}$: partial non-renormalization (Eden&Petkou&ES)

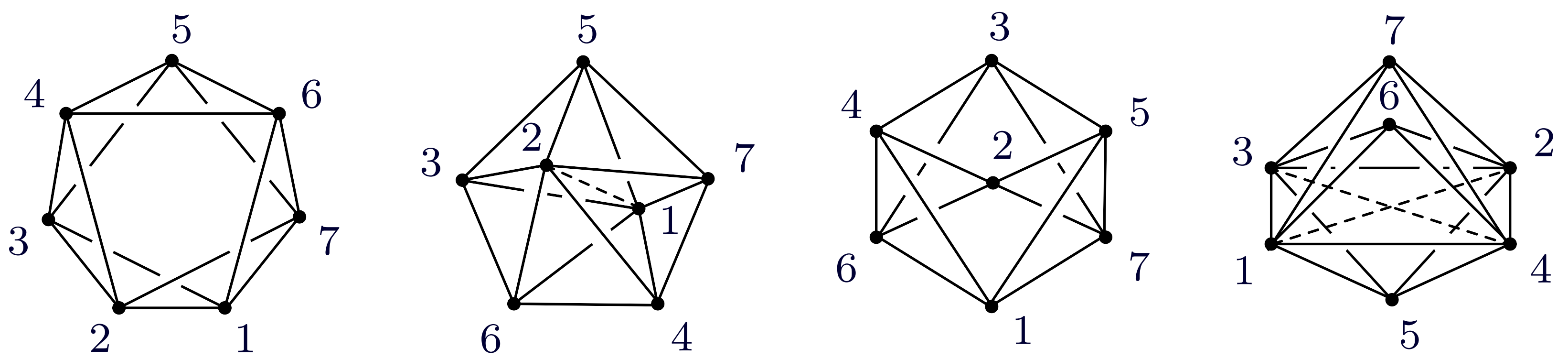
The 3-loop 4-point correlator has so far resisted all attempts to be calculated from Feynman graphs. Here we show how to do it by just drawing pictures!

- ✔ A graph-theoretical problem: How to construct permutation invariant numerators?
- ✔ P graphs at 3 loops:



(a) (b) (c) (d)

- ✔ f graphs at 3 loops:



(a) (b) (c) (d)

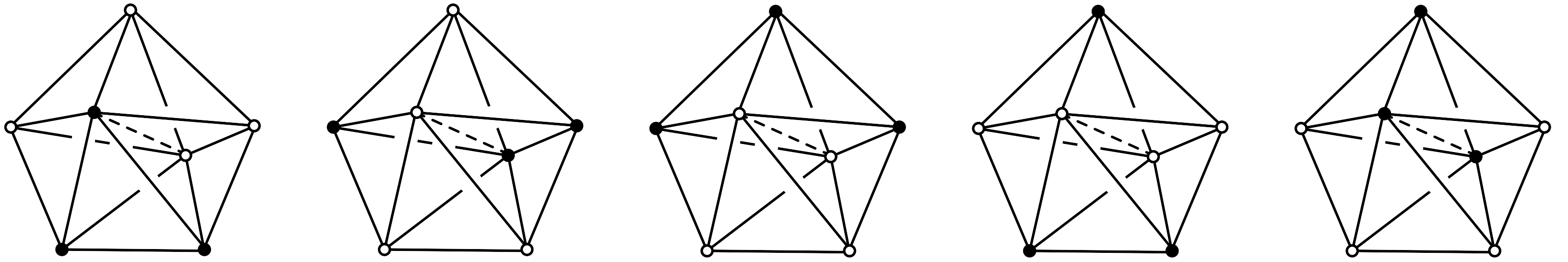
Three-loop correlator II

- ✔ We found 4 permutation-symmetric classes of graphs, but only graph (b) is planar, in the sense of the tree-level correlator

$$G^{(0)}_{4+\ell;\ell}(1,\ldots,4+\ell) \sim \mathcal{I}_{4+\ell} \times f^{(\ell)}(x_1,\ldots,x_{4+\ell})$$

It is also planar in the sense of the 4-gluon amplitude, after restricting to the light cone

- ✔ Choose 4 external and 3 internal (integration) points:



(a) (b) (c) (d) (e)

- ✔ Finally, add the prefactor in

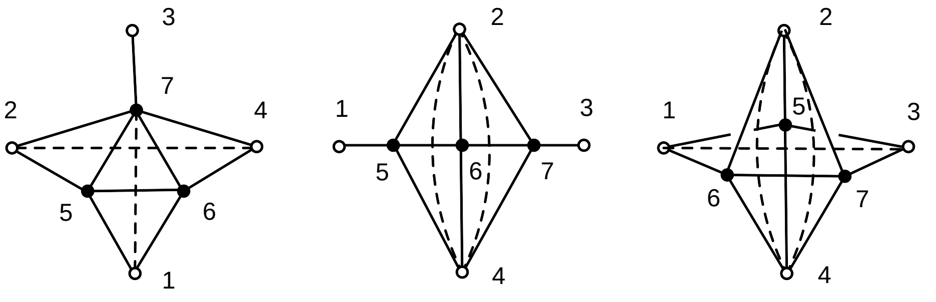
$$[F^{(3)}]_{\text{integrand}} = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{3!\,(4\pi^2)^3} \times f^{(3)}(x_1,\ldots,x_7)$$

These steps break the permutation symmetry of the integrand.

Three-loop correlator III

We find two types of 4-point 3-loop integrals:

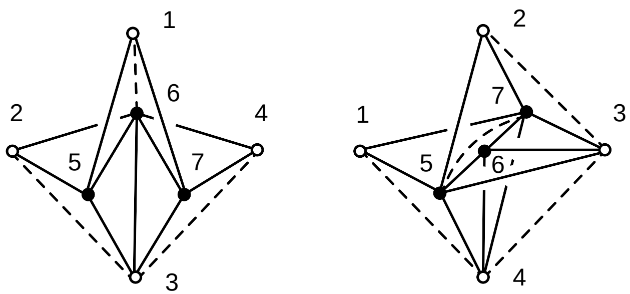
- ✔ Those which survive in the light-cone (or on-shell) limit $x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 = 0$:



$T(1,3;2,4)$ $L(1,3;2,4)$ $g \times h(1,3;2,4)$

T and L are dual to the “tennis court" and “ladder" diagrams in the on-shell 4-gluon amplitude

- ✔ Those which vanish in the light-cone limit (thus not seen in the 4-gluon amplitude):



$E(1;2,4;3)$ $H(1,2;3,4)$

These integrals are new. They are conformal, hence depend on two cross-ratio variables. Who can tell what the functions (symbols) look like? Are they maximally transcendental?

Fixing the coefficients

- ✔ We found the general form of the $\ell-$loop integrand

$$f^{(\ell)}(x_i) = \sum_{\alpha=1}^{n_r} c_\alpha \frac{P_\alpha^{(\ell)}(x_i)}{\prod_{1\le i<j\le 4+\ell} x_{ij}^2}$$

where the sum goes over all permutation invariant topologies. How to fix the coefficients c_α?

- ✔ Softening of the singularity of $\ln G_4$ in the light-cone limit $x_{i,i+1}^2 \to 0$ $(u, v \to 0)$:

$$\ln G_4 \sim \ln\Big(1 + 2\sum_{\ell\ge 1} a^\ell F^{(\ell)}(x_i)\Big) = \Big(-\frac{1}{4}a + \frac{1}{8}a^2\zeta_2\Big)\ln u \ln v + \sum_{\ell \ge 2} b^{(\ell)}\left[(a\ln u)^\ell + (a \ln v)^\ell\right] + \dots$$

- ✔ Example at 2 loops:

$$\ln G_4 \;\to\; a^2\left(\mathcal{F}^{(2)} - (\mathcal{F}^{(1)})^2\right) \;\to\; a^2 \int d^4x_5 d^4x_6 \frac{x_{13}^2 x_{24}^2}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2x_{16}^2x_{26}^2x_{36}^2x_{46}^2x_{56}^2}$$
$$\times\left[(c^{(2)}-2)x_{13}^2x_{24}^2x_{56}^2 + c^{(2)}x_{13}^2(x_{25}^2x_{46}^2 + x_{45}^2x_{26}^2) + c^{(2)}x_{24}^2(x_{15}^2x_{36}^2 + x_{35}^2x_{16}^2)\right]$$

Divergences come from integration over x_5 (or x_6) approaching a light-like edge, e.g., $[x_1, x_2]$:

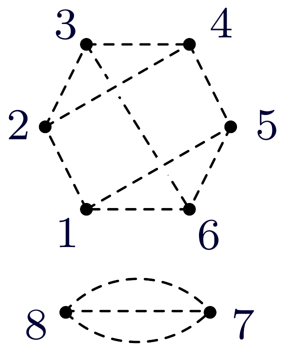
$$x_5^\mu \to (1-\alpha)x_1^\mu + \alpha x_2^\mu \qquad \Rightarrow \qquad x_{i5}^2 \to (1-\alpha)x_{1i}^2 + \alpha x_{2i}^2$$

- ✔ Requiring that the numerator vanish in this limit fixes $c^{(2)} = 1$
  - ✗ This criterion fixes all coefficients c_α in the planar sector
  - ✗ Checked to 6 loops, see also Spradlin et al up to 7 loops for the amplitude

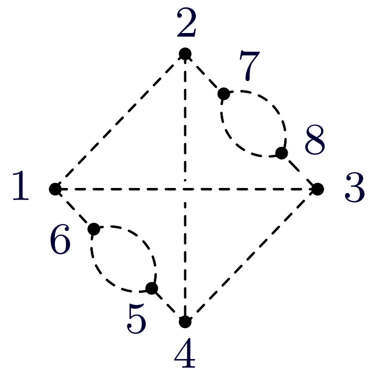
Four-loop correlator (planar)

We can play the same game at higher loops. At four loops we find

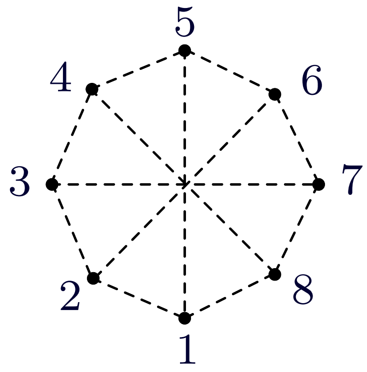
- ✔ 3 planar numerator topologies $P^{(4)}$



P_A

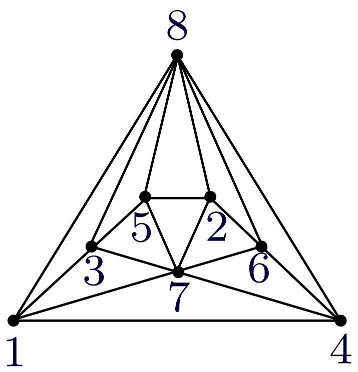


P_B

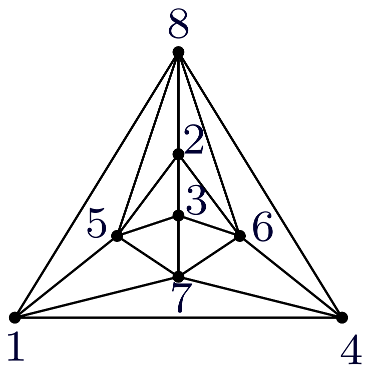


P_C

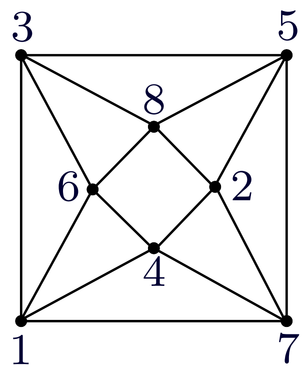
- ✔ and the corresponding permutation invariant integrands $f^{(4)}$



f_A



f_B

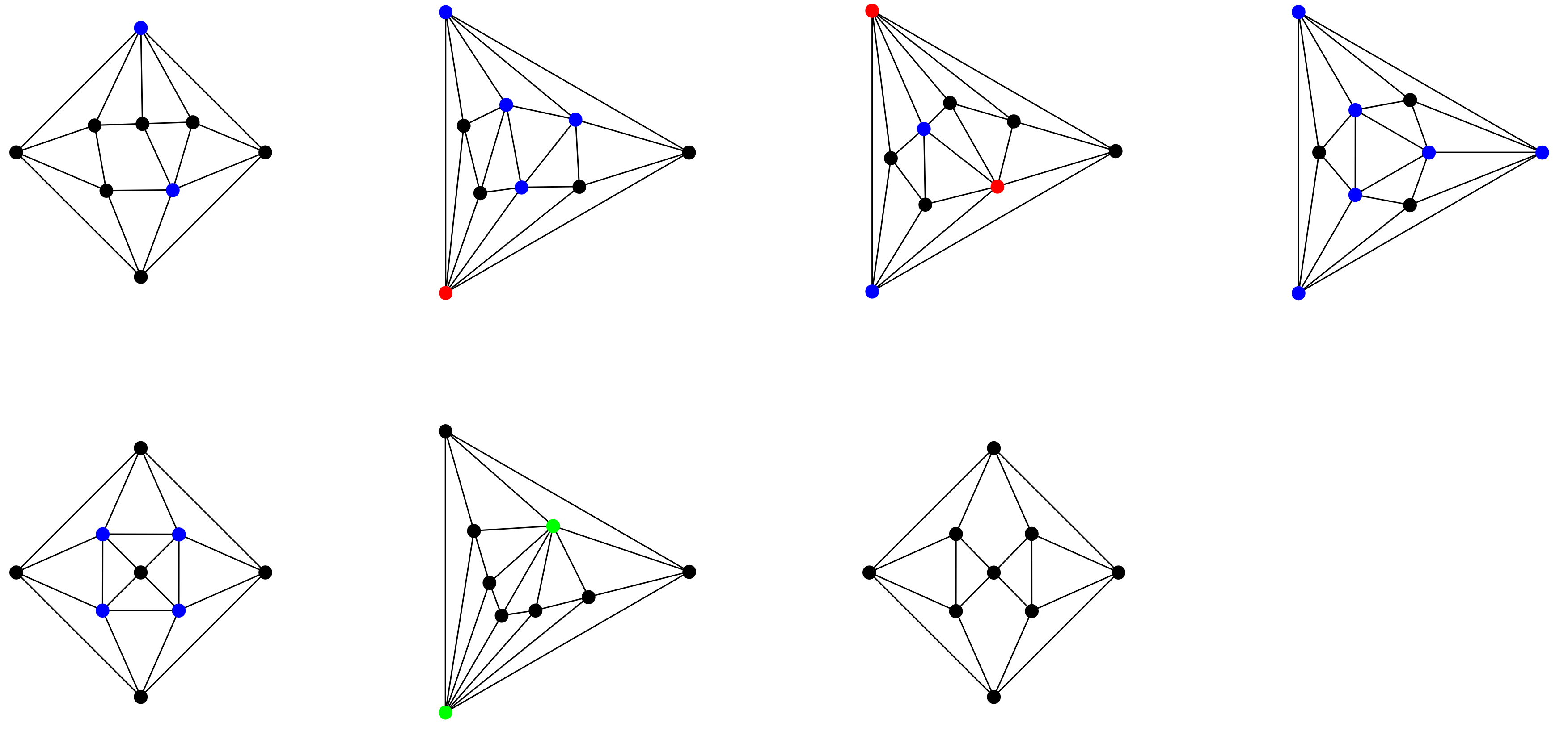


f_C

- ✔ The light-cone limit fixes $c_A = c_B = -c_C = 1$, exactly as in the amplitude.

Five loops (planar)

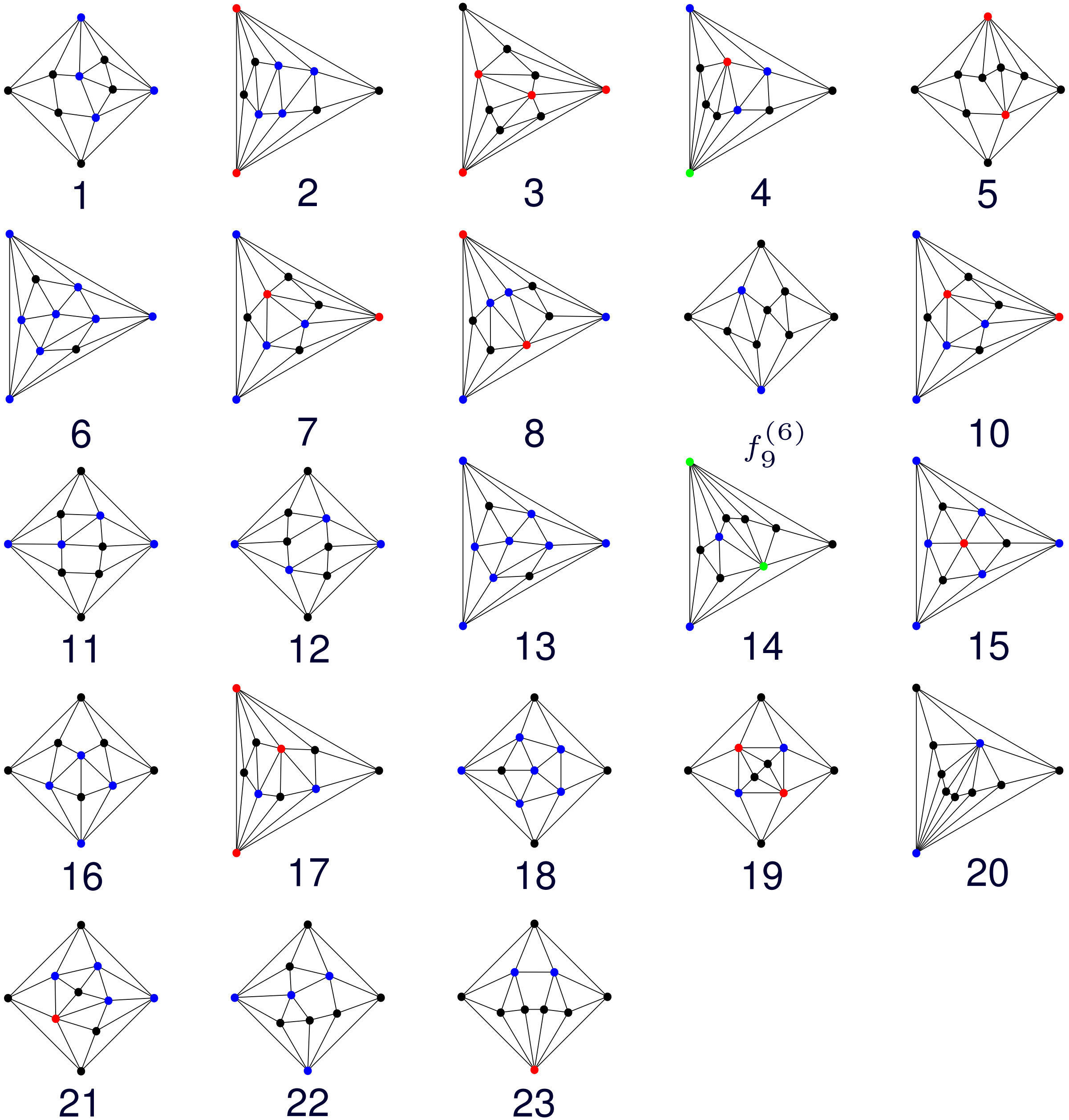
- We find only 7 planar $f-$graphs:



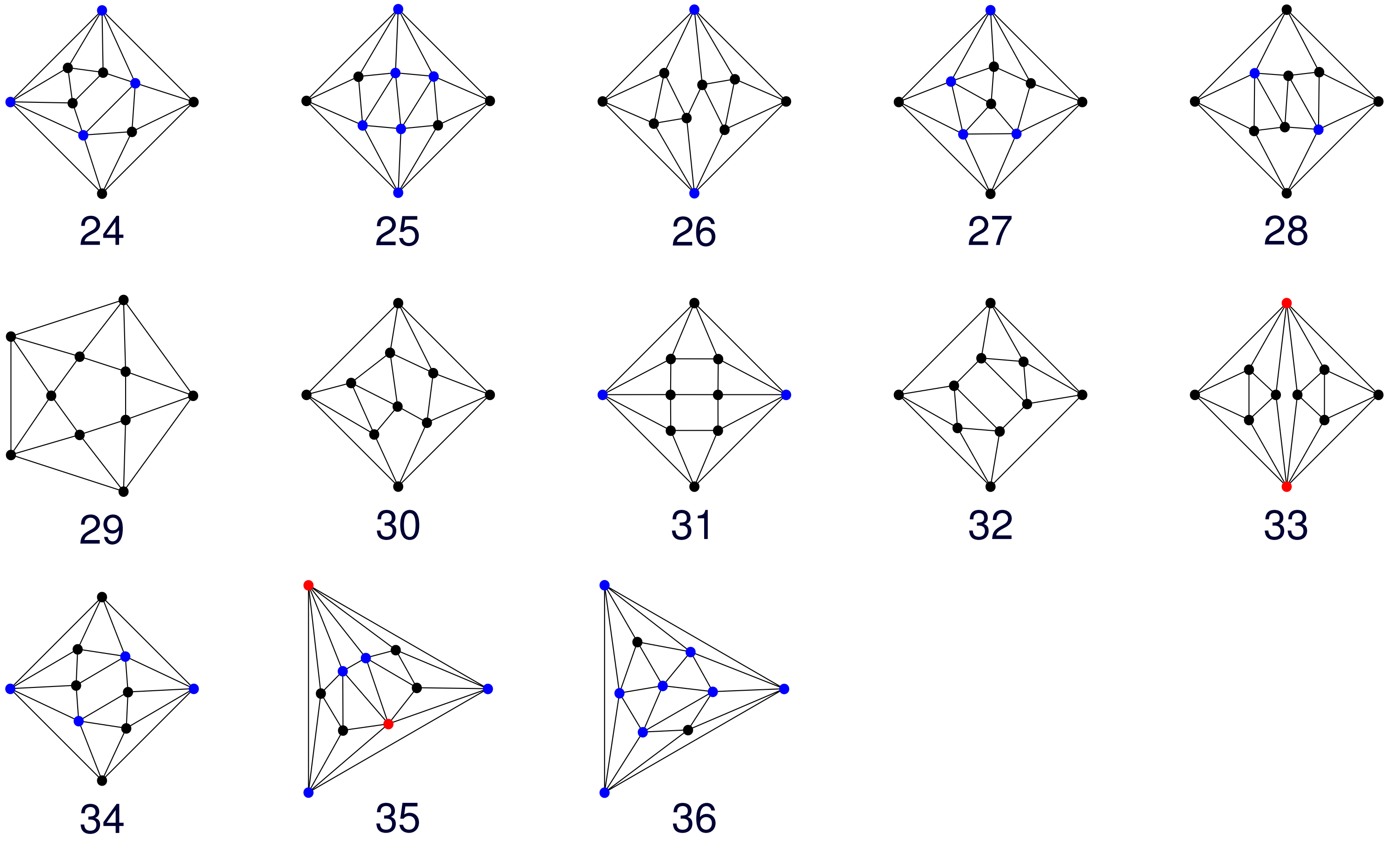
- All coefficients are fixed by the log singularity criterion.

Six loops (planar)

✔ 23 rung-rule six-loop $f-$graphs



- 13 potential non-rung-rule six-loop $f-$graphs



- In fact, only $f_{28}^{(6)}$, $f_{29}^{(6)}$ and $f_{31}^{(6)}$ contribute.
- All coefficients are fixed by the log singularity criterion.

Konishi anomalous dimension

- OPE of half-BPS operators

$$\mathcal{O}(x_1,y_1)\mathcal{O}(x_2,y_2) = c_{\mathbb{I}}\frac{y_{12}^4}{x_{12}^4}\mathbb{I} + c_{\mathcal{K}}(a)\frac{y_{12}^4}{(x_{12}^2)^{1-\gamma_{\mathcal{K}}/2}}\mathcal{K}(x_2) + c_{\mathcal{O}}\frac{y_{12}^2}{x_{12}^2}\mathcal{O}_{\mathbf{20'}}(x_2,y_2) + (\mathbf{84}+\mathbf{105}+\mathbf{175})$$

  with the unprotected Konishi operator $\mathcal{K} = \mathrm{tr}\left(\Phi^I\Phi^I\right)$.

- $\mathcal{K}$ has the minimal scaling dimension among the unprotected operators, so it dominates the double short-distance expansion of the log of the correlator:

$$\ln\left(1 + 6x_{13}^2 x_{34}^2 \sum_{\ell\geq 1} a^\ell F^{(\ell)}(x_i)\right) \overset{u\to 0}{\underset{v\to 1}{\longrightarrow}} \frac{1}{2}\gamma_{\mathcal{K}}(a)\ln u + \ln\left(c_{\mathcal{K}}^2(a)\right) + O(u) + O((1-v))$$

- The values of $\gamma_{\mathcal{K}}^{(1)}$ and $\gamma_{\mathcal{K}}^{(2)}$ were extracted from the explicit form of $F^{(1)}$ and $F^{(2)}$ (Eden&Schubert&ES; Bianchi et al; Dolan&Osborn)

- We propose a new method which bypasses the evaluation of the higher-loop 4-point integrals in $F^{(\ell)}$. Instead, we need to compute only standard two-point propagator type integrals.

Konishi anomalous dimension II

- ✔ Idea of the method:
  - ✗ At one loop, in the double coincidence Euclidean limit $x_{12}^2 = x_{34}^2 = \delta \to 0$ we have

$$\hat{F}^{(1)} = \lim_{x_{12},x_{34}\to 0} x_{13}^4 F^{(1)} = -\frac{1}{4\pi^2} \lim_{x_{12},x_{34}\to 0} \int \frac{x_{13}^4 \, d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \frac{1}{4} \ln \delta - \frac{1}{2} + \ldots$$

  - ✗ Different regulator: identify the points in the 4-point integral and regularize dimensionally:

$$\hat{F}_{\epsilon}^{(1)} = -\frac{\mu^{2\epsilon}}{4\pi^2} \int \frac{x_{13}^4 \, d^{4-2\epsilon} x_5}{x_{15}^4 x_{35}^4} = (x_{13}^2/\mu^2)^{-\epsilon} \left(\frac{1}{2\epsilon} + \frac{1}{2} + O(\epsilon^2) \right)$$

  - ✗ Both singular limits give the same value for

$$\gamma_{\mathcal{K}}^{(1)} = 12 \frac{d}{d \ln \delta} \hat{F}^{(1)} = 6 \frac{d}{d \ln \mu^2} \hat{F}_{\epsilon}^{(1)} = 3$$

- ✔ At higher loops the log of the correlator always has a simple pole, e.g., at two loops

$$\ln G_4 \sim \hat{F}_{\epsilon}^{(2)} - 3 \, (\hat{F}_{\epsilon}^{(1)})^2 = (x_{13}^2/\mu^2)^{-2\epsilon} \Big(-\frac{1}{4\epsilon} - \frac{3}{4} + O(\epsilon) \Big)$$

- ✔ Two-point integrals of propagator type can be computed by standard methods up to five loops ⇒ Full agreement with integrability (Bajnik&Janik, Arutyunov&Frolov, Gromov&Kazakov&Vieira).

Conclusions

- ✔ Using only known basic properties of the four-point correlator of $\mathcal{N} = 4$ stress-tensor multiplets, we unveiled a hidden, highly symmetric structure.
- ✔ This structure allows to find the off-shell integrand of G_4 at any loop level.
- ✔ Two ingredients were essential for this:
  - ✗ $\mathcal{N} = 4$ SUSY. It is known that the 2-loop correlator in a generic $\mathcal{N} = 2$ conformal theory does not posses the permutation symmetry of the integrand.
  - ✗ The number of point is 4. For $n > 4$ the nilpotent superconformal invariant $\mathcal{I}_n$ is not unique, so we have to find many functions $F^{(\ell)}$ and the full permutation symmetry is lost. Still, we might be able to make some limited predictions in this case.
- ✔ The recently discovered triality (Alday&Eden&Korchemsky&Maldacena&ES)

$$\lim_{x_{i,i+1}^2 \to 0} \ln G_n = 2 \ln \mathcal{A}_n = 2 \ln W_n$$

  between correlators in the singular light-cone limit, on-shell scattering amplitudes and light-like Wilson loops allows us to predict the integrand of the four-gluon amplitude $\mathcal{A}_4$. The results are the same as in the momentum twistor approach (Arkani-Hamed et al). It would be interesting to understand the intimate connection between the two, seemingly very different constructions.
- ✔ The highly predictable structure of G_4 is undoubtedly related to the integrability of $\mathcal{N} = 4$ SYM. In particular, the 4-point integrals that we find should have some hidden structure, at the level of their symbols, for example.