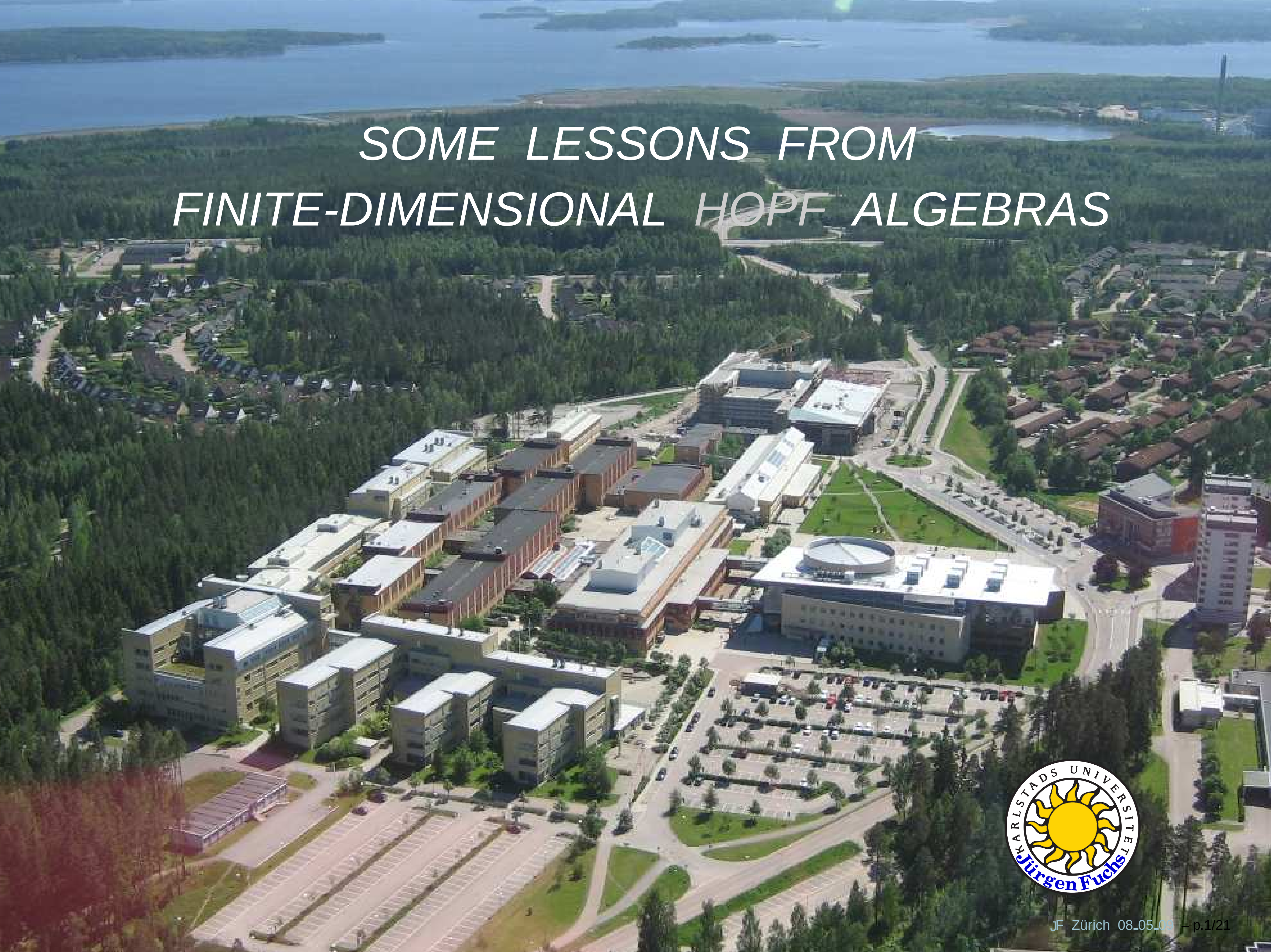

SOME LESSONS FROM FINITE-DIMENSIONAL HOPF ALGEBRAS

Plan

- Structure of algebras
- Basic algebras
- Chains of subalgebras
- Frobenius map
- Drinfeld map
- Quantum Fourier transform

Warning : largely textbook material

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- Verlinde formula for fusion simple $\times$ projective [Cohen & Westreich 2008]

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Conventions :

- ▷ A finite-dimensional $\Bbbk$-algebra
- ▷ $\Bbbk$ algebraically closed of characteristic 0
- ▷ later on: A in addition symmetric Frobenius
- ▷ H in addition Hopf
- ▷ later on: H in addition quasitriangular ribbon

But : various results valid in more general situation

- ▷ modules $=$ left-modules

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- ▷ " categories $\rightsquigarrow$ rep categories $\rightsquigarrow$ algebras "

$\boldsymbol{A}$

▷ has finitely many simple modules S_i up to isomorphism

▷ every simple module S_i has a projective cover P_i

▷ every indecomposable projective module is isomorphic to one of the P_i

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Structure of A

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- ▷ A as A-module: $\boldsymbol{{}_A A \;\cong\; \bigoplus_{i\in\mathcal{I}} \dim(S_i)\, P_i}$
- ▷ $S_i = P_i \,/\, \big(P_i \cap J(A)\big) = P_i \,/\, J(A) P_i$

  $J(A)$ Jacobson radical (intersection of all maximal left ideals)

  $\overline{A} \;\equiv\; A \,/\, J(A) \;\cong\; \bigoplus_{i\in\mathcal{I}} \dim(S_i)\, S_i$
- ▷ ${}_A A = \bigoplus_\alpha P_\alpha = \bigoplus_\alpha A\, e_\alpha$ $\qquad e_\alpha \in A$ primitive orthogonal idempotents, $\sum_\alpha e_\alpha = 1$

  for each $i \in \mathcal{I}$ have $\dim(S_i)$ values of α s.t. $P_\alpha \cong P_i$
- ▷ *Cartan matrix*: $\mathtt{C}_A = \big(\, c_{i,j} \,\big)$ with $c_{i,j} = [\, P_i : S_j \,]$

Basic algebras

study the rep theory of A — i.e. A-mod as abelian category — via the rep theory of the basic algebra $B \equiv B_A$

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▷ $B_A := e\,A\,e$ *reduced form / basic algebra* of A

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▷▷ $B_A \cong \mathrm{End}(A\,e) \cong \mathrm{End}(\bigoplus_{i\in\mathcal{I}} A\,e_i)$ as algebras

▷▷ $J(B_A) = e\,J(A)\,e$

▷▷ $\overline{B_A} \cong \bigoplus_{i\in\mathcal{I}} \Bbbk$

▷▷ composition series $B_A = eA_1e \supset eA_2e \supset \cdots \supset eA_\ell e \supset \{0\}$

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▷ Rep theory : e.g. $\dim_\Bbbk(\mathrm{Ext}^1_B(S_i, S_j)) = \dim_\Bbbk(e_j J(B)/J^2(B) e_i)$

Relevance : B_A-mod $\simeq$ A-mod as abelian categories

▷ Interpolating bimodules : Ae and eA , i.e. equivalence functor given by

T : B_A-mod $\longrightarrow$ A-mod : $M \mapsto A\,e \otimes_{B_A} M$

▷ Left adjoint and quasi-inverse to T : Restriction functor $\mathrm{Res}:$ A-mod $\longrightarrow$ B_A-mod

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NB : $B \cong \Bbbk Q_B/I$ bound quiver algebra $\equiv$ path algebra of (Q_B, I)

▷ quiver Q_B with vertices $\{i\}$ in bijection with complete set $\{e_i\}$ of idempotents and arrows $i \to j$ in bijection with a basis of $e_i\,(J(A)/J^2(A))\,e_j$

▷ I an (admissible) ideal contained in ideal generated by paths of length ≥ 2

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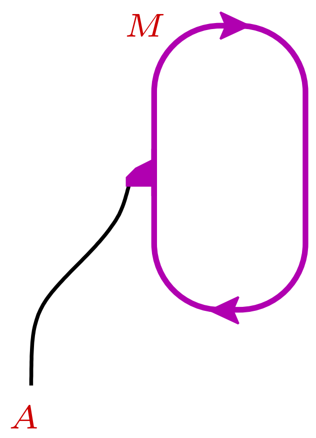

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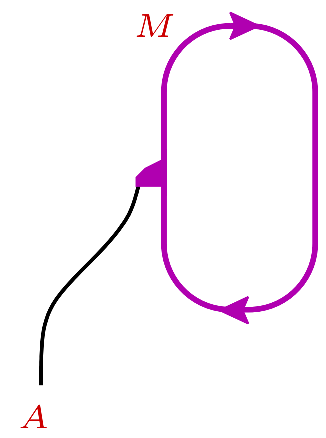


$\dim(M) = \chi_M \circ \eta$

Notation : $A \equiv \mathrm{Hom}(\Bbbk, A) \equiv \mathrm{Hom}(\mathbf{1}, A)$ and $A^\star \equiv \mathrm{Hom}(A, \Bbbk) \equiv \mathrm{Hom}(A, \mathbf{1})$

Warning : pictures in $\mathcal{V}ect_{\Bbbk}$ (not in A-mod)

Chains of subalgebras

Ingredient : symmetric Frobenius algebras (assumed from now on)

▷ A *symmetric* $\Longleftrightarrow$ existence of a *symmetrizing form* $\mathrm{t} \in A^\star$

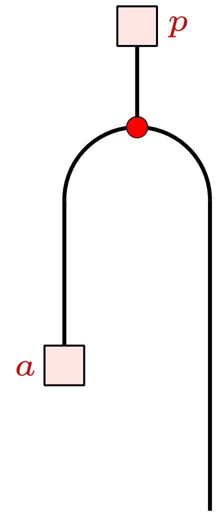
s.t. $\mathrm{t} \circ m = \mathrm{t} \circ m \circ c_{A,A}$

▷ A *Frobenius* $\Longleftrightarrow$ existence of $\mathrm{t} \in A^\star$ s.t. $\mathrm{t} \circ m$ non-degenerate

$\Longleftrightarrow$ also a coalgebra , with coproduct a bimodule morphism

$\Longleftrightarrow$ Φ_t: $a \mapsto \mathrm{t} \leftharpoondown a$ *iso*morphism

with $\leftharpoondown$ right action of A on the dual $A^\star$:

$p \leftharpoondown a \;=\; p \circ \mathrm{L}_a \;=$  $\in A^\star$

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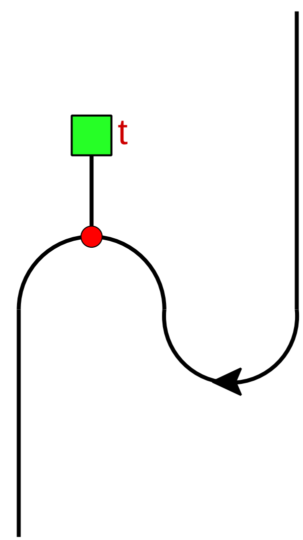
$\Phi_{\mathrm{t}} =$  $\in \mathrm{Hom}(A, A^\star)$

Chains of subalgebras

- pair of *dual bases* of A w.r.t. t:

  subsets $\{a_l\}$ and $\{b_l\}$ s.t. $\sum_l a_l\,(\mathsf{t}\circ \mathsf{L}_{b_l}) = \mathrm{id}_A = \sum_l (\mathsf{t}\circ \mathsf{R}_{a_l})\,b_l$

- *trace map* τ: $x \mapsto \sum_l b_l\,x\,a_l$

  - τ is zero on $J(A)$
  - Ae simple A-module iff $\tau(e)$ not nilpotent

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$$\boldsymbol{Z_0(A) \subseteq \mathrm{Hig}(A) \subseteq \mathrm{Rey}(A) \subseteq Z(A)}$$

▷▷ *Reynolds ideal* $\mathrm{Rey}(A) = \mathrm{Soc}(A) \cap Z(A)$

▷▷ *Higman ideal* / projective center $\mathrm{Hig}(A) = \mathrm{im}(\tau)$

▷▷ $Z_0(A) =$ span of those central primitive idempotents e for which Ae is simple

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- Chain of subalgebras in $A^\star$: $\boldsymbol{C_0(A) \subseteq I(A) \subseteq R(A) \subseteq C(A)}$

  - *central forms* / class functions / symmetric linear functions

    $C(A) = \{\, x\in A^\star \,|\, x\circ m = x\circ m\circ c_{A,A} \,\}$

  - $R(A)$ = span of characters of all A-modules
  - $I(A)$ = span of characters of all projective A-modules
  - $C_0(A)$ = span of characters of all simple projective A-modules

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- ▷ $\dim(C(A)) = \dim(A/[A,A])$ $\qquad$ $\dim(I(A)) = \mathrm{rank}(\mathtt{C}_A)$

- ▷ $R(\overline{A}) = R(A) = C(\overline{A}) \equiv C(A) \cap \{\, p\in A^\star \,|\, p\circ J(A) = 0 \,\}$

- ▷ $[M] \mapsto \chi_M$ is group homomorphism from Grothendieck group of A-mod to $C(\overline{A})$

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- Φ_{t} furnishes bijections

  $Z_0(A) \xrightarrow{\Phi_{\mathrm{t}}} C_0(A)$ $\qquad$ $\mathrm{Hig}(A) \xrightarrow{\Phi_{\mathrm{t}}} I(A)$

  $\mathrm{Rey}(A) \xrightarrow{\Phi_{\mathrm{t}}} R(A)$ $\qquad$ $Z(A) \xrightarrow{\Phi_{\mathrm{t}}} C(A)$

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from now on consider Hopf algebras H

- ▷ there is an isomorphism $\Psi :\ H \to H^\star$ of left H-modules and right $H^\star$-modules

  s.t. $\Lambda = \varepsilon \circ \Psi^{-1}$ is a left integral for H

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▷▷ called *Frobenius map* (inverse also called *Radford map*)

▷▷ unique up to scalar

▷ Ingredient : integrals

Integrals on Hopf algebras

▷ *Left integral on* H : $\Lambda \in H$ s.t. $m \circ (\mathrm{id}_A \otimes \Lambda) = \Lambda \circ \varepsilon$

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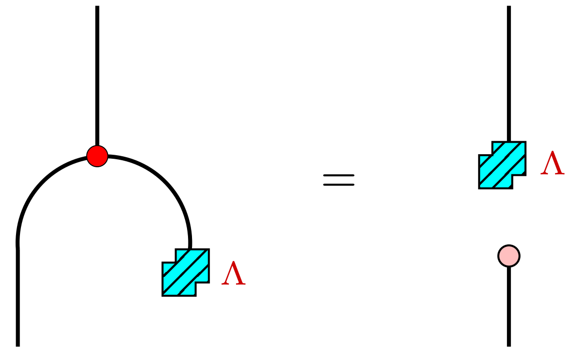
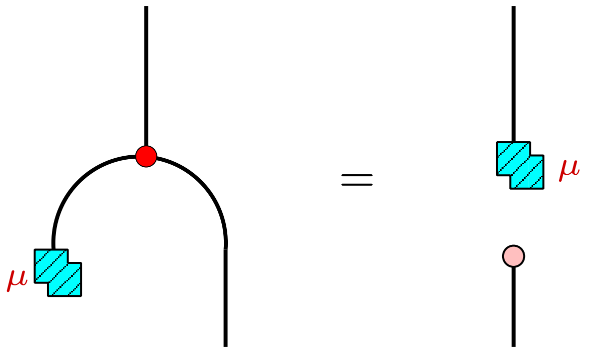


▷ *Right integral* :



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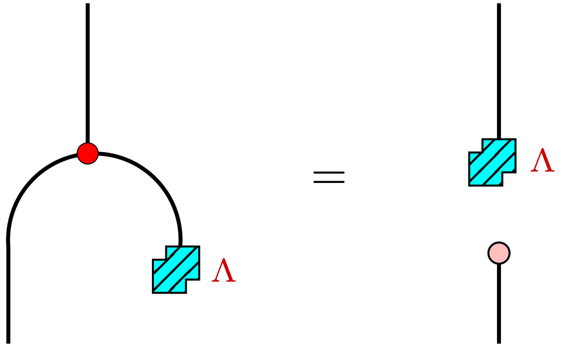

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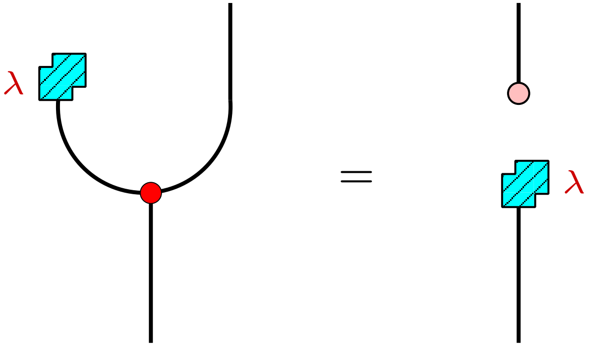

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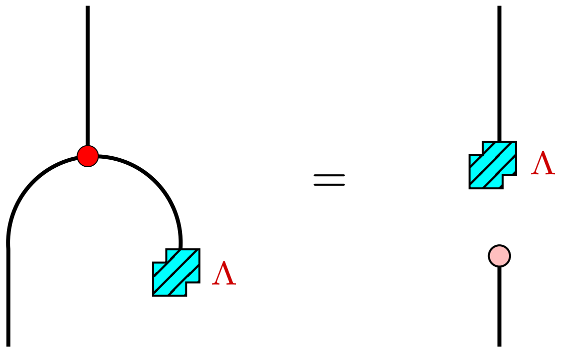

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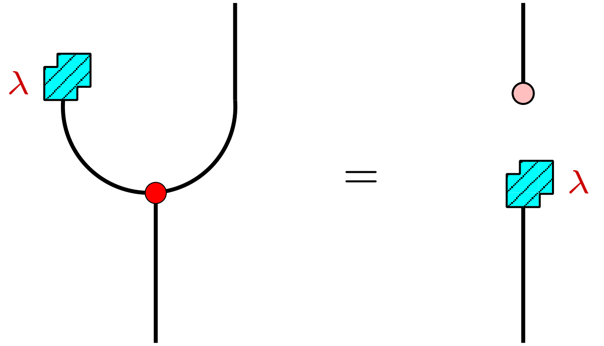

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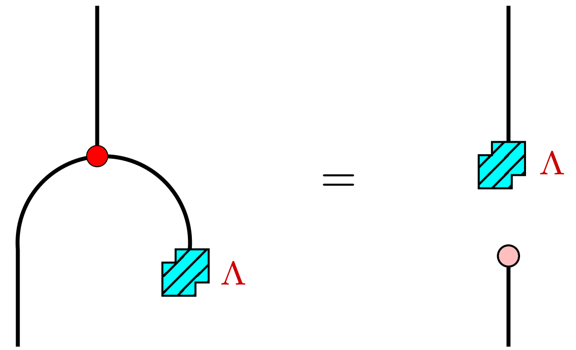

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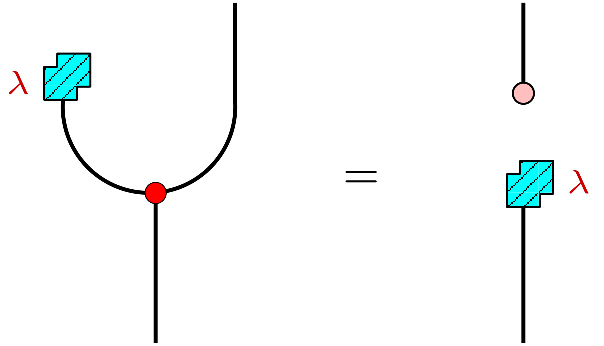

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▷ normalize such that $\lambda \circ \Lambda = 1$

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- ▷ *Right integral on* $H^\star$: $\lambda \in H^\star$ s.t. $(\lambda \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \lambda$
- ▷ one application : $\varepsilon \circ \Lambda \neq 0 \iff H$ semisimple

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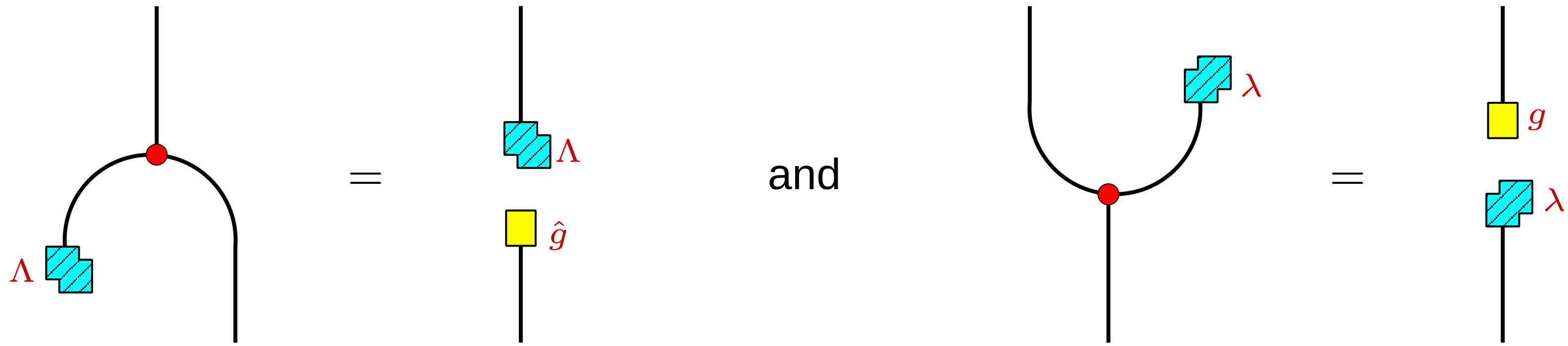
- ▷ *distinguished group-like elements* / right/left modular elements / comodulus/modulus

  $g \in G(H) = \{a \in H \,|\, \Delta \circ a = a \otimes a\} \setminus \{0\}$ and

  $\hat{g} \in G(H^\star) = \{p \in H^\star \,|\, p \circ m = p \otimes p\} \setminus \{0\}$ s.t.



- ▷ one application : $(P(\Bbbk_\varepsilon))^\vee = P(\Bbbk_{\hat{g}})$

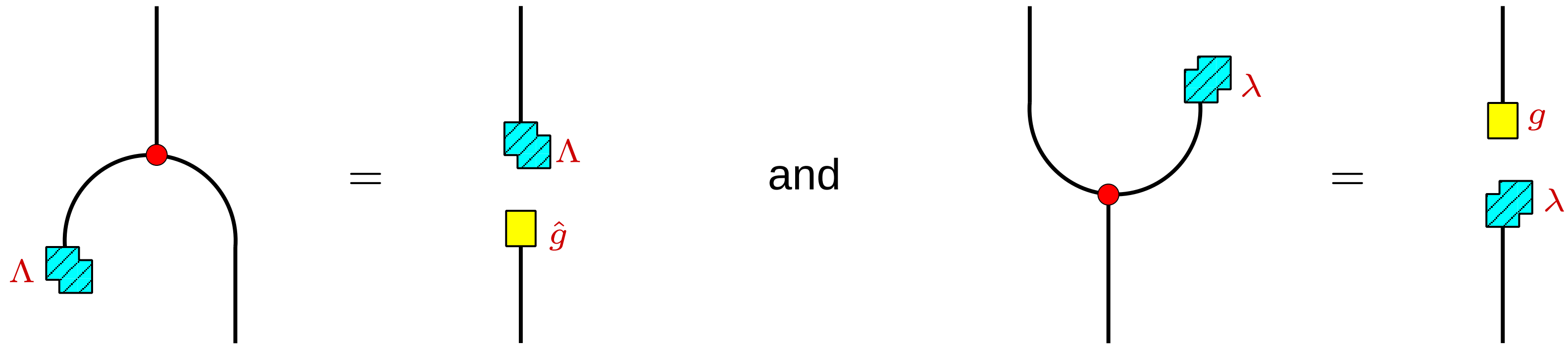
  module dual to projective cover of the one-dim. H-module associated to the counit

  $=$ projective cover of the one-dim. module associated to $\hat{g}$

- ▷ *Left integral on* H : $\Lambda \in H$ s.t. $m \circ (\mathrm{id}_A \otimes \Lambda) = \Lambda \circ \varepsilon$
- ▷ *Right integral on* $H^\star$: $\lambda \in H^\star$ s.t. $(\lambda \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \lambda$
- ▷ one application : $\varepsilon \circ \Lambda \neq 0 \iff H$ semisimple
- ▷ *distinguished group-like elements* / right/left modular elements / comodulus/modulus

  $g \in G(H) = \{a \in H \,|\, \Delta \circ a = a \otimes a\} \setminus \{0\}$ and

  $\hat g \in G(H^\star) = \{p \in H^\star \,|\, p \circ m = p \otimes p\} \setminus \{0\}$ s.t.



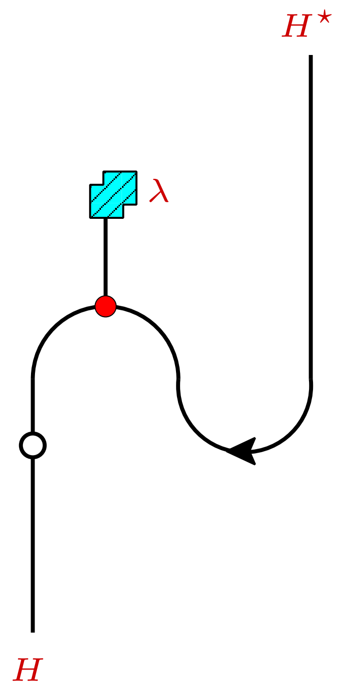
- ▷ another one : *Radford formula* $\mathrm{S}^4 = \mathrm{ad}_g \circ \mathrm{ad}^{\hat g^{-1}}$
- ▷ *balancing element* : if a square root of g exists in H

  then have group-like element $b \in H$ s.t. $\mathrm{S}^2 = \mathrm{ad}_b$ and $b^2 = g$

▷ Frobenius map and its inverse :

$$\Psi : \; H \to H^\star : \quad h \mapsto \lambda \leftharpoonup \mathrm{S}(h) \; = h \rightharpoonup \lambda$$

$$\Psi^{-1} : \; H^\star \to H : \quad p \mapsto \Lambda \leftharpoonup p$$

$\Psi =$  $\in \mathrm{Hom}(H, H^\star)$

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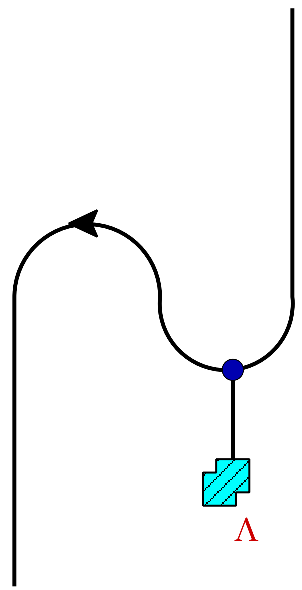
$\Psi^{-1} =$  $\in \mathrm{Hom}(H^\star, H)$

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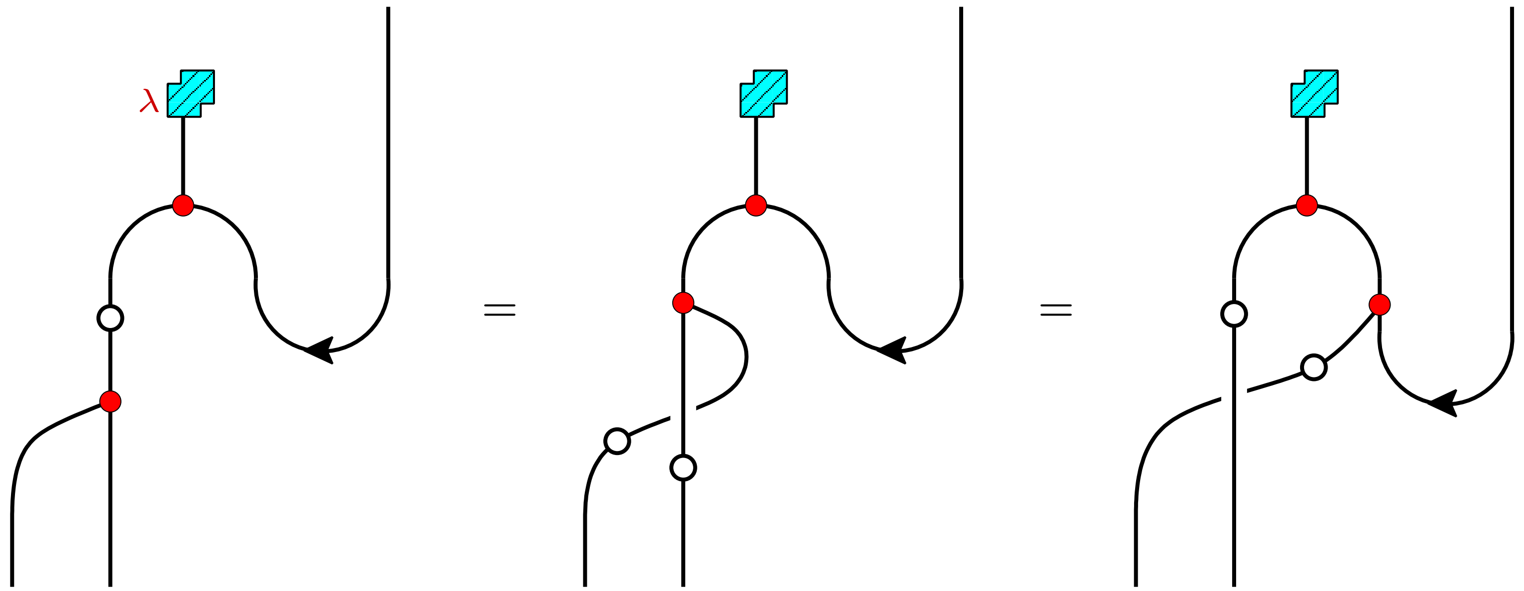
- ▷ Ψ morphism of left H-modules and of right $H^\star$-modules :



Frobenius map

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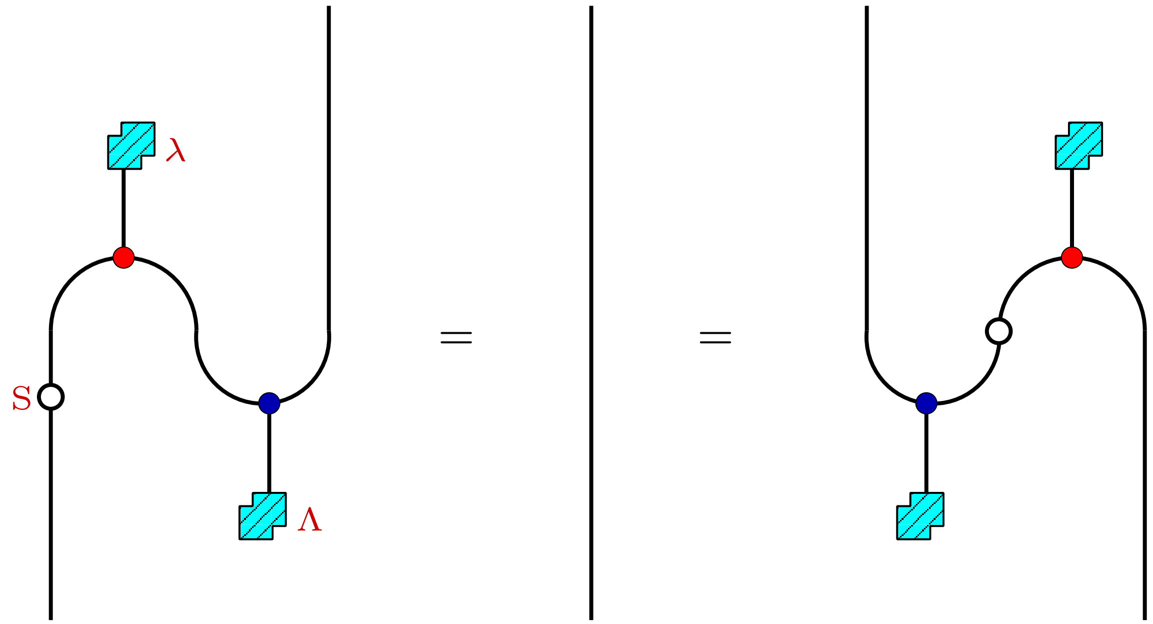
▷ Ψ and Ψ^{-1} inverse to each other :



Frobenius map

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▷ one application: H is Frobenius

$$\varepsilon_{\mathrm{Fr}} = \lambda$$

$$\Delta_{\mathrm{Fr}} = [\mathrm{id}_H \otimes (m \circ [(\mathrm{S} \circ \Psi^{-1}) \otimes \mathrm{id}_H])] \otimes (b_H \otimes \mathrm{id}_H)$$

▷ Frobenius map and its inverse :

$\Psi \; : \quad H \;\longrightarrow\; H^\star : \quad h \;\mapsto\; \lambda \leftharpoonup \mathrm{S}(h) \;= h \rightharpoonup \lambda$

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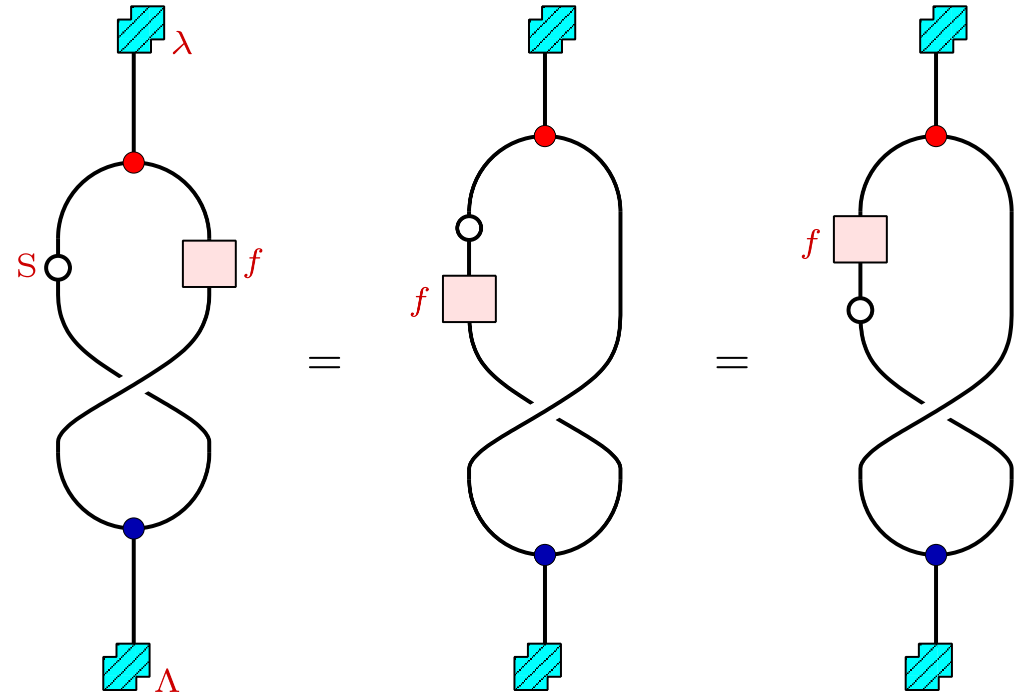
▷ one application : H is Frobenius

▷ another one : express trace of an endomorphism of H through integrals

$\mathrm{tr}(f) \;=$ 

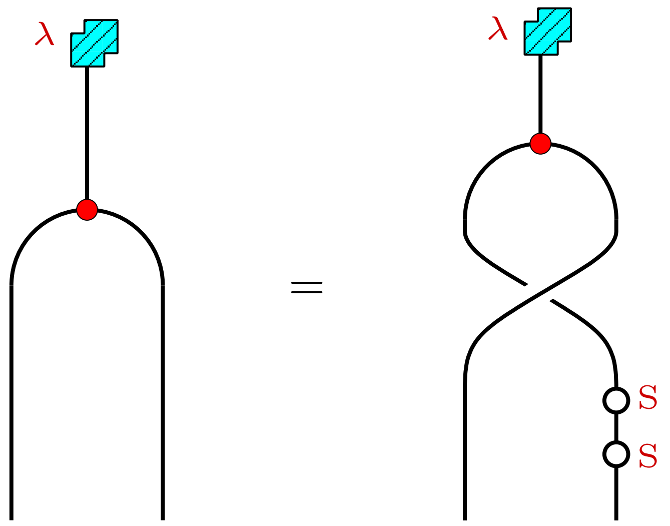
Unimodular Hopf algebras

▷ *unimodular* Hopf algebra : left integral Λ also a right integral

Unimodular Hopf algebras

- *unimodular* Hopf algebra: left integral Λ also a right integral
- Radford formula for unimodular H: $\mathrm{S}^4 = \mathrm{ad}_g$
- H unimodular $\Longleftrightarrow$ $\varepsilon \circ \mathrm{S}^2 = \varepsilon$ $\Longleftrightarrow$ S^2 a Nakayama automorphism
- H unimodular

  $\Longrightarrow$ right integral of $H^\star$ satisfies



$\Longrightarrow$ $\mathrm{S}|_{Z(H)} = \mathrm{id}_{Z(H)}$

$\Longrightarrow$ $Z(H) = \Psi^{-1}(O_{\mathrm{S}^2}(H))$ with $O_{\mathrm{S}^2}(H) = \{p \in H^\star \,|\, p(ab) = p((\mathrm{S}^2(b))a) \, \forall\, a, b \in H\}$

Unimodular Hopf algebras

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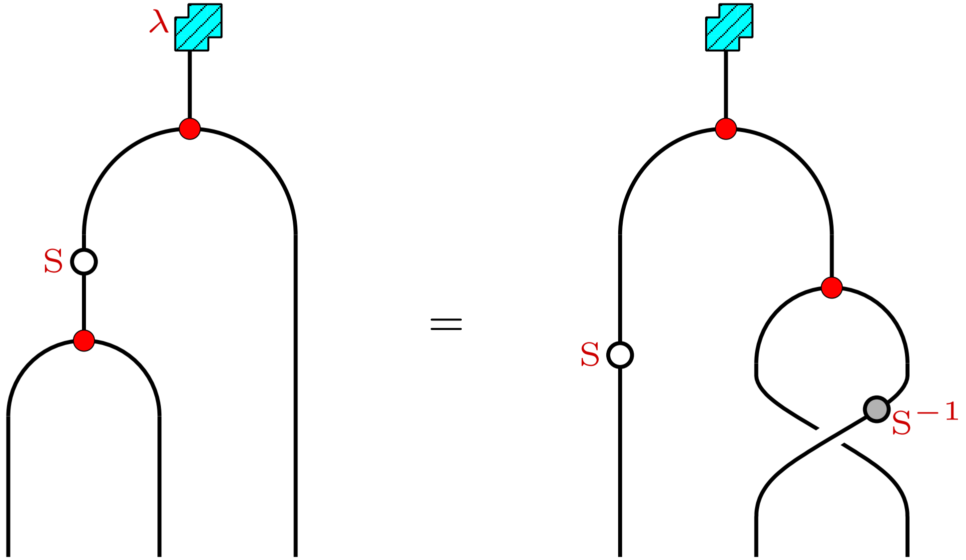
$\Longrightarrow$



$\Longrightarrow$ ...

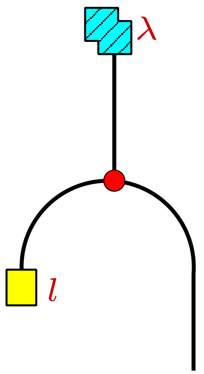
$\Longrightarrow$ $I(H)$ an ideal of $C(H)$ and stable under the antipode of $H^\star$

▷ H unimodular and S^2 inner $\Longleftrightarrow$ H symmetric

Take now H symmetric and set $\mathrm{S}^2 = \mathrm{ad}_l$

$\Longrightarrow$ symmetrizing form on H given by

$$\mathrm{t} \;=\; \lambda \leftharpoonup l \;=\; \lambda \circ \mathsf{L}_l \;=$$



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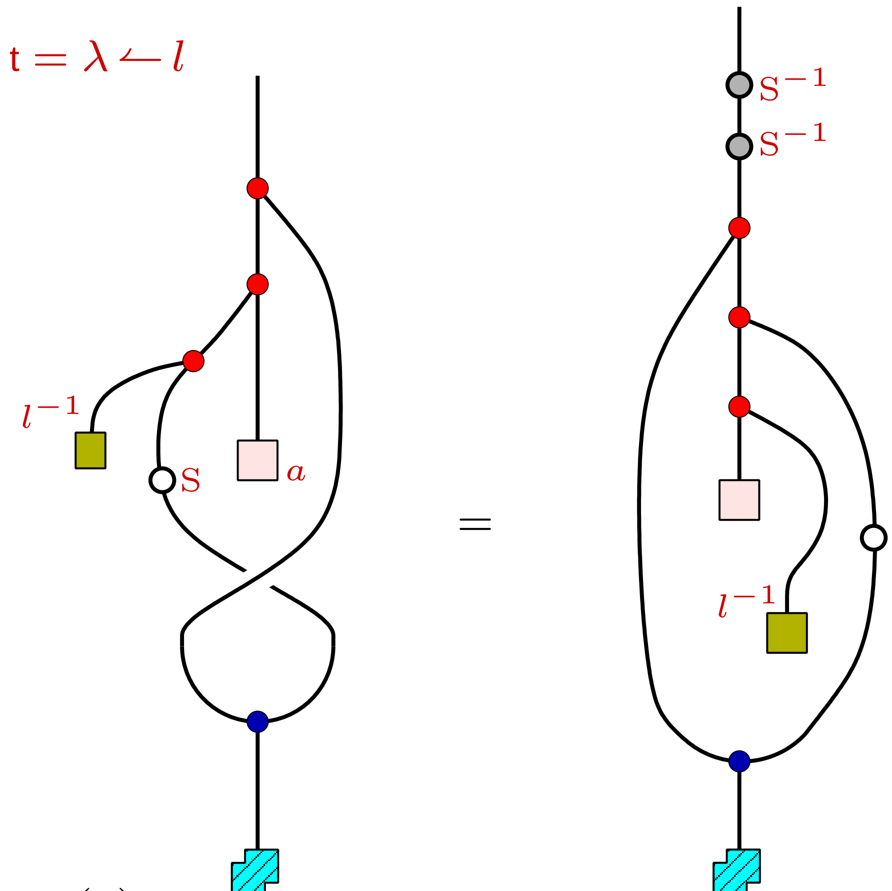
$\sum_n a_n \otimes b_n = (\mathrm{id}_H \otimes (\mathsf{R}_{l^{-1}} \circ \mathrm{S})) \circ \Delta \circ \Lambda$

$\Longrightarrow$ trace map

$\tau(a) =$



$= \mathrm{S}^{-2} \circ \ell\mathrm{ad}_\Lambda \circ \mathsf{R}_{l^{-1}}(a)$

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$\tau = \mathrm{S}^{-2} \circ \ell\mathrm{ad}_\Lambda \circ \mathsf{R}_{l^{-1}}$ with $\ell\mathrm{ad}_x$ *left adjoint action*

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$\Longrightarrow$ $\mathrm{Hig}(H) = \ell\mathrm{ad}_\Lambda(H)$

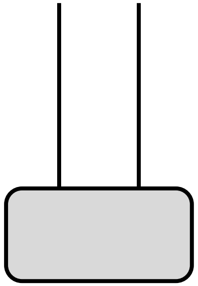
▷ based on this characterization of $\mathrm{Hig}(H)$ get $\widehat{\Psi}(\ell\mathrm{ad}_\Lambda(H)) = I(H)$

with *modified Frobenius map* $\widehat{\Psi}: H \to H^\star \quad h \mapsto \mathrm{t} \leftharpoonup \mathrm{S}^{-1}(h)$

Drinfeld map

from now on : H *quasitriangular*

▷ thus *R-matrix* $R \in H \otimes H$

$R =$ 

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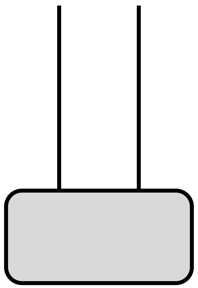
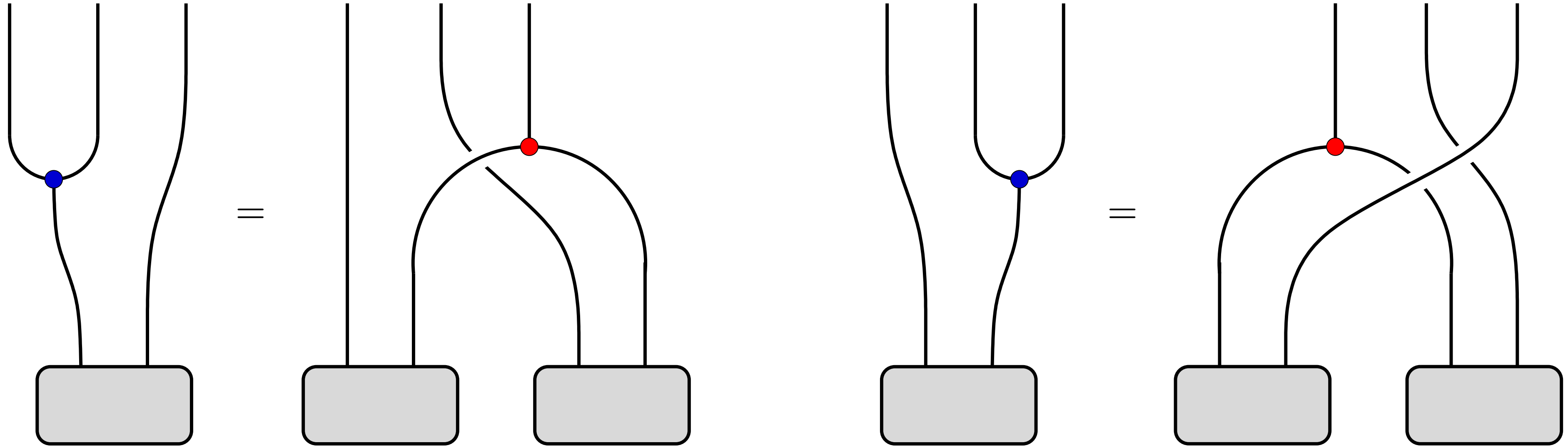
$R =$ 

invertible and satisfying $\Delta^{\mathrm{op}} = \mathrm{ad}_R \circ \Delta$ and



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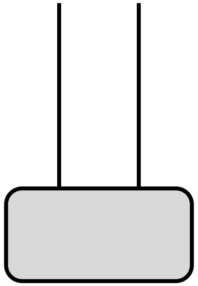
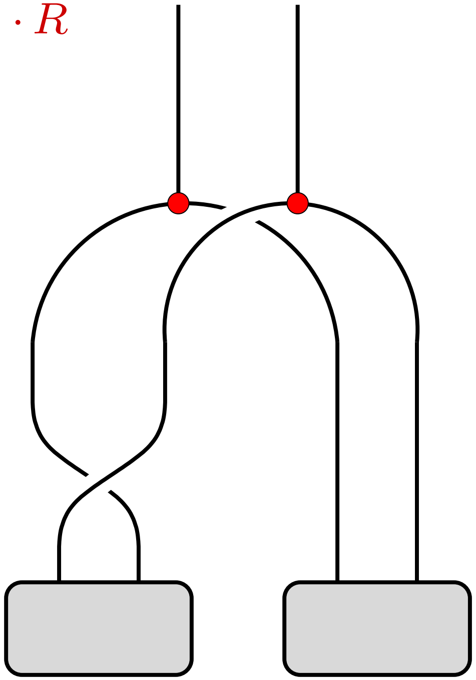
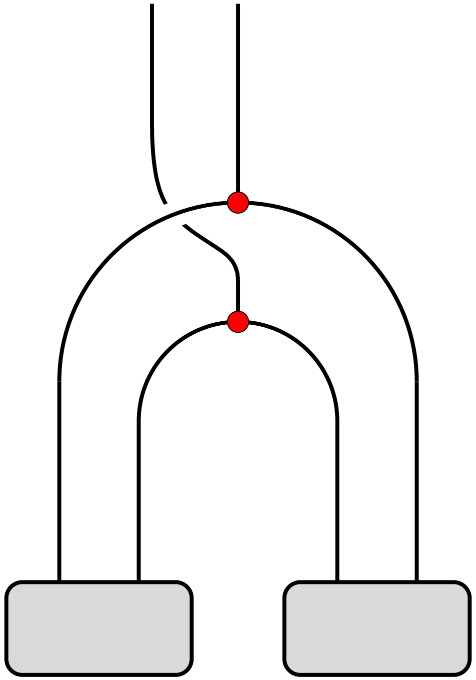
$R =$ 

▷ *monodromy matrix* $Q = R_{21} \cdot R$

$Q =$  $\equiv$ 

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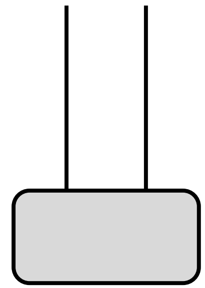
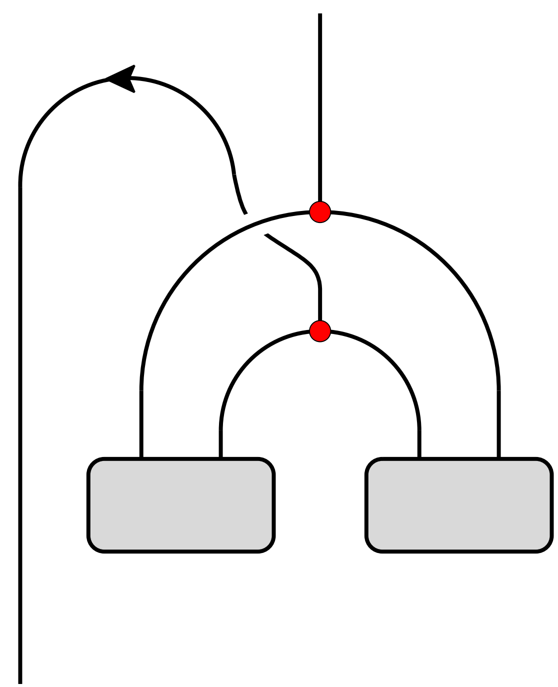
$R =$ 

▷ *monodromy matrix* $Q = R_{21} \cdot R$

▷ *Drinfeld map* $f_Q = (d_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_{H^\vee} \otimes Q)$

$H^\star \to H$

$f_Q =$ 

Drinfeld map

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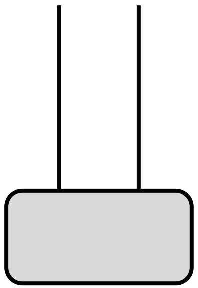
▷ thus *R-matrix* $R \in H \otimes H$

$R \;=$



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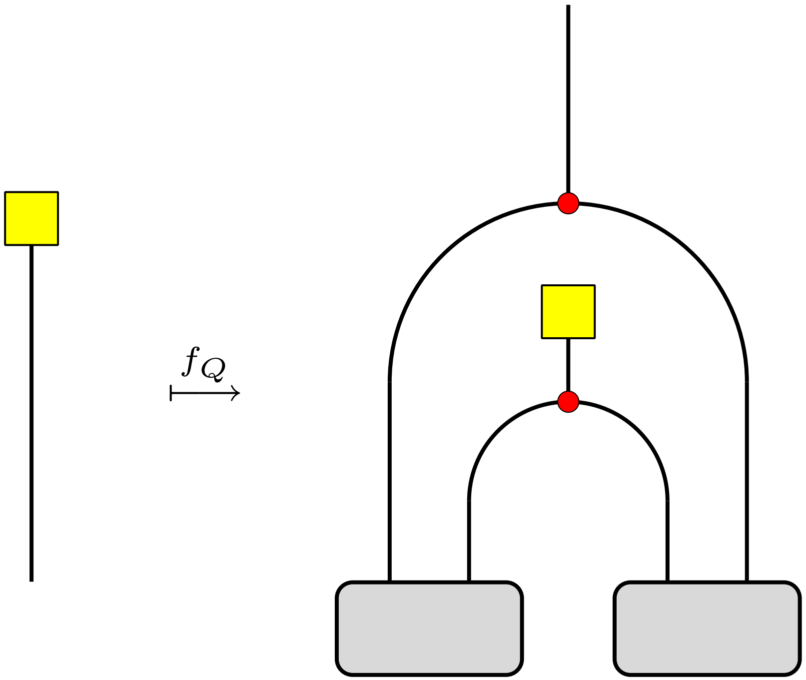
equivalently



Drinfeld map

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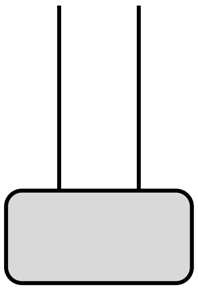
$R =$ 

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$H^\star \to H$

▷▷ restriction to $O_{S^2}(H)$ is isomorphism to $Z(H)$ as associative algebras

$$f_Q\big|_{O_{S^2}(H)} : \quad O_{S^2}(H) \xrightarrow{\cong} Z(H)$$

Drinfeld map

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- ▷ *monodromy matrix* $Q = R_{21} \cdot R$
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- ▷ for *semisimple* H : *S-matrix* $S_{i,j} = \chi_j \circ (f_Q(\chi_i))$

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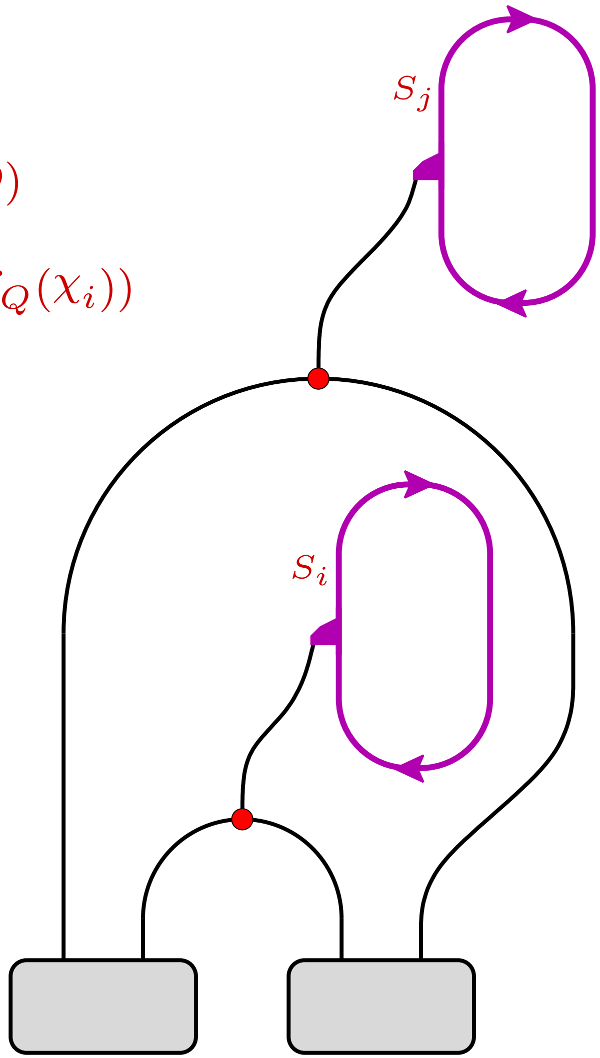
$$S_{i,j} \;=$$



- *R-matrix* $R \in H \otimes H$
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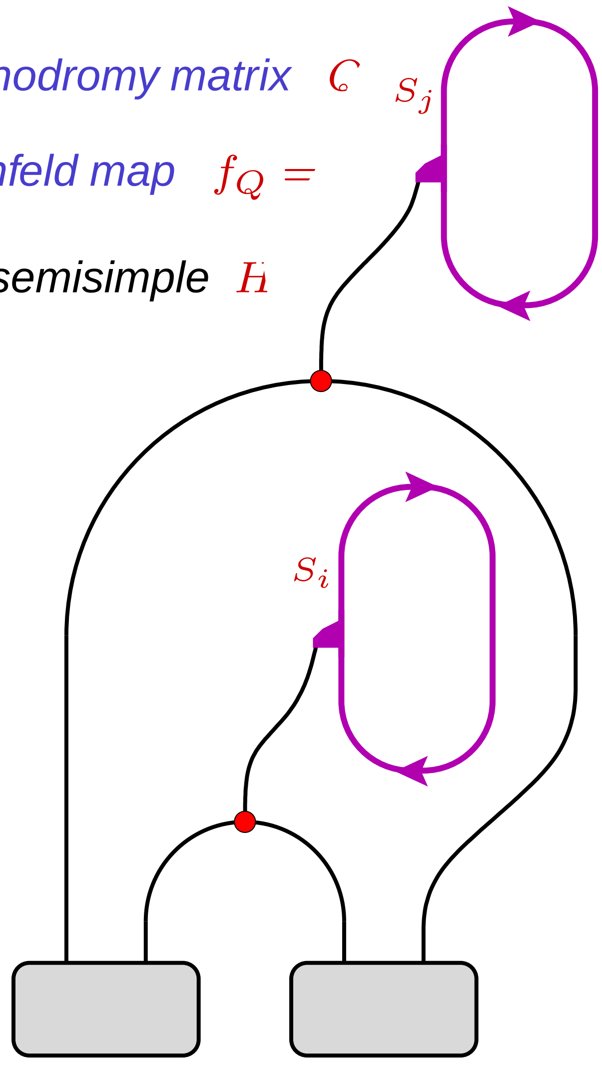

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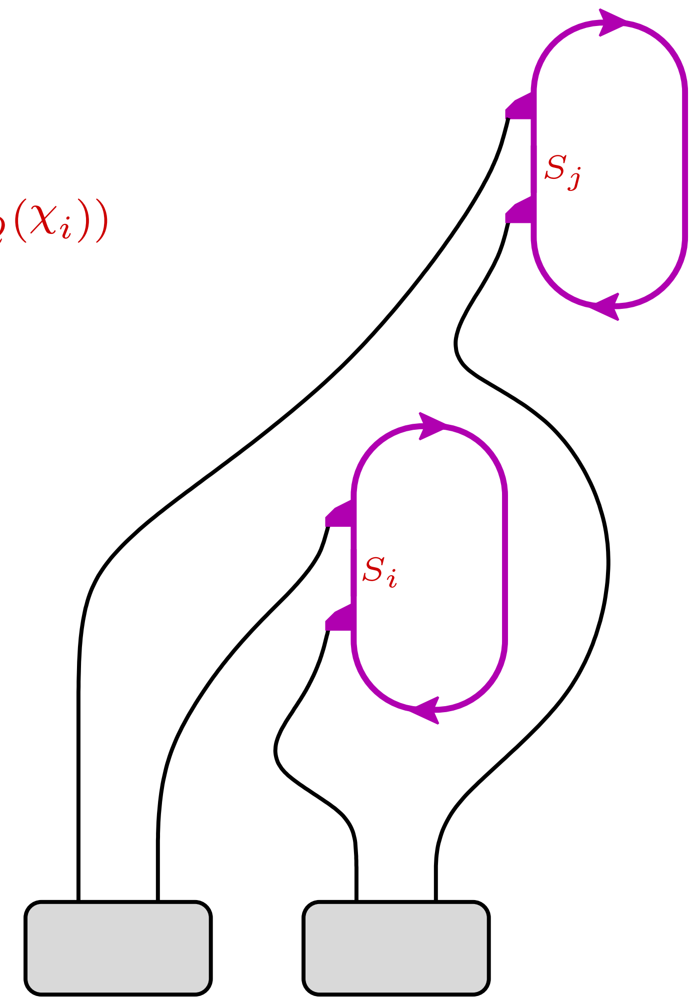

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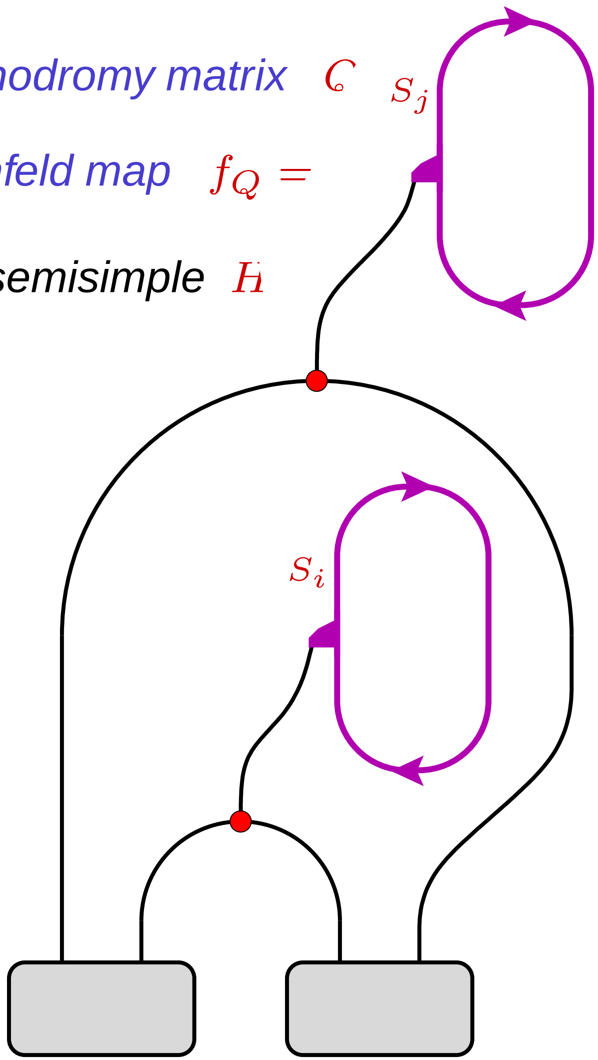

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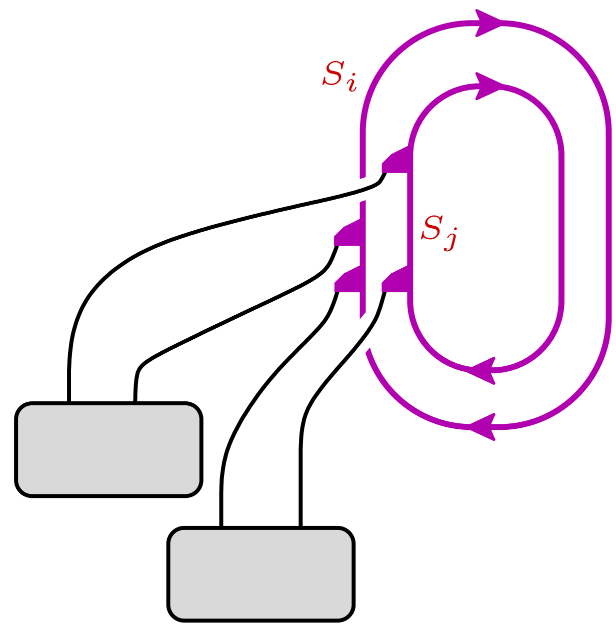

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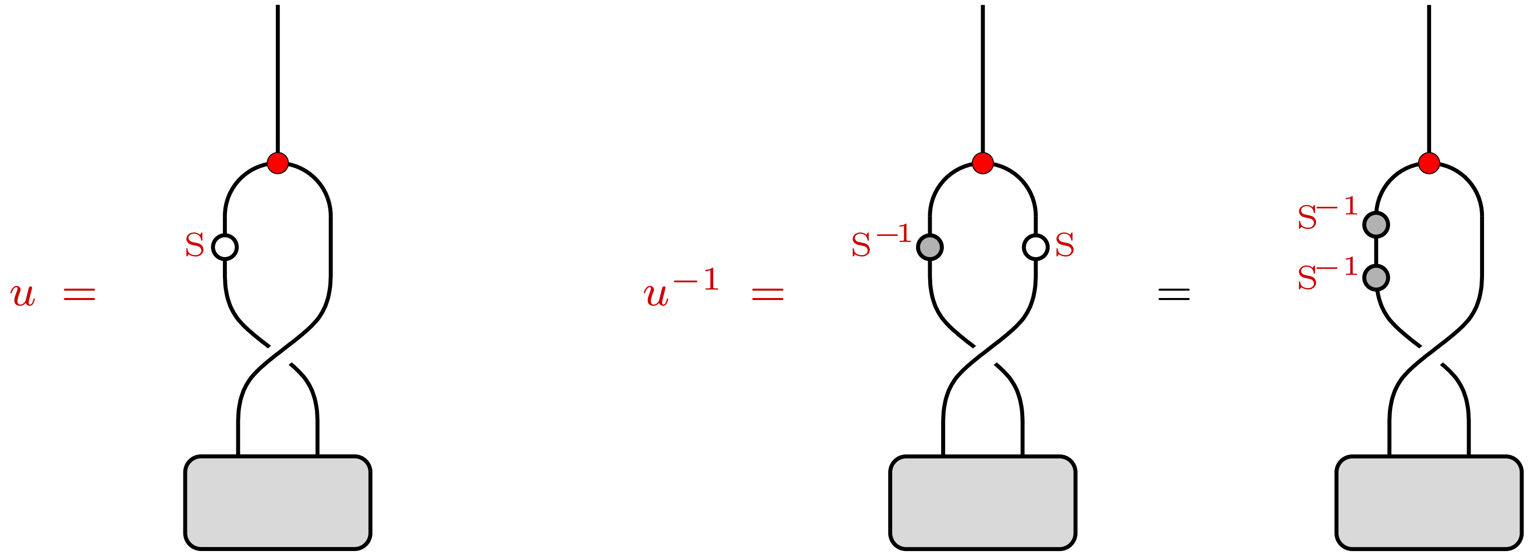
- for *semisimple* H : *S-matrix* $S_{i,j} = \chi_j \circ (f_Q(\chi_i))$
  - diagonalizes the fusion rules of H-mod
  - relation with RCFT Verlinde formula : new braiding on H-mod by combining the symmetric braiding (flip) of $\mathcal{V}ect_{\Bbbk}$ with R

▷ *canonical element* / Drinfeld element of H : $u = m \circ (\mathrm{S} \otimes \mathrm{id}_H) \circ R_{21}$



▷▷ $\mathrm{S}^2 = \mathrm{ad}_u$

▷▷ $\mathrm{ad}_t = \mathrm{S}^2 \implies t = t' * u$ with t' an invertible central element

▷▷ $\mathrm{S}^4 = \mathrm{ad}_{\widetilde{g}}$ with $\widetilde{g} = (\mathrm{S} \circ u^{-1}) * u$

▷▷ H unimodular $\implies \widetilde{g} = g$

Ribbon Hopf algebras

- *canonical element* / Drinfeld element of H : $u = m \circ (\mathrm{S} \otimes \mathrm{id}_H) \circ R_{21}$
- *Ribbon Hopf algebra* : existence of $v \in Z(H)$ s.t.

  $\mathrm{S} \circ v = v \qquad \varepsilon \circ v = 1 \qquad \Delta \circ v = (v \otimes v) * Q^{-1}$ (*ribbon element*)

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  - ▷▷ $v * v = u * (\mathrm{S} \circ u)$
  - ▷▷ v invertible
  - ▷▷ H ribbon $\Longrightarrow$ H-mod a ribbon category : acting with v^{-1} is the twist
  - ▷▷ $b = v^{-1} * u$ balancing element , $b^2 = g$

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- *factorizable* ribbon Hopf algebra : Drinfeld map f_Q invertible

from now on : H factorizable

- f_Q intertwines the left coadjoint and adjoint actions
- $\mathrm{Hig}(H)$ contains a non-zero idempotent

  $\Longrightarrow$ existence of a simple projective module
- H unimodular

▷ for any Hopf algebra H-mod is (equivalent to) a sovereign monoidal category and has absolutely simple tensor unit $\mathbf{1} = \mathbb{k}_\varepsilon$

▷ *categorical dimension* q-$\mathrm{dim}(M)$ of a H-module M is morphism $\mathbf{1} \to M \otimes M^\vee \to \mathbf{1}$

$\Longrightarrow$ if q-$\mathrm{dim}(P) \neq 0$ for a projective P then $\mathbf{1}$ is a retract of P and thus itself projective

$\Longrightarrow$ every projective module P over non-semisimple H has q-$\mathrm{dim}(P) = 0$

▷ in particular ordinary ($\mathcal{V}ect_{\mathbb{k}}$-) dimension / trace / dualities and categorical (H-mod-) dimension / trace / dualities differ

q-characters

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▷ for a *ribbon* Hopf algebra with balancing element b: q-trace obtained by twisting with b^{-1}: q-tr$(f) = \mathrm{tr}(\,\rho_M \circ (b^{-1} \otimes f)\,)$ for $f \in \mathrm{End}(M)$

▷ e.g. *q-character* $\chi_M^{b^{-1}} =$ q-tr$_M(\rho_M) = \chi_M \circ \llcorner_{b^{-1}} \equiv \chi_M \leftharpoonup b^{-1}$

▷ *space of q-characters* coincides with $O_{S^2}(H)$

▷ ring homomorphism $[M] \longmapsto \chi_M^{b^{-1}}$ from the Grothendieck ring of H-mod to $O_{S^2}(H)$

Quantum Fourier transform

▷ *Quantum Fourier transform* F : composition of Frobenius and Drinfeld maps

$F = f_Q \circ \Psi \; : \; H \to H$

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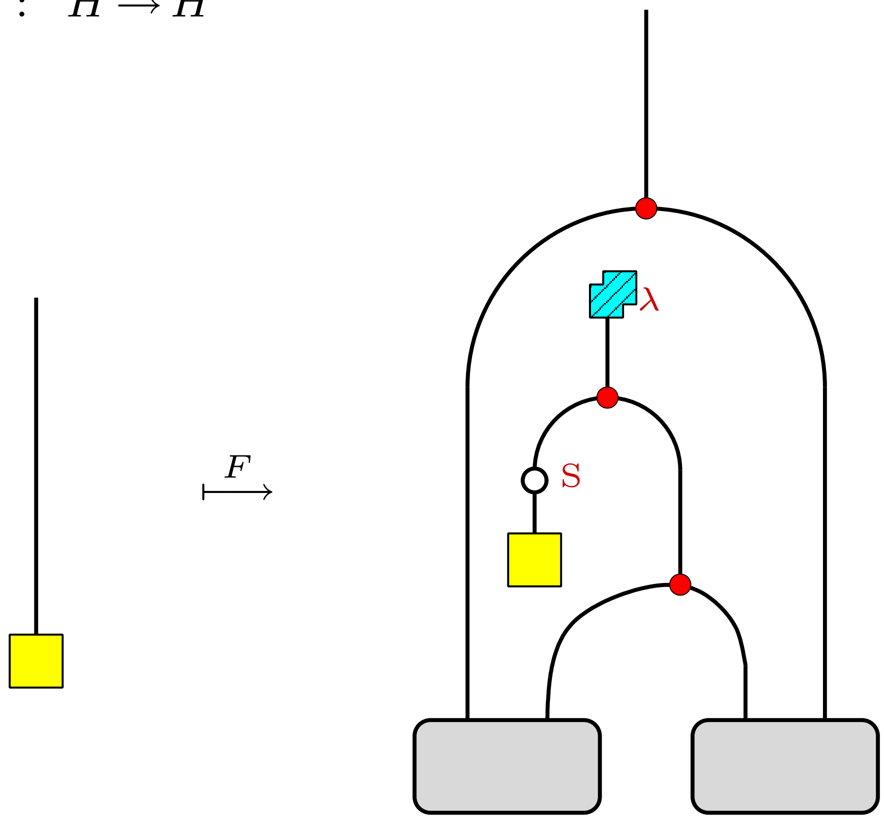


- ▷ *Quantum Fourier transform* F : composition of Frobenius and Drinfeld maps

  $F = f_Q \circ \Psi \;:\; H \to H$

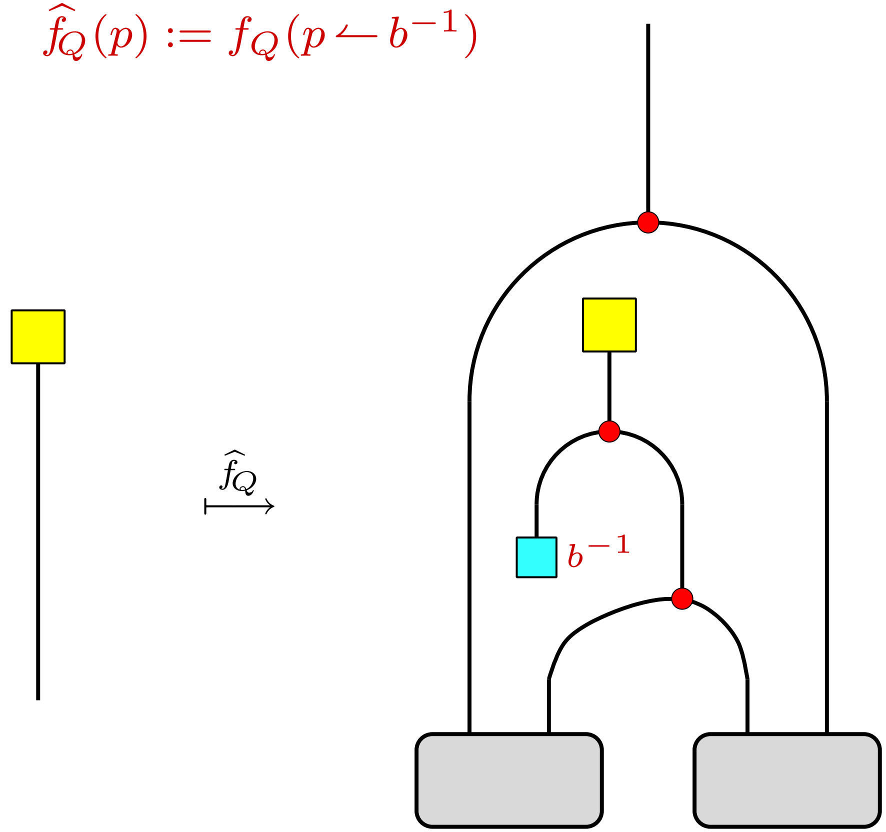
- ▷ F commutes with left adjoint action

- ▷ $F^2\big|_{Z(A)} = \mathrm{S}$

▷ *Quantum Fourier transform* F : composition of Frobenius and Drinfeld maps

$F = f_Q \circ \Psi \; : \; H \to H$

▷ *modified Drinfeld map* : $\widehat{f_Q}(p) := f_Q(p \leftharpoonup b^{-1})$



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- ▷ $(\widehat{\Psi} \circ \widehat{f_Q})^2 |_{C(H)} = \mathrm{S}$

- ▷ $\widehat{f_Q}$ is algebra isomorphism $C(H) \xrightarrow{\cong} Z(H)$

- ▷ $\mathrm{Hig}(H)$ is stable under the quantum Fourier transform

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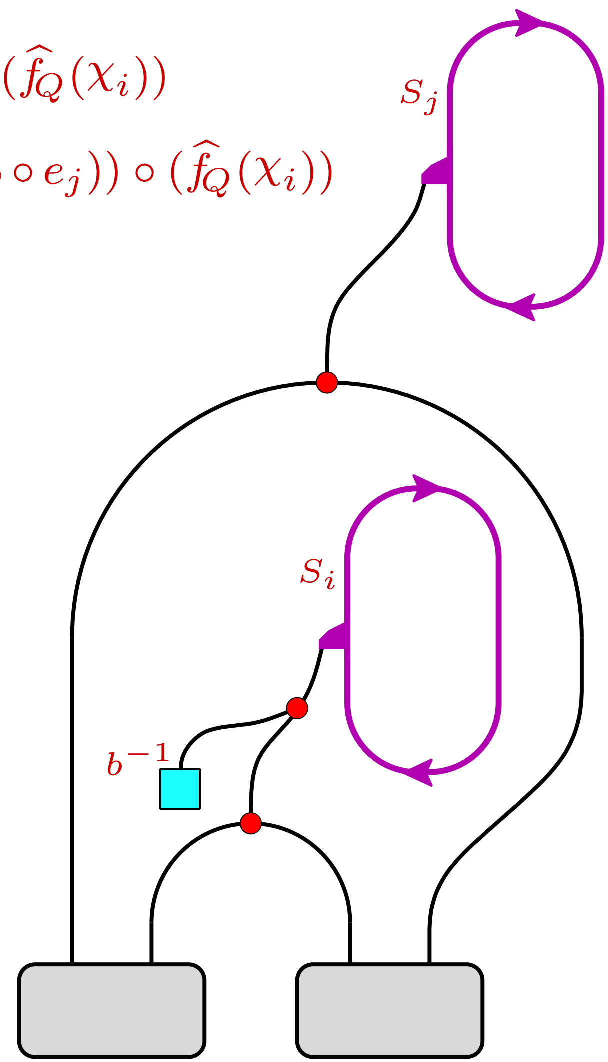

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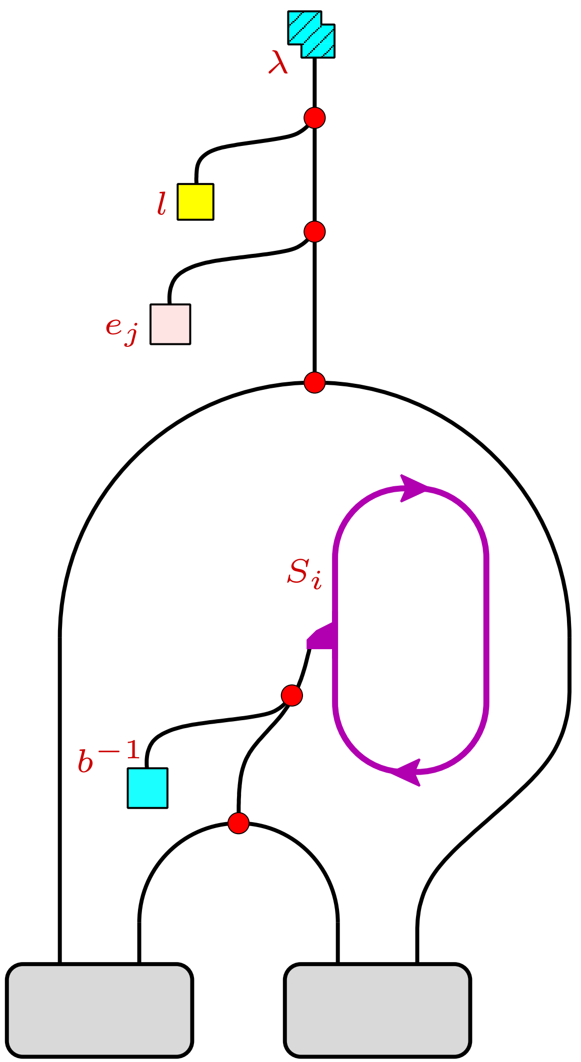

Cohen-Westreich formula

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- define $\mathcal{F} =$ matrix for the basis transformation between bases

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- define coefficients $\widehat{N}_{ij}^{\ \ k}$ by $\chi_{S_i}\, \chi_{P_j} = \sum_{k=1}^m \widehat{N}_{kj}^{\ \ i}\, \chi_{P_k}$

  for $i \in \mathcal{I}$ and $j, k \in \{1, 2, ..., m\} =: \mathcal{I}_\circ$

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Then $\mathcal{F} = (\mathtt{c}_H|_m)^{-1}\, (\mathtt{c}_H \widehat{S})|_m$ diagonalizes the matrices $\widehat{N}_i$