Hybrid integrable structure of squashed sigma models

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Based on the collaboration with

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Refs: JHEP1206 (2012) 082 [arXiv:1203.3400],
JHEP1204 (2012) 115 [arXiv:1201.3058].

0. Introduction

- Motivation & Background -

One of the most important progress in string theory

Integrability in AdS/CFT (see Beisert et. al. 1012.3982)

Next Step: Integrable deformations of AdS/CFT

[Basso's talk, Torrielli's talk, Hoare's talk, van Tongeren's talk]

NOTE AdS/CFT belongs to the rational class (like XXX-model)

➡ Our motive an XXZ-like deformation of AdS/CFT

As an example, we will concentrate on squashed S^3.

What is squashed S^3 ?

Round S^3 with the radius *L*

$$ds^2 = \frac{L^2}{4}[\underbrace{d\theta^2 + \cos^2\theta d\phi^2}_{S^2} + \underbrace{(d\psi + \sin\theta d\phi)^2}_{S^1\text{-fibration}}]$$

3 angles (θ, ϕ, ψ)

Isometry: $SU(2)_L \times SU(2)_R$

a deformation of the round S^3

C=0

Squashed S^3

$$ds^2 = \frac{L^2}{4}[d\theta^2 + \cos^2\theta d\phi^2 + (1 + C)(d\psi + \sin\theta d\phi)^2]$$

squashing parameter (C)

Isometry: $SU(2)_L \times U(1)_R$

Group element representation of squashed S^3

Let us introduce the *SU(2)* group element: $g = \mathrm{e}^{\phi T_1}\mathrm{e}^{\theta T_2}\mathrm{e}^{\psi T_3} \in SU(2)$

Here θ, ϕ, ψ are the angles of S^3 and T_A's are the SU(2) generators:

$$[T_A, T_B] = \varepsilon_{AB}{}^{C} T_C \ , \qquad \mathrm{Tr}(T_A T_B) = -\frac{1}{2}\delta_{AB}$$

Then the left-invariant 1-form is expanded as

$$J = g^{-1}dg = J^1 T_1 + J^2 T_2 + J^3 T_3$$

Finally the metric of squashed S^3 is rewritten as

$$\begin{aligned} ds^2 &= \frac{L^2}{4}[(J^1)^2 + (J^2)^2 + \underline{(1+C)}(J^3)^2] \\ &= -\frac{L^2}{2}\left[\mathrm{Tr}[(J)^2] - 2C(\mathrm{Tr}(JT_3))^2\right] \end{aligned}$$

(XXZ-like deformation)

Sigma model action on squashed S^3 (squashed sigma model)

$$S = \int dt dx \, [\mathrm{Tr}(J_\mu J^\mu) - 2C \mathrm{Tr}(T_3 J_\mu) \mathrm{Tr}(T_3 J^\mu)]$$

$x^\mu = (t, x), \quad \eta_{\mu\nu} = (-1, 1)$: 2D Minkowski spacetime

Benjamin's talk!

Boundary condition: $g(t,x) \to g_\infty : \ \text{const.} \quad (x \to \pm\infty)$

That is, $J_\mu(x)$ vanishes rapidly as $x \to \pm\infty$.

Global symmetry: $SU(2)_\mathrm{L} \times U(1)_\mathrm{R}$

$$\delta^{L,a} g = \epsilon_L T^a g \,, \qquad \delta^{R,3} g = -\epsilon_R \, g T^3 \,.$$

Classical EOM: $\partial^\mu J_\mu - 2C \mathrm{Tr}\left(T^3 J_\mu\right) \left[J^\mu, T^3\right] = 0$

We discuss the classical integrable structure of squashed sigma model.

For quantum integrability, see [Wiegmann, Balog-Forgacs-Palla, Basso-Rej]

Our claim

There are two descriptions to describe the classical dynamics based on the global symmetry

1) Trigonometric description based on $U(1)_R$

2) Rational description based on $SU(2)_L$

Lax pair and monodromy matrix can be constructed for each of them.

→
1) Two kinds of Lax pairs lead to the identical EOM.
2) The monodromy matrices are gauge-equivalent.

Hybrid integrable structure !

Plan of the talk

1. Trigonometric description
2. Rational description
3. Equivalence of two descriptions
4. Summary & Discussions

1. Trigonometric description

I. Kawaguchi and K.Y., PLB705 (2011) 251 [arXiv: 1107.3662].

I. Kawaguchi, T. Matsumoto and K.Y., JHEP1204 (2012) 115 [arXiv:1201.3058].

c.f. Cherednik, Theor. Math. Phys. 47 (1981) 422,
Faddeev-Reshetikhin, Ann. Phys. 167 (1986) 227.

Trigonometric Lax pair

[Cherednik, 1981] [Faddeev and Reshetikhin,1986]

$$L_t^R(x;\lambda_R) = -\sum_{a=1}^{3} \left[w_a(\lambda_R+\alpha)\,(J_t^a + J_x^a) - w_a(\lambda_R-\alpha)\,(J_t^a - J_x^a) \right] T^a$$

$$L_x^R(x;\lambda_R) = -\sum_{a=1}^{3} \left[w_a(\lambda_R+\alpha)\,(J_t^a + J_x^a) + w_a(\lambda_R-\alpha)\,(J_t^a - J_x^a) \right] T^a$$

$$w_1(\lambda_R) = w_2(\lambda_R) = \frac{\sinh\alpha}{\sinh\lambda_R}\,,\quad w_3(\lambda_R) = \frac{\tanh\alpha}{\tanh\lambda_R} \qquad C = -\tanh^2\alpha$$

reflect $U(1)_R$ symmetry.

Monodromy matrix

$$U^R(\lambda_R) = \mathrm{P}\exp\left[\int_{-\infty}^{\infty} dx\, L_x^R(x;\lambda_R)\right] \quad \Rightarrow \quad \frac{d}{dt}U^R(\lambda) = 0$$

conserved

NOTE The classical *r*-matrix is of trigonometric type.

Enhancement of $U(1)_R$

[$SU(2)_R$ is broken to $U(1)_R$ due to the squashing]

$U(1)_R$ current: $j_\mu^{R,3} = -2(1+C)\mathrm{Tr}\left(T^3 J_\mu\right)$ (Noether current)

The broken components of $SU(2)_R$ are realized as non-local conserved currents.

[I. Kawaguchi and K.Y., 1107.3662]

$$j_\mu^{R,\pm} = -2\mathrm{e}^{\gamma\chi}\left(\eta_{\mu\nu} \pm i\sqrt{C}\epsilon_{\mu\nu}\right)\mathrm{Tr}\left(T^\pm J^\nu\right) \qquad \left[T^\pm \equiv \frac{1}{\sqrt{2}}(T^1 + iT^2)\right]$$

$$\chi(x) \equiv \frac{1}{2}\int_{-\infty}^{\infty} dy\, \epsilon(x-y) j_t^{R,3}(y) \qquad \gamma \equiv \frac{\sqrt{C}}{1+C}$$

non-local

$$\epsilon(x-y) \equiv \theta(x-y) - \theta(y-x)$$

Conserved charges

$$Q^{R,3} = \int dx\, j_t^{R,3}(x), \qquad Q^{R,\pm} = \left(\frac{\gamma}{\sinh\gamma}\right)^{1/2}\int dx\, j_t^{R,\pm}(x)$$

Current algebra

$$\left\{ j_t^{R,\pm}(x),\, j_t^{R,\mp}(y) \right\}_{\mathrm{P}} = \pm i\, \mathrm{e}^{2\gamma\,\chi(x)}\, j_t^{R,3}(x)\delta(x-y)$$

$$= \boxed{\pm \frac{i}{2\gamma}\partial_x \left[\mathrm{e}^{2\gamma\,\chi(x)}\right] \delta(x-y)\,,}$$

$$\left\{ j_t^{R,\pm}(x),\, j_t^{R,\pm}(y) \right\}_{\mathrm{P}} = \pm i\,\gamma\,\epsilon(x-y)\, j_t^{R,\pm}(x) j_t^{R,\pm}(y)\,,$$

$$\left\{ j_t^{R,3}(x),\, j_t^{R,\pm}(y) \right\}_{\mathrm{P}} = \pm i\, j_t^{R,\pm}(x)\delta(x-y)\,.$$

q-deformed *SU(2)*$_{\mathrm{R}}$ algebra

$$\{Q^{R,+}, Q^{R,-}\}_{\mathrm{P}} = i\frac{q^{Q^{R,3}} - q^{-Q^{R,3}}}{q-q^{-1}}\,, \quad \{Q^{R,3}, Q^{R,\pm}\}_{\mathrm{P}} = \pm i Q^{R,\pm}$$

A deformation parameter : $q = \mathrm{e}^{\gamma} = \exp\left(\frac{\sqrt{C}}{1+C}\right)$

The classical analog of quantum affine algebra

There are the other non-local currents [I.Kawaguchi-T.Matsumoto-K.Y., 1201.3058]

$$\tilde{j}_\mu^{R,\pm} \equiv -2\mathrm{e}^{-\gamma\chi} \left(\eta_{\mu\nu} \pm i\sqrt{C}\,\epsilon_{\mu\nu}\right) \mathrm{Tr}\left(T^\mp J^\nu\right)$$

$$\widetilde{Q}^{R,3} \equiv -Q^{R,3}, \quad \widetilde{Q}^{R,\pm} \equiv \left(\frac{\gamma}{\sinh\gamma}\right)^{1/2} \int dx\; \tilde{j}_t^{R,\pm}(x)$$

c.f., the previous ones

slightly different!

$$j_\mu^{R,\pm} = -2\underline{\mathrm{e}^{\gamma\chi}} \left(\eta_{\mu\nu} \pm i\sqrt{C}\,\epsilon_{\mu\nu}\right) \mathrm{Tr}\left(\underline{T^\pm} J^\nu\right)$$

$$Q^{R,3} \equiv \int dx\; j_t^{R,3}(x), \quad Q^{R,\pm} \equiv \left(\frac{\gamma}{\sinh\gamma}\right)^{1/2} \int dx\; j_t^{R,\pm}(x)$$

The defining relations of quantum affine algebra: $U_q(\widehat{sl(2)_R})$

$$\{Q^{R,\pm}, Q^{R,\mp}\}_{\mathrm{P}} = \pm i \frac{q^{Q^{R,3}} - q^{-Q^{R,3}}}{q - q^{-1}},$$

$$\left\{\widetilde{Q}^{R,\pm}, \widetilde{Q}^{R,\mp}\right\}_{\mathrm{P}} = \pm i \frac{q^{\widetilde{Q}^{R,3}} - q^{-\widetilde{Q}^{R,3}}}{q - q^{-1}},$$

$$\{Q^{R,\pm}, Q^{R,3}\}_{\mathrm{P}} = \mp i Q^{R,\pm}, \qquad \left\{Q^{R,\pm}, \widetilde{Q}^{R,3}\right\}_{\mathrm{P}} = \pm i Q^{R,\pm},$$

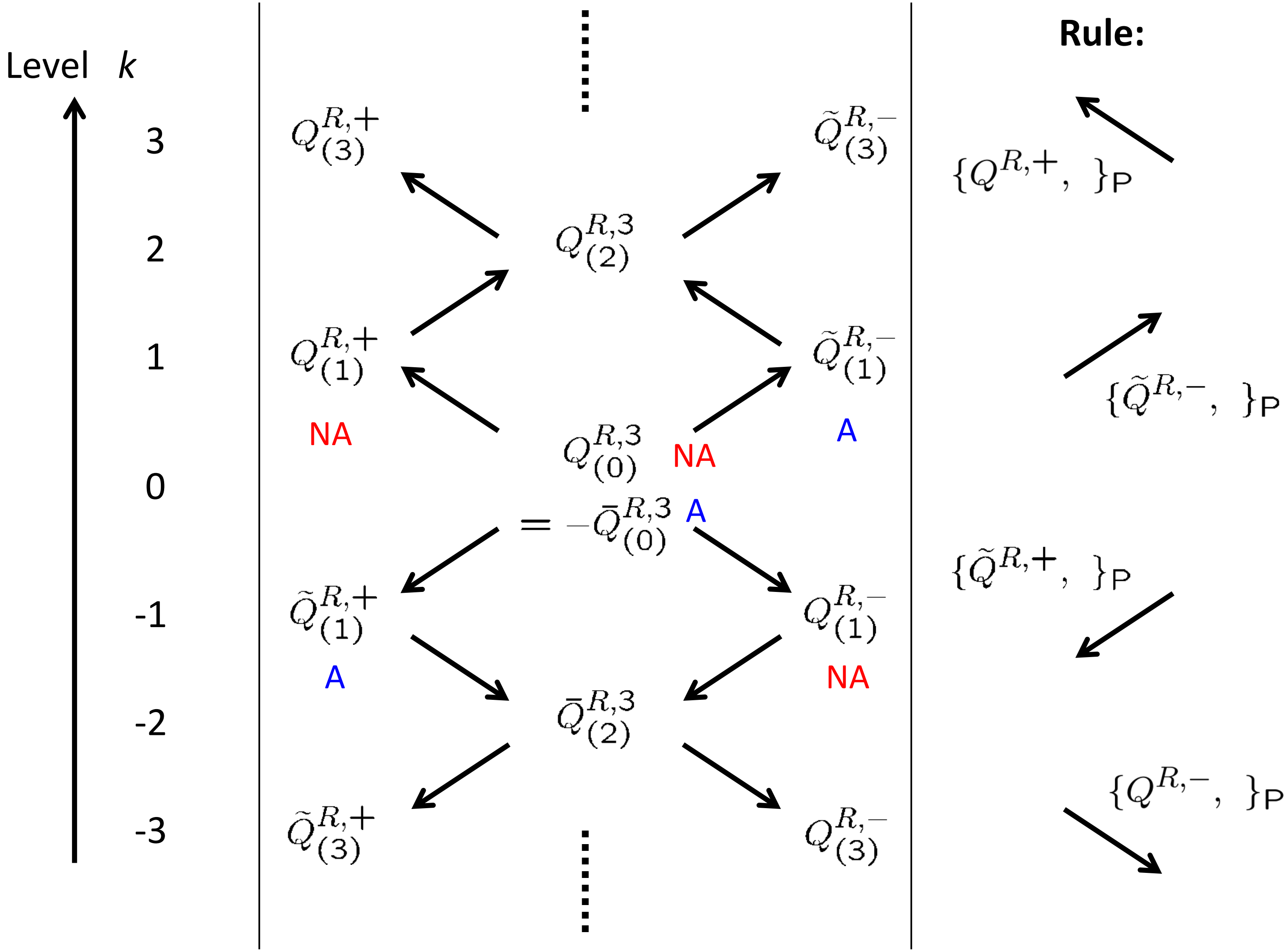
$$\left\{\widetilde{Q}^{R,\pm}, \widetilde{Q}^{R,3}\right\}_{\mathrm{P}} = \mp i \widetilde{Q}^{R,\pm}, \qquad \left\{\widetilde{Q}^{R,\pm}, Q^{R,3}\right\}_{\mathrm{P}} = \pm i \widetilde{Q}^{R,\pm},$$

The classical analog of Serre relations:

$$\left\{Q^{R,\pm}, \left\{Q^{R,\pm}, \left\{Q^{R,\pm}, \widetilde{Q}^{R,\pm}\right\}_{\mathrm{P}}\right\}_{\mathrm{P}}\right\}_{\mathrm{P}} = -\gamma^2 \left\{Q^{R,\pm}, \widetilde{Q}^{R,\pm}\right\}_{\mathrm{P}} (Q^{R,\pm})^2,$$

$$\left\{\widetilde{Q}^{R,\pm}, \left\{\widetilde{Q}^{R,\pm}, \left\{\widetilde{Q}^{R,\pm}, Q^{R,\pm}\right\}_{\mathrm{P}}\right\}_{\mathrm{P}}\right\}_{\mathrm{P}} = -\gamma^2 \left\{\widetilde{Q}^{R,\pm}, Q^{R,\pm}\right\}_{\mathrm{P}} (\widetilde{Q}^{R,\pm})^2.$$

Infinite tower of conserved charges



where $\quad Q^{R,3}_{(0)} = Q^{R,3}, \quad Q^{R,\pm}_{(1)} \equiv Q^{R,\pm}, \quad \widetilde{Q}^{R,\pm}_{(1)} \equiv \widetilde{Q}^{R,\pm}$

Concrete expressions of conserved charges

$$Q_{(0)}^{R,3} = \int_{-\infty}^{\infty} dx \; j_t^{R,3}(x) = -\bar{Q}_{(0)}^{R,3} \,,$$

$$Q_{(1)}^{R,+} = \int_{-\infty}^{\infty} dx \; j_t^{R,+}(x) \,, \qquad \widetilde{Q}_{(1)}^{R,-} = \int_{-\infty}^{\infty} dx \; \widetilde{j}_t^{R,-}(x) \,,$$

$$Q_{(2)}^{R,3} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \epsilon(x-y) j_t^{R,+}(x) \widetilde{j}_t^{R,-}(y) \; -2i \int_{-\infty}^{\infty} dx \, j_x^{R,3}(x) - \frac{1-C}{\sqrt{C}} Q_{(0)}^{R,3} \,,$$

$$\vdots$$

$$Q_{(1)}^{R,-} = \int_{-\infty}^{\infty} dx \, j_t^{R,-}(x) \,, \qquad \widetilde{Q}_{(1)}^{R,+} = \int_{-\infty}^{\infty} dx \; \widetilde{j}_t^{R,+}(x) \,,$$

$$\bar{Q}_{(2)}^{R,3} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \epsilon(x-y) j_t^{R,-}(x) \widetilde{j}_t^{R,+}(y) + 2i \int_{-\infty}^{\infty} dx j_x^{R,3}(x) + \frac{1-C}{\sqrt{C}} \bar{Q}_{(0)}^{R,3} \,,$$

$$\vdots$$

Another derivation: expanding the monodromy matrix

Expansion in $|z_R| > 1$ → $Q_{(0)}^{R,3},\ Q_{(1)}^{R,+},\ \widetilde{Q}_{(1)}^{R,-},\ Q_{(2)}^{R,3},\cdots$

$z_R \equiv \mathrm{e}^{-\lambda_R}$

(the upper half of the tower)

Expansion in $|z_R| < 1$ → $\bar{Q}_{(0)}^{R,3},\ Q_{(1)}^{R,-},\ \widetilde{Q}_{(1)}^{R,+},\ \bar{Q}_{(2)}^{R,3},\cdots$

(the lower half of the tower)

Yangian limit

$$\lim_{C\to 0}\frac{1}{2i\sqrt{C}}\left(Q_{(1)}^{R,+}-\widetilde{Q}_{(1)}^{R,+}\right) = \int dx\, J_x^+(x) - \frac{i}{2}\iint dxdy\,\epsilon(x-y)\,J_t^+(x)J_t^3(y)$$

$$\lim_{C\to 0}\frac{1}{2i\sqrt{C}}\left(\widetilde{Q}_{(1)}^{R,-}-Q_{(1)}^{R,-}\right) = \int dx\, J_x^-(x) + \frac{i}{2}\iint dxdy\,\epsilon(x-y)\,J_t^-(x)J_t^3(y)$$

$$\lim_{C\to 0}\frac{i}{4}\left(Q_{(2)}^{R,3}-\bar{Q}_{(2)}^{R,3}\right) = \int dx\, J_x^3(x) + \frac{i}{2}\iint dxdy\,\epsilon(x-y)\,J_t^+(x)J_t^-(y)$$

SU(2)$_R$ Yangian generators are reproduced.

2. Rational description

I. Kawaguchi and K.Y., JHEP1011 (2010) 032, 1008.0776.

I. Kawaguchi, D. Orlando and K.Y., PLB701 (2011) 475, 1104.0738.

c.f. J. Balog, P. Forgacs and L. Palla, PLB484 (2000) 367, hep-th/0004180.

The key ingredient

SU(2)$_L$ Noether current: $$j^L_\mu = \partial_\mu g\, g^{-1} - 2C\,\mathrm{Tr}(T_3 J_\mu) g T_3 g^{-1}$$

Conserved ⟵ EOM

BIZZ construction

[Brezin-Itzykson-ZinnJustin-Zuber PLB82 (1979) 442]

If the conserved current j_μ satisfies the flatness condition,

$$\epsilon^{\mu\nu}(\partial_\mu j_\nu - j_\mu j_\nu) = 0$$

then an infinite number of conserved non-local charges can be constructed.

The flatness condition for the *SU(2)*$_L$ current :

$$\epsilon^{\mu\nu}(\partial_\mu j^L_\nu - j^L_\mu j^L_\nu) = -C\epsilon^{\mu\nu}\partial_\mu(gT_3g^{-1})\partial_\nu(gT_3g^{-1})$$

Non-vanishing, because $C \neq 0$. But total derivative!

Improved currents

[J. Balog, P. Forgacs, L. Palla, 2000,
I. Kawaguchi , K.Y., 1008.0776]

$$j_\mu^{L\pm} \equiv j_\mu^L \mp \sqrt{C}\,\epsilon_{\mu\nu}\partial^\nu(gT_3g^{-1})$$

Improvement term

const.

Ambiguity to add a topological term

These currents satisfy the flatness condition

$$\epsilon^{\mu\nu}(\partial_\mu j_\nu^{L\pm} - j_\mu^{L\pm} j_\nu^{L\pm}) = 0$$

BIZZ

→ An infinite number of non-local charges (straightforward)

An infinite number of conserved non-local charges:

0-th $$Q^A_{(0)} = \int dx\, j_t^{L\pm,A}(x)$$ ($SU(2)_L$ Noether charge)

1-st $$Q^A_{(1)} = \int dx\, j_x^{L\pm,A}(x) + \frac{1}{4}\iint dx dy\, \epsilon(x-y)\varepsilon_{BC}{}^{A} j_t^{L\pm,B}(x) j_t^{L\pm,C}(y)$$

Non-local

⋮

$$\epsilon(x-y) \equiv \theta(x-y) - \theta(y-x)$$

NOTE the sign of improvement is irrelevant at the charge level

↳ Two copies of infinite sets of conserved charges

What is the charge algebra ?

Current algebra

with $A^2 = C$

$$\{j_t^{L\pm,A}(x), j_t^{L\pm,B}(y)\}_{\rm P} = \varepsilon^{AB}{}_C j_t^{L\pm,C}(x)\delta(x-y)$$

$$\{j_t^{L\pm,A}(x), j_x^{L\pm,B}(y)\}_{\rm P} = \varepsilon^{AB}{}_C j_x^{L\pm,C}(x)\delta(x-y) + \underline{(1+C)}\delta^{AB}\partial_x\delta(x-y)$$

[Magro's talk]

$$\{j_x^{L\pm,A}(x), j_x^{L\pm,B}(y)\}_{\rm P} = \underline{-C}\,\varepsilon^{AB}{}_C j_t^{L\pm,C}(x)\delta(x-y)$$

The current algebra is deformed due to the improvement.

⇒ Is Yangian algebra still realized?

(non-trivial question)

A pair of *SU(2)*$_L$ Yangian algebras

$$\{Q^{L,A}_{(0)}, Q^{L,B}_{(0)}\}_{\rm P} = \varepsilon^{AB}{}_C Q^{L,C}_{(0)}$$

$$\{Q^{L,A}_{(0)}, Q^{L,B}_{(1)}\}_{\rm P} = \varepsilon^{AB}{}_C Q^{L,C}_{(1)}$$

Non-trivial modification only for this part.

$$\{Q^{L,A}_{(1)}, Q^{L,B}_{(1)}\}_{\rm P} = \varepsilon^{AB}{}_C [Q^{L,C}_{(2)} + \frac{1}{12} Q^{L,C}_{(0)} Q^{L,D}_{(0)} Q^{L}_{(0)\,D} - \underline{C Q^{L,C}_{(0)}}]$$

Serre relations are also satisfied.

In summary,

Yangian algebra is realized even after the squashing.

Lax pairs and monodromy matrices

Two kinds of Lax pairs

$\lambda_{L_\pm}$: spectral parameters

$$L_\mu^{L_\pm}(\lambda_{L_\pm}) \equiv \frac{1}{1-\lambda_{L_\pm}^2}\left[j_\mu^{L_\pm} - \epsilon_{\mu\nu} j^{\nu,L_\pm}\right]$$

$$\left[\partial_t - L_t^{L_\pm}, \partial_x - L_x^{L_\pm}\right] = 0$$

due to EOM and flatness condition

Monodromy matrices

$$U^{L_\pm}(\lambda_{L_\pm}) = \mathrm{P}\exp\left[\int_{-\infty}^{\infty} dx\, L_x^{L_\pm}(x;\lambda_{L_\pm})\right] \quad \Rightarrow \quad \frac{d}{dt}U^{L_\pm}(\lambda) = 0$$

conserved

NOTE classical *r*-matrix is of rational type

The list of symmetries and integrable classes

Global symm.	$SU(2)_L$	$U(1)_R$
Hidden symm.	Yangian	quantum affine
Class (Lax pair)	rational	trigonometric

Squashed sigma models can be described as two different classes.

⇒ **Hybrid integrable structure**

3. Equivalence of two descriptions

[I. Kawaguchi and K.Y., PLB705 (2011) 251, 1107.3662]

[I. Kawaguchi-T. Matsumoto-K.Y., arXiv:1203.3400]

Equivalence of monodromy matrices

[Kawaguchi-Matsumoto-K.Y., 1203.3400]

$$\widetilde{g}_+^{-1} \cdot U^{L_+}(\lambda_{L_+}) \cdot \widetilde{g}_+ = U^R(\lambda_R) \qquad (\mathrm{Re} z_R \geq 0)$$

$$\widetilde{g}_-^{-1} \cdot U^{L_-}(\lambda_{L_-}) \cdot \widetilde{g}_- = U^R(\lambda_R) \qquad (\mathrm{Re} z_R < 0)$$

$$z_R = \mathrm{e}^{-\lambda_R}$$

where $\widetilde{g}_\pm \equiv g_\infty \cdot \mathrm{e}^{\pm i T^3 \lambda_R}$ ← Rescaling of sl(2) generators

Isomorphism of sl(2)

$$T^\pm \to \mathrm{e}^{\mp\lambda_R} T^\pm \quad \text{for} \quad U^{L_+}(\lambda_{L_+})$$

$$T^\pm \to \mathrm{e}^{\pm\lambda_R} T^\pm \quad \text{for} \quad U^{L_-}(\lambda_{L_-})$$

The relation between spectral parameters

$$z_R = \begin{cases} \left(\dfrac{\lambda_{L_+} - i\sqrt{C}}{\lambda_{L_+} + i\sqrt{C}}\right)^{1/2} & (\mathrm{Re}\, z_R > 0) \\ -\left(\dfrac{\lambda_{L_-} - i\sqrt{C}}{\lambda_{L_-} + i\sqrt{C}}\right)^{1/2} & (\mathrm{Re}\, z_R < 0) \end{cases} .$$

(The derivation will be explained later)

The gauge equivalence of Lax pair

Start from a left Lax pair with + sign,

$$L_{\pm}^{L_+}(x;\lambda_{L_+}) \ = \frac{1}{1\pm\lambda_{L_+}} g\left[T^+\left(1\mp i\sqrt{C}\right)J_{\pm}^- + T^-\left(1\pm i\sqrt{C}\right)J_{\pm}^+ + T^3(1+C)J_{\pm}^3\right]g^{-1}\,.$$

The gauge transformation is given by

$$\begin{aligned}
&\left[L_{\pm}^{L_+}(x;\lambda_{L_+})\right]^g \equiv g^{-1}L_{\pm}^{L_+}(x;\lambda_{L_+})g - g^{-1}\partial_{\pm}g \\
&= -J_{\pm} + \frac{1}{1\pm\lambda_{L_+}}\left[T^+\left(1\mp i\sqrt{C}\right)J_{\pm}^- + T^-\left(1\pm i\sqrt{C}\right)J_{\pm}^+ + T^3(1+C)J_{\pm}^3\right] \\
&= -\frac{\pm\lambda_{L_+}}{1\pm\lambda_{L_+}}\left[T^+\left(1+\frac{i\sqrt{C}}{\lambda_{L_+}}\right)J_{\pm}^- + T^-\left(1-\frac{i\sqrt{C}}{\lambda_{L_+}}\right)J_{\pm}^+ + T^3\left(1\mp\frac{C}{\lambda_{L_+}}\right)J_{\pm}^3\right]\,.
\end{aligned}$$

By using the relation between the spectral parameters,

$$\lambda_{L_\pm} = \frac{\tanh\alpha}{\tanh\lambda_R}\,, \qquad \text{(inverse relation)}$$

we obtain that

$$\left[L_\pm^{L_+}(x;\lambda_{L_+})\right]^g = -\frac{\sinh\alpha}{\sinh(\alpha\pm\lambda_R)}\left[T^+\underline{\mathrm{e}^{\lambda_R}}J_\mp^- + T^-\underline{\mathrm{e}^{-\lambda_R}}J_\pm^+ + T^3\frac{\cosh(\alpha\pm\lambda_R)}{\cosh\alpha}J_\pm^3\right].$$

By rescaling the generators,

$$T^\pm \;\to\; \mathrm{e}^{\mp\lambda_R}T^\pm \qquad \text{for} \quad U^{L_+}(\lambda_{L_+})\,,$$

we can show that

$$\left[L_\pm^{L_+}(x;\lambda_{L_+})\right]^g \simeq L_\pm^R(x;\lambda_R)\,. \qquad \text{(trigonometric Lax pair!)}$$

Thus we can show the equivalence at the monodromy matrix level,

$$g_\infty^{-1}\cdot U^{L_+}(\lambda_{L_+})\cdot g_\infty \simeq U^R(\lambda_R)\,.$$

How to derive the spectral parameter relation

The relation is rewritten as

$$z_R^2 = \frac{\lambda_{L_\pm} - i\sqrt{C}}{\lambda_{L_\pm} + i\sqrt{C}}$$ (Möbius trans.)

The data of monodromy matrix expansions

	Charges \ Monodromies	$U^R(\lambda_R)$	$U^{L_+}(\lambda_{L_+})$	$U^{L_-}(\lambda_{L_-})$
quantum affine	$Q^{R,3}_{(0)}, Q^{R,-}_{(1)}, \widetilde{Q}^{R,+}_{(1)}$	0	$+i\sqrt{C}$	$+i\sqrt{C}$
	$Q^{R,3}_{(0)}, Q^{R,+}_{(1)}, \widetilde{Q}^{R,-}_{(1)}$	∞	$-i\sqrt{C}$	$-i\sqrt{C}$
Yangian	$Q^{L,a}_{(0)}, Q^{L,a}_{(1)}$	± 1	∞	∞
	local charges	$\pm e^{\alpha}, \pm e^{-\alpha}$	± 1	± 1

Note U^R is expanded in terms of z_R

[blue box] has been discussed already

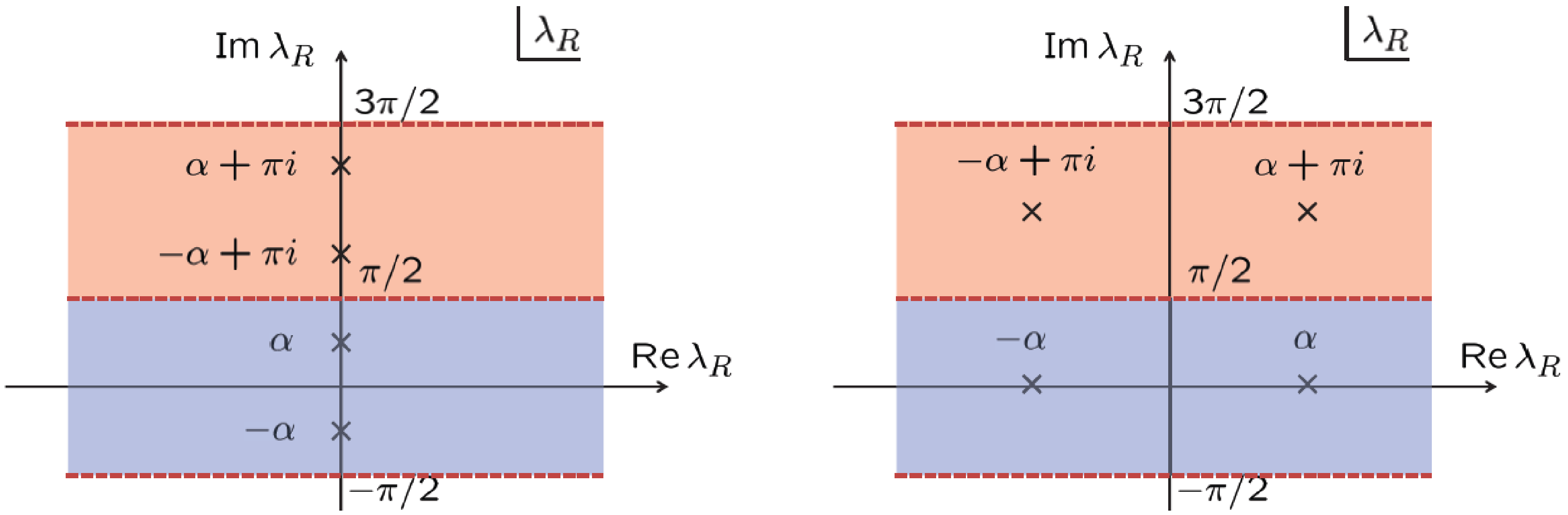
[red box] might be a bit surprising,

1) $SU(2)_L$ Yangian algebra can be reproduced from $U^R(\lambda_R)$

2) Quantum affine algebra can be reproduced from $U^{L_\pm}(\lambda_{L_\pm})$

What is the geometrical meaning of the relation?

The space of spectral parameter in the trigonometric description

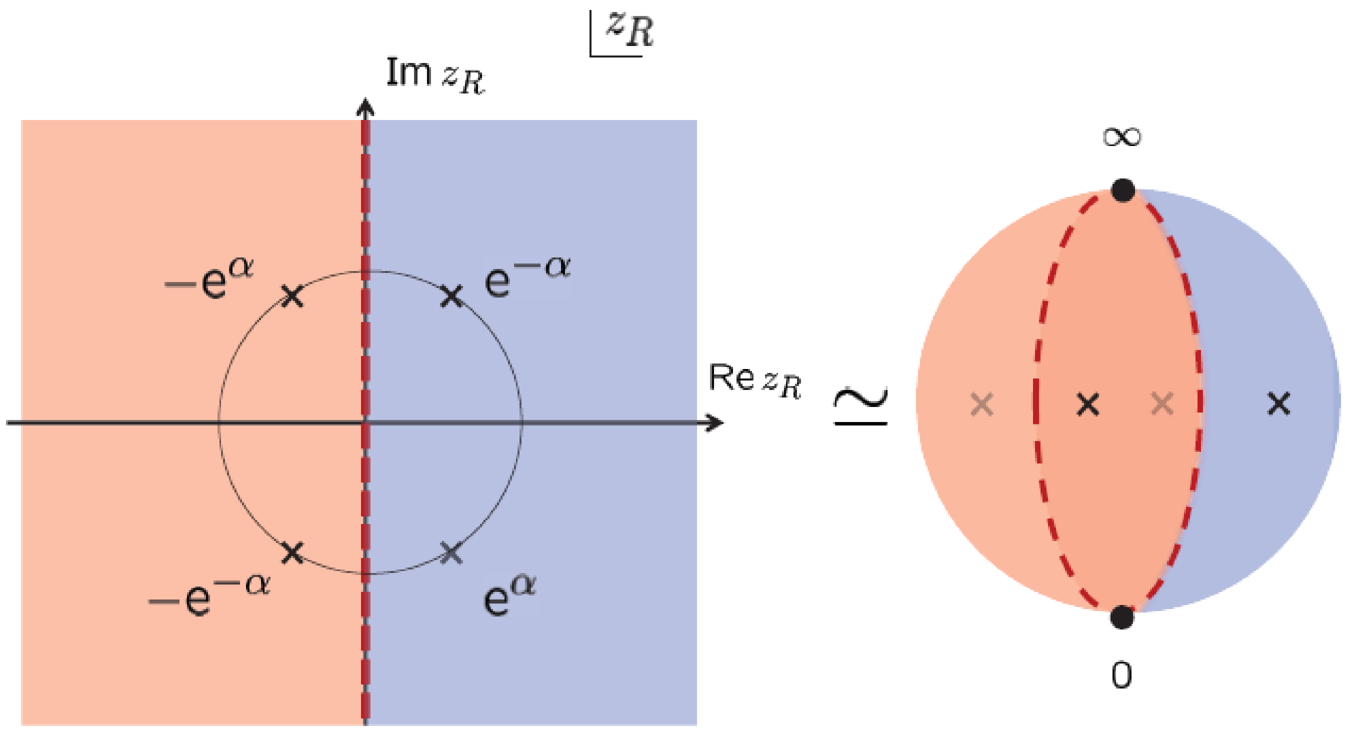


a) For $C > 0$ b) For $-1 < C < 0$

There are four poles in the trigonometric Lax pair.

The position of poles depends on the value of *C*.

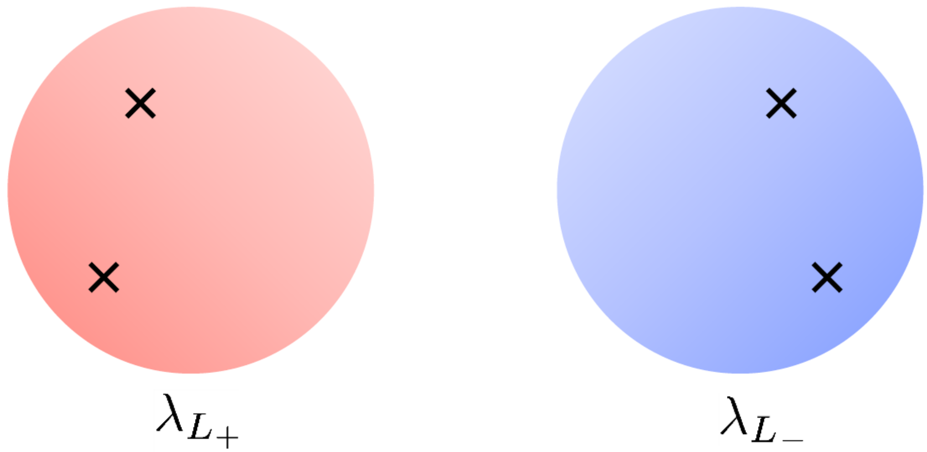
According to the map $z_R = e^{-\lambda_R}$



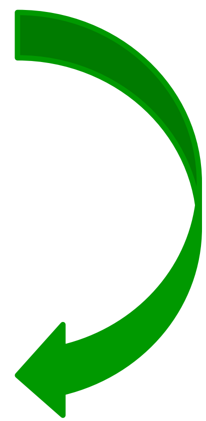
C > 0

Riemann sphere with four punctures

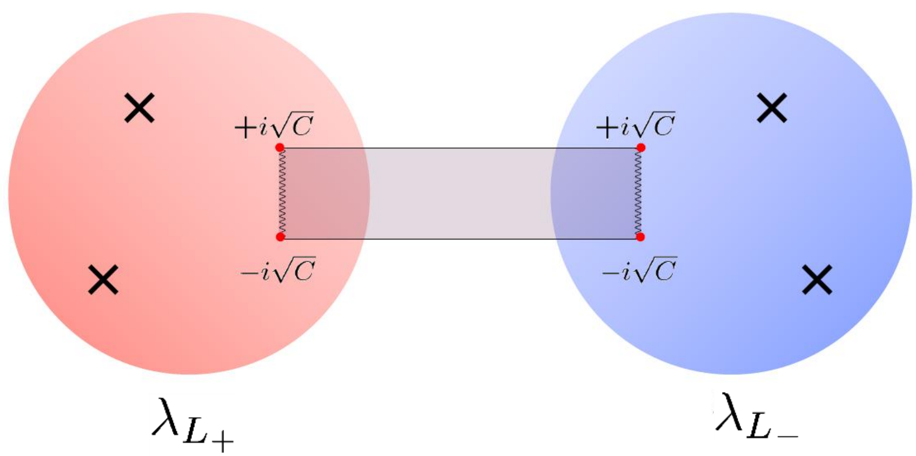
The space of spectral parameter in the rational description



Two Riemann spheres with two punctures



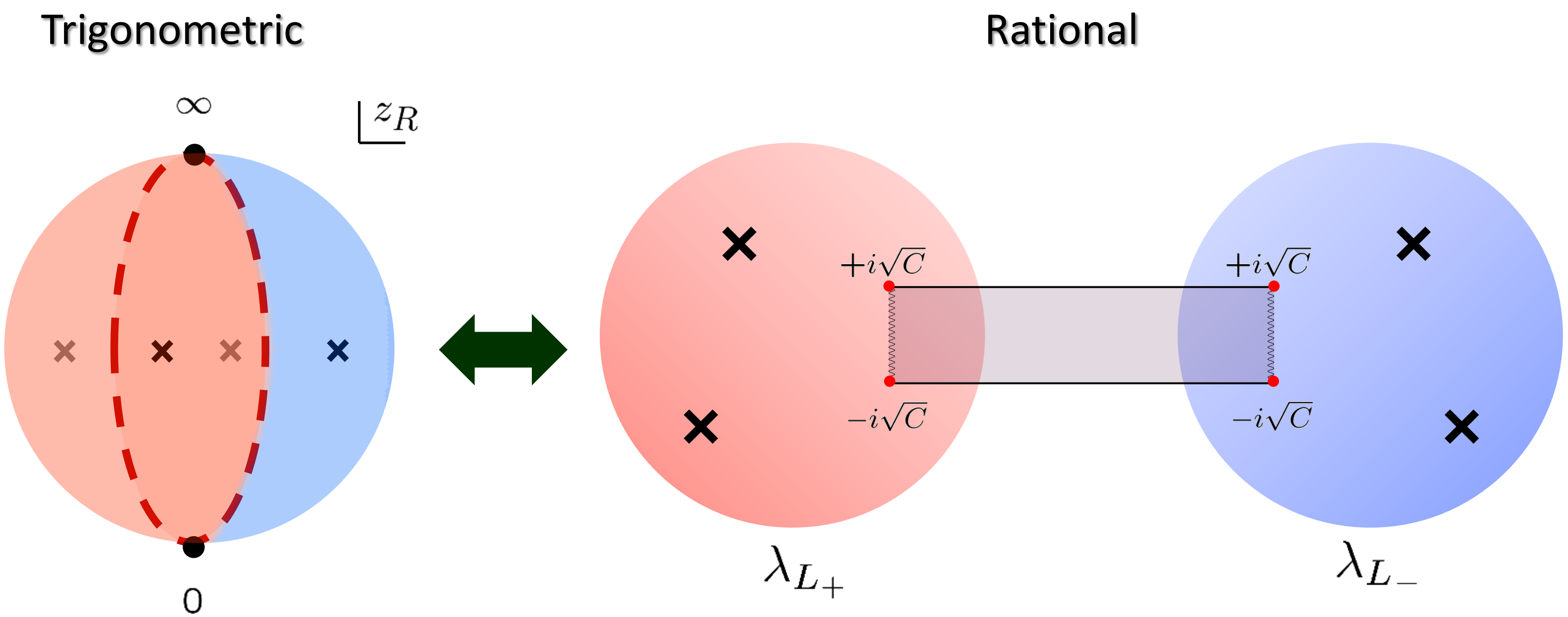
join



The two Riemann spheres are joined on the cut.

This is regarded as a Riemann sphere with four punctures

The spaces of spectral parameters are identical



The imaginary axis on the z_R -plane corresponds to the cut on $\lambda_{L_\pm}$-plane.

A trigonometric description = a pair of rational descriptions

$C \to 0$ limit : The cut shrinks to a point (SU(2)$_R$ Yangian point)

Inversely speaking, the breaking of SU(2)$_R$ opens up the cut.

4. Summary & Discussions

Summary

We have discussed the classical integrable structure of squashed sigma models .

- $SU(2)_L$ Yangians & quantum affine algebra

- Two descriptions

  1) rational one 2) trigonometric one

  ➡ Hybrid classical integrable structure

- Equivalence of two descriptions at the monodromy matrix level

Note The present argument is applicable to warped AdS_3 at classical level.

Application to AdS/condensed matter physics? [D'Hoker-Kraus, 2009]

Application to warped AdS_3/dipole CFT_2 ? [El-Showk-Guica, Song-Strominger, 2011]

Other directions

1) Schrödinger sigma models [Kawaguchi and K. Y., 1109.0872]

The same argument is applicable. Kawaguchi's poster

➡ q-deformed Poincare symmetry affine extension?

2) Squashed Wess-Zumino-Novikov-Witten model

The trigonometric description becomes rational at certain values of the coefficient of WZ term (in progress)

3) Other integrable deformations of S^3 c.f., [Fateev, NPB473 (1996) 509]

" trigonometric & trigonometric" " rational & elliptic " etc.

4) Half line case twisted quantum affine, twisted Yangian?

c.f., [MacKay-Short, hep-th/0104212]

Regelskis' poster

Thank you !

Backup

One may see a larger hidden structure by deforming integrable systems.

EX XXX model → XXZ model

Hamiltonian of XXZ model

$$H_{\rm XXZ} = -J\sum_j \left[S^x_{j+1}S^x_j + S^y_{j+1}S^y_j + \Delta S^z_{j+1}S^z_j\right]$$

$\Delta = 1$: XXX SU(2) symmetry (isotropic)

↓ broken

$\Delta \neq 1$: XXZ U(1) symmetry (uniaxial)

But the remaining U(1) is enhanced to *q*-deformed SU(2) $U_q(su(2))$

The *q*-deformed SU(2) is further enhanced to quantum affine algebra $U_q(\widehat{su(2)})$

→ Symmetry is enhanced by integrable deformation

The universality classes of classical integrable systems

[Belavin-Drinfeld, 1982]

1) Rational class - XXX model (No deformation parameter)

Hidden symmetry : Yangian algebra

2) Trigonometric class - XXZ model (1 deformation parameters)

Hidden symmetry : quantum affine algebra

3) Elliptic class - XYZ model (2 deformation parameters)

Hidden symmetry : elliptic algebra

BIZZ construction

[Brezin, Itzykson, Zinn-Justin, Zuber, 1979]

Assume that we have a flat conserved current j_μ

Let's introduce the covariant derivative: $$D_\mu = \partial_\mu - j_\mu$$

satisfies:

$$\partial^\mu D_\mu = D_\mu \partial^\mu \quad \leftarrow \quad \partial^\mu j_\mu = 0$$

$$\epsilon^{\mu\nu} D_\mu D_\nu = 0 \quad \leftarrow \quad \epsilon^{\mu\nu}(\partial_\mu j_\nu - j_\mu j_\nu) = 0$$

With the covariant derivative, one can construct an infinite number of non-local charges recursively.

NOTE If there is a flat conserved current, then *M* is not needed to be symmetric.

Let's take the Noether current as the 0th current :

$$J_{(0)\mu} = j_\mu = D_\mu \chi_{(0)} \longrightarrow \partial^\mu J_{(0)\mu} = 0$$ Conserved by definition.

$$(\chi_{(0)} = -1)$$

$$\Rightarrow \quad J_{(0)\mu} = \epsilon_{\mu\nu}\partial^\nu \chi_{(1)}$$

$$\epsilon(x-y) \equiv \theta(x-y) - \theta(y-x)$$

$$\chi_{(1)}(x) = \frac{1}{2}\int dy\, \epsilon(x-y) J_{(0)t}(y)$$

Then the next current is defined as $J_{(1)\mu} \equiv D_\mu \chi_{(1)}$:conserved

---

$$(\because) \quad \partial^\mu J_{(1)\mu} = \partial^\mu D_\mu \chi_{(1)} = D_\mu \partial^\mu \chi_{(1)} = \epsilon^{\mu\nu} D_\mu J_{(0)\nu}$$

$$= \epsilon^{\mu\nu} D_\mu D_\nu \chi_{(0)} = 0$$

Repeat the same step $\Rightarrow$ Infinite number of non-local charges

q-deformed Poincare algebra

The Poisson brackets we obtained :

$$\{Q^{R,+}, Q^{R,-}\}_{\rm P} = -Q^{R,2}\,,$$
$$\{Q^{R,+}, Q^{R,2}\}_{\rm P} = -Q^{R,+}\cosh\left(\frac{\sqrt{C}}{2}Q^{R,-}\right)\,,$$
$$\{Q^{R,-}, Q^{R,2}\}_{\rm P} = \frac{2}{\sqrt{C}}\sinh\left(\frac{\sqrt{C}}{2}Q^{R,-}\right)\,.$$

$C \to 0$ ➡ $SU(2)_R$

By rescaling the charge as $Q^{R,+} \to \frac{\sqrt{C}}{2}Q^{R,+}$

q-deformed Poincare algebra :

$$\{Q^{R,+}, Q^{R,-}\}_{\rm P} = -\frac{\sqrt{C}}{2}Q^{R,2}\,,$$
$$\{Q^{R,+}, Q^{R,2}\}_{\rm P} = -Q^{R,+}\cosh\left(\frac{\sqrt{C}}{2}Q^{R,-}\right)\,,$$
$$\{Q^{R,-}, Q^{R,2}\}_{\rm P} = \frac{2}{\sqrt{C}}\sinh\left(\frac{\sqrt{C}}{2}Q^{R,-}\right)$$

$C \to 0$ ➡ Poincare

The classical action revisited

$$S = \frac{1}{1+C}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx\, \eta^{\mu\nu}\, \mathrm{Tr}(j_\mu^{L+} j_\nu^{L-})$$

The classical action can be expressed in terms of the improved currents.

Dipole-like form!

Warped AdS_3 = a double Wick rotation of squashed S^3

$$S^3 \to AdS_3\,, \quad SU(2) \to SL(2,R)$$

1) space-like warped AdS_3 : $\theta \to i\sigma,\ \phi \to iu,\ \psi \to \tau$

$$ds^2 = \frac{L^2}{4}[-\cosh^2\sigma d\tau^2 + d\sigma^2 + (1+C)(du + \sinh\sigma d\tau)^2]$$

2) time-like warped AdS_3 : $\theta \to i\sigma,\ \phi \to \tau,\ \psi \to iu$

$$ds^2 = \frac{L^2}{4}[-(1+C)(d\tau - \sinh\sigma du)^2] + d\sigma^2 + \cosh^2\sigma du^2$$

The difference between warped AdS_3 and squashed S^3 is just signature

at least at classical level.

Equivalence of two descriptions

Two types of Lax pair coexist in squashed sigma models .

In order to understand the relation between them , it would be helpful to compare the present case with principal chiral models.

SU(2) principal chiral models

$SU(2)_L$ symmetry	rational
$SU(2)_R$ symmetry	rational

↓ deformation (squashing)

Sigma models on squashed S^3

$SU(2)_L$ symmetry	rational
q-deformed $SU(2)_R$ symmetry	trigonometric

Let us see the relation between the two descriptions.

SU(2)$_L$ x SU(2)$_R$

$C = 0$

SU(2) principal chiral models $\quad (j^R_\mu = dg \cdot g^{-1}, \quad j^L = g^{-1} dg)$

→ Two rational descriptions are equivalent.

The relationship : $j^R_\mu = g^{-1} j^L_\mu g$ (left-right symm.)

↓ deformation (squashing)

$C \neq 0$

A similar relation holds even after the squashing

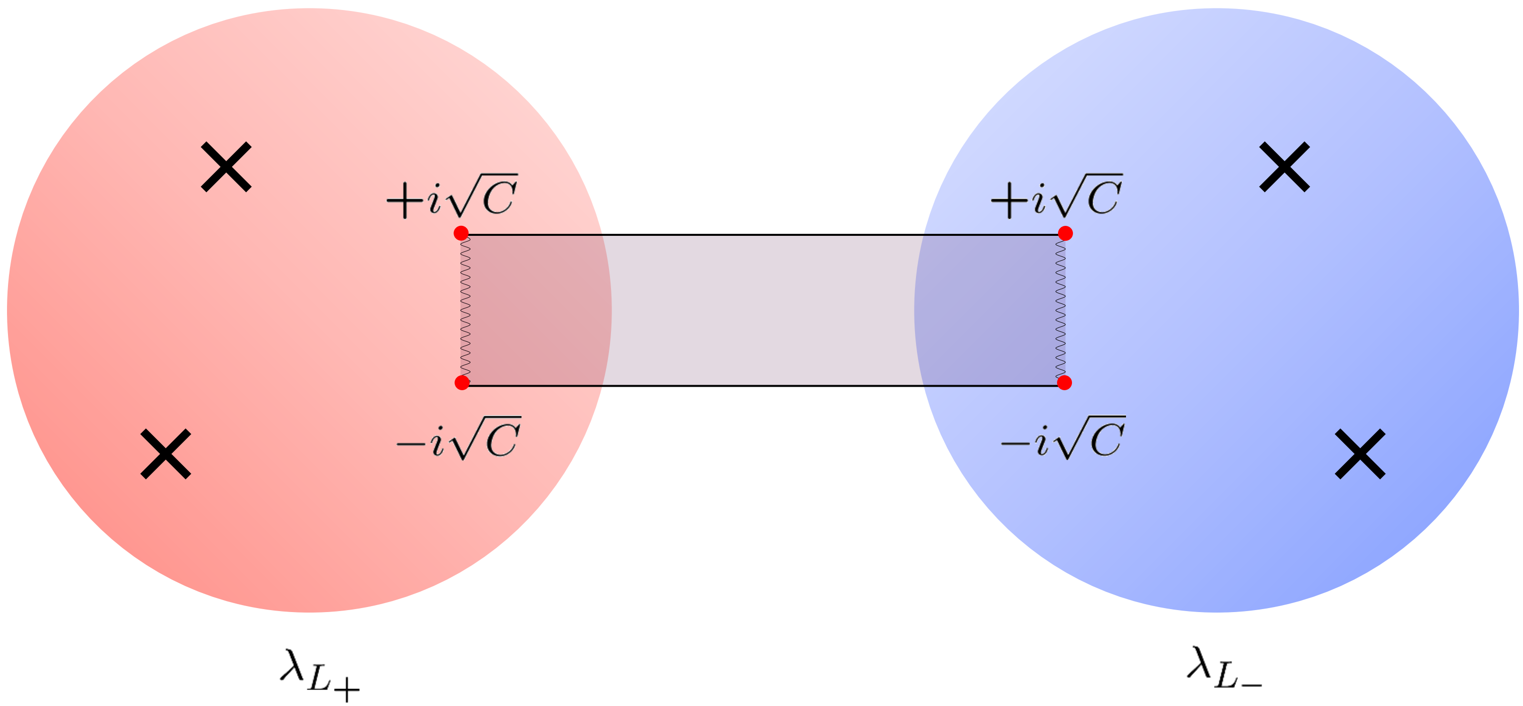
$$j^{R,3}_\mu = -2\mathrm{Tr}\left(T^3 g^{-1} j^{L+}_\mu g\right)$$

$$j^{R,\pm}_\mu = -2\mathrm{e}^{\gamma\chi}\mathrm{Tr}\left(T^\pm g^{-1} j^{L+}_\mu g\right)$$

Non-local map

[I. Kawaguchi and K.Y., arXiv: 1107.3662]

NOTE j^{L-}_μ corresponds to affine generators.



$+i\sqrt{C}$
$+i\sqrt{C}$
$-i\sqrt{C}$
$-i\sqrt{C}$
λ_{L_+}
λ_{L_-}