Percolation as a LCFT: logarithmic couplings and dualities

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with

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Context

- Minimal models describe the local observables at the critical point

- Non-local observables (crossing probabilities, fractal dimensions) appear to probe representations outside the Kac table

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Context

- Minimal models describe the local observables at the critical point
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- Problem considered here:

  how to deform the irreducible modules to probe (by fusion) the exterior of the Kac table

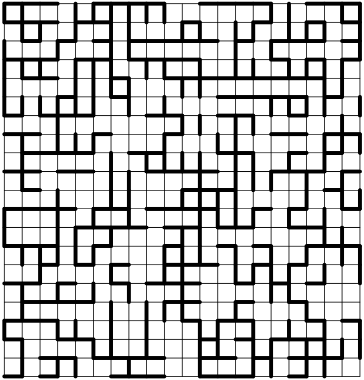
  guided by physical considerations
- Focus: percolation and the Cardy's formula

Percolation: the Cardy formula for crossing probability

Bound percolation and crossing probabilities

Bound open (p) or closed ($1-p$):

$$p_c = \frac{1}{2}: \qquad \pi_h = f(r) \qquad (r = \text{aspect ratio})$$



Percolation as a limiting Q-state Potts model

- Q-state Potts model: $\sigma_i \in \{1, \cdots, Q\}$

$$E = -J \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}$$

- $$Z = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \left((1-p) + p\delta_{\sigma_i, \sigma_j} \right)$$

  where

$$p = 1 - e^{-\beta J}$$

- $$Z = \sum p^{B_o} (1-p)^{B-B_o} Q^{N_c}$$

  B: # bonds; B_o: # open bonds; N_c: #clusters

- Percolation:

$$Q = 1 \quad \Rightarrow \quad Z = 1 \quad \Rightarrow \quad c = 0$$

Percolation as a CFT with $c = 0$

Continuum version of the Q-state Potts model is CFT with

$$c = 1 - \frac{6}{m(m+1)}$$

with

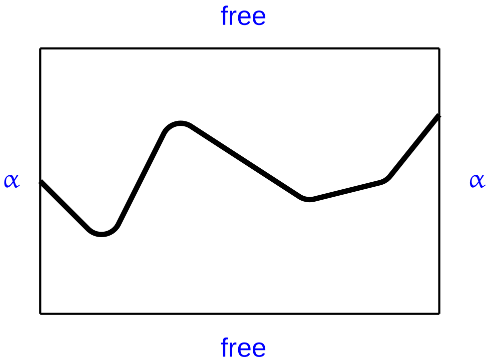
$$Q = 4\cos^2\left(\frac{\pi}{(m+1)}\right)$$

and

$$Q = 1 \quad \leftrightarrow \quad m = 2 \qquad \Rightarrow \qquad c = 0$$

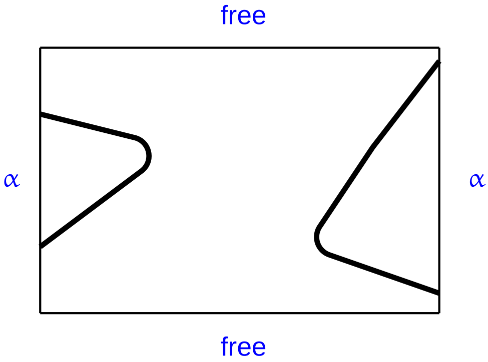
Crossing probability

Count the configurations



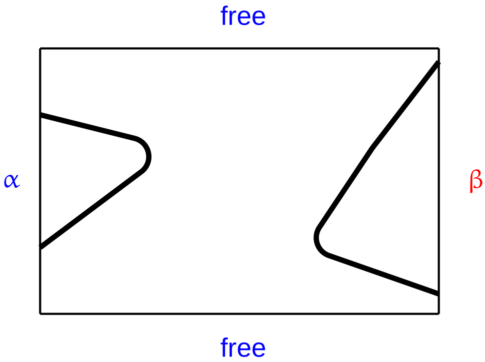
Crossing probability

Subtract



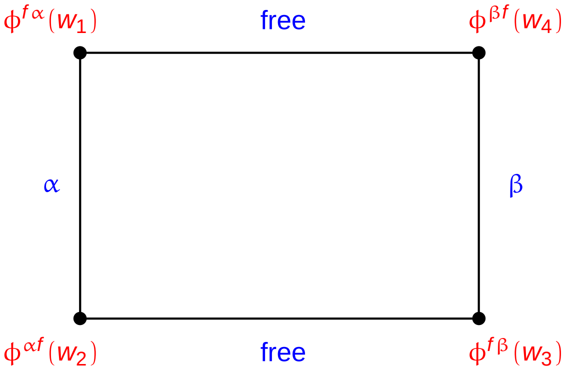
Crossing probability

Equivalently: subtract (with $\beta \neq \alpha$: excludes crossing)



Crossing probability as a four-point function

Introduce fields



Crossing probability as a four-point function

Cardy: crossing probability (mapped to UHP) is

$$\boxed{\pi_h(r) = \lim_{Q\to 1} (Z_{\alpha\alpha} - Z_{\alpha\beta})}$$

where

$$Z_{\alpha\beta} = \langle \phi^{f\alpha}(z_1)\phi^{\alpha f}(z_2)\phi^{f\beta}(z_3)\phi^{\beta f}(z_4)\rangle Z_f$$

Physical imput (Cardy)

- Scale invariance of $Z_{\alpha\alpha}/Z_f$ requires that

$$\phi^{f\alpha} \quad \text{has} \quad h = 0$$

- Boundary changing operator in Q-state Potts models:

$$\phi^{f\alpha} = \phi_{1,2}$$

- SV at level 2 $\quad \Rightarrow \quad$ ODE for $\langle \cdots \rangle$ $\quad \Rightarrow \quad$ fixes $\pi_h(r)$

$$\pi_h(r) = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2} x^{\frac{1}{3}} F(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; x)$$

Test of the Cardy formula

- Match perfectly the numerical data
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  (1997)

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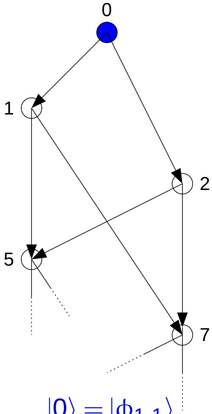
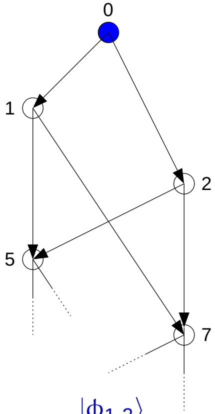
  (1997)
- Proved by Smirnov and by SLE techniques (2001)

Percolation as a logaritmic conformal field theory

Cardy formula does not fit within a minimal model

Module content of $M(2,3)$





$$|0\rangle = |\phi_{1,1}\rangle \quad = \quad |\phi_{1,2}\rangle$$

In fact the $M(2,3)$ model is trivial: only has $|0\rangle$

$$L_{-1}|0\rangle = L_{-2}|0\rangle = 0 \quad \Rightarrow \quad L_{-n}|0\rangle = 0 \qquad \forall n > 0$$

Minimal deformation of $M(2,3)$ that fits Cardy's result

To make the theory non-trivial, need to break

$$|\phi_{1,1}\rangle = |\phi_{1,2}\rangle$$

Need to modify the structure of the modules

But what needs to be kept?

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(global conformal invariance of the vacuum)

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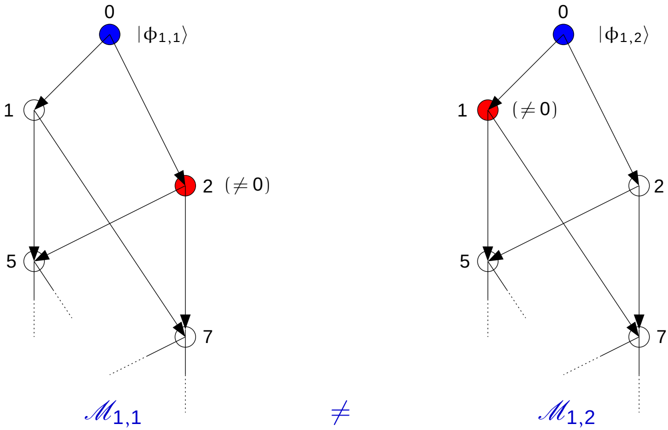
- $|\phi_{1,2}\rangle$ must have a vanishing SV at level 2:

$$\left(L_{-2} - \frac{3}{2}L_{-1}^{2}\right)|\phi_{1,1}\rangle = 0$$

  (SV $\Rightarrow$ ODE for the 4-pt function)

New modules $\mathscr{M}_{1,r}$: reducible but indecomposable

Minimal deformation of the modules (red SV $\neq 0$)



$T(z) \neq 0$ $\qquad$ $\partial\phi_{1,2}(z) \neq 0$

Constructing the deformed M(2,3) model

Building the theory:

- Take multiple fusions of the two basic modules $\mathscr{M}_{1,1}$ and $\mathscr{M}_{1,2}$

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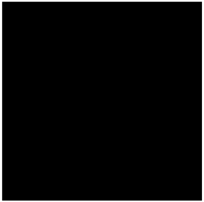
Nahm-Gaberdiel-Kausch algorithm



Fusion rules

- $\mathscr{M}_{1,1} \times \mathscr{M}_{1,1} = \mathscr{M}_{1,1}$
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The presence of $\mathscr{M}_{1,3}$

$\Downarrow$

this deformation forces us to leave the Kac table

- Consider $\mathscr{M}_{1,2} \times \mathscr{M}_{1,3}$; natural guess

$$\mathscr{M}_{1,2} \times \mathscr{M}_{1,3} = \mathscr{M}_{1,2} \oplus \mathscr{M}_{1,4}$$

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- Wrong: the correct result is

  $$\mathscr{M}_{1,2} \times \mathscr{M}_{1,3} = \mathscr{I}_{1,4}$$

  where $\mathscr{I}_{1,4}$ is indecomposable $\quad ([\mathscr{I}_{1,4}]_{v.s} \approx \mathscr{M}_{1,2} \oplus \mathscr{M}_{1,4})$

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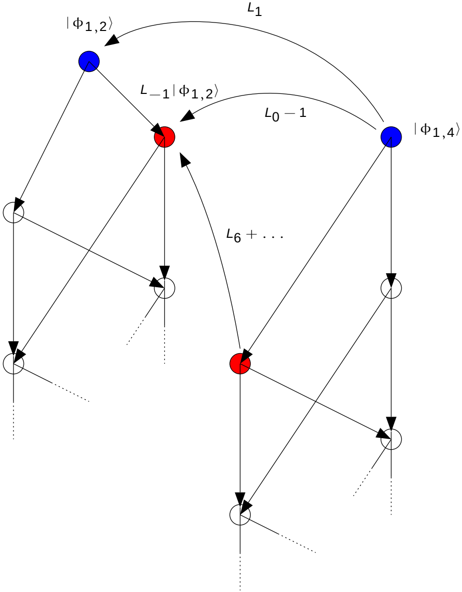
  The action of L_0 displays a Jordan cell structure:

$$L_0|\phi_{1,4}\rangle = |\phi_{1,4}\rangle + L_{-1}|\phi_{1,2}\rangle$$

  ... the defining property of a logarithmic CFT

The module $\mathscr{I}_{1,4}$



$$L_0|\phi_{1,4}\rangle = |\phi_{1,4}\rangle + L_{-1}|\phi_{1,2}\rangle$$

$$L_1|\phi_{1,4}\rangle \neq 0$$

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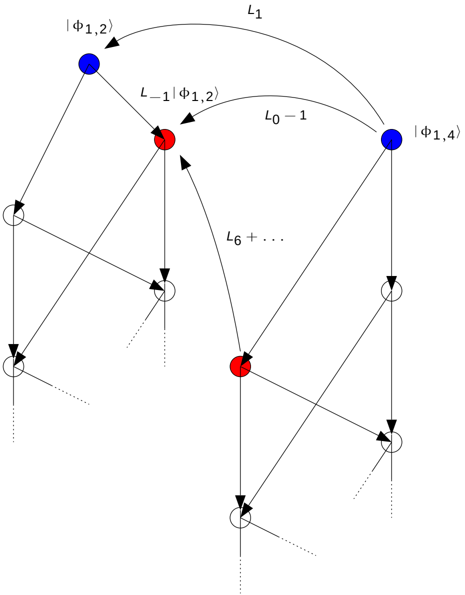

$$L_0|\phi_{1,4}\rangle = |\phi_{1,4}\rangle + L_{-1}|\phi_{1,2}\rangle$$

$$L_1|\phi_{1,4}\rangle = -\frac{1}{2}|\phi_{1,2}\rangle$$

Scalar products in $\mathscr{I}_{1,4}$

- Normalization:

$$\langle\phi_{1,2}|\phi_{1,2}\rangle = 1$$

- $L_{-1}|\phi_{1,2}\rangle$ is a SV $(\neq 0)$ but:

$$\langle\phi_{1,2}|L_1 L_{-1}|\phi_{1,2}\rangle = 0$$

- More generally

$$\langle\psi|L_{-1}|\phi_{1,2}\rangle = 0 \qquad \forall\ |\psi\rangle \in \mathscr{M}_{1,2}$$

the SV is orthogonal to all states in $\mathscr{M}_{1,2}$ (as usual)

- The ‘linking relation’

$$L_1|\phi_{1,4}\rangle = -\frac{1}{2}|\phi_{1,2}\rangle$$

implies that

$$\langle\phi_{1,4}|L_{-1}|\phi_{1,2}\rangle = -\frac{1}{2}$$

i.e., $L_{-1}|\phi_{1,2}\rangle$ is not orthogonal to states in $\mathscr{M}_{1,4}$

This is the where the effect of having a SV $\neq 0$ enters

- The two linking relations:

$$L_0|\phi_{1,4}\rangle = |\phi_{1,4}\rangle + L_{-1}|\phi_{1,2}\rangle \qquad \text{and} \qquad L_1|\phi_{1,4}\rangle = -\frac{1}{2}|\phi_{1,2}\rangle$$

imply

$$T(z)\phi_{1,4}(w) = -\frac{1}{2}\frac{\phi_{1,2}(w)}{(z-w)^3} + \frac{\phi_{1,4}(w) + \partial\phi_{1,2}(w)}{(z-w)^2} + \frac{\partial\phi_{1,4}(w)}{z-w} + \dots$$

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- With global conformal invariance this implies

$$(z\partial_z + w\partial_w + 2)\langle\phi_{1,4}(z)\phi_{1,4}(w)\rangle = \frac{1}{(z-w)^2}$$

with solution

$$\langle\phi_{1,4}(z)\phi_{1,4}(w)\rangle = \frac{A + \ln(z-w)}{(z-w)^2}$$

$$\langle \phi_{1,4}(z)\phi_{1,4}(w)\rangle = \frac{A + \ln(z-w)}{(z-w)^2}$$

- Correlation functions contain logs: a genuine LCFT

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- A is arbitrary – we can set $A = 0$ in using a gauge transformation

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- Correlation functions contain logs: a genuine LCFT
- A is arbitrary – we can set $A = 0$ in using a gauge transformation
- The scalar product diverges

$$\langle \phi_{1,4}|\phi_{1,4}\rangle = \lim_{z\to\infty} z^2 \langle \phi_{1,4}(z)\phi_{1,4}(0)\rangle$$
$$\to \infty$$

Fusion rules up to this point

- $\mathscr{M}_{1,1} \times \mathscr{M}_{1,1} = \mathscr{M}_{1,1}$
- $\mathscr{M}_{1,1} \times \mathscr{M}_{1,2} = \mathscr{M}_{1,2}$
- $\mathscr{M}_{1,2} \times \mathscr{M}_{1,2} = \mathscr{M}_{1,1} \oplus \mathscr{M}_{1,3}$
- $\mathscr{M}_{1,2} \times \mathscr{M}_{1,3} = \mathscr{I}_{1,4}$

Another fusion rule

- $$\mathscr{M}_{1,3} \times \mathscr{M}_{1,3} = \mathscr{M}_{1,3} \oplus \mathscr{I}_{1,5}$$

  where
  $$[\mathscr{I}_{1,5}]_{v.s.} \approx \mathscr{M}_{1,1} \oplus \mathscr{M}_{1,5}$$

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- $|\phi_{1,5}\rangle$ is coupled to the energy-momentum tensor

  $$L_0|\phi_{1,5}\rangle = 2|\phi_{1,5}\rangle + L_{-2}|0\rangle$$

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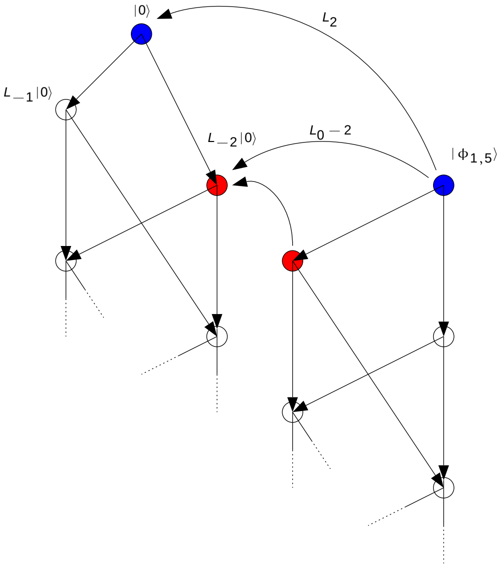
  $$L_0|\phi_{1,5}\rangle = 2|\phi_{1,5}\rangle + L_{-2}|0\rangle$$

- Also

  $$L_2|\phi_{1,5}\rangle = -\frac{5}{8}|0\rangle$$

The module $\mathscr{I}_{1,5}$



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Logarithmic couplings

Logarithmic coupling $\beta_{1,4}$

$|\chi_{1,2}\rangle \equiv L_{-1}|\phi_{1,2}\rangle \neq 0$

The linking relation

$$L_1|\phi_{1,4}\rangle = -\frac{1}{2}|\phi_{1,2}\rangle$$

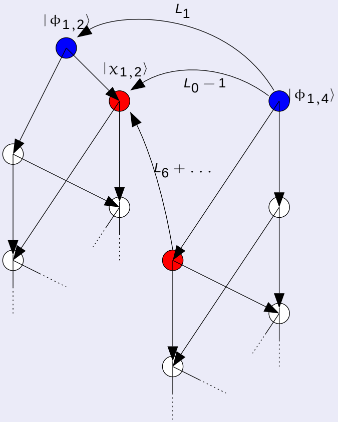
implies

$$\langle\phi_{1,2}|L_1|\phi_{1,4}\rangle = -\frac{1}{2}\langle\phi_{1,2}|\phi_{1,2}\rangle = -\frac{1}{2}$$

i.e.,

$$\langle\chi_{1,2}|\phi_{1,4}\rangle \equiv \beta_{1,4} = -\frac{1}{2}$$

The module $\mathscr{I}_{1,4}$



$\beta_{1,4}$ is gauge invariant

$$\beta_{1,4} = \langle \chi_{1,2} | \phi_{1,4} \rangle$$

is invariant under the gauge transformation

$$|\phi_{1,4}\rangle \rightarrow |\phi'_{1,4}\rangle = |\phi_{1,4}\rangle + \alpha |\chi_{1,2}\rangle$$

that preserves the Jordan cell structure

$$L_0 |\phi_{1,4}\rangle = |\phi_{1,4}\rangle + L_{-1} |\phi_{1,2}\rangle$$

$$\Downarrow$$

$$L_0 |\phi'_{1,4}\rangle = |\phi'_{1,4}\rangle + L_{-1} |\phi_{1,2}\rangle$$

Gauge invariance and correlators

A is arbitrary in

$$\langle \phi_{1,4}(z)\phi_{1,4}(w)\rangle = \frac{A + \ln(z-w)}{(z-w)^2}$$

We can set $A = 0$ in using a gauge transformation

$$|\phi_{1,4}\rangle \to |\phi'_{1,4}\rangle = |\phi_{1,4}\rangle + \alpha|\chi_{1,2}\rangle$$

Logarithmic coupling $\beta_{1,5}$

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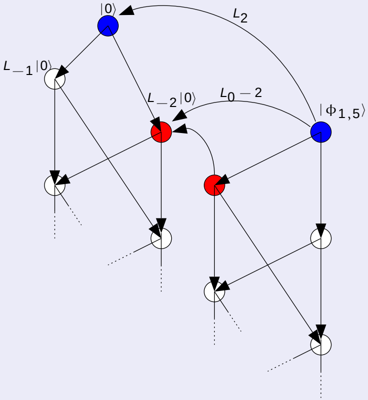


$|\chi_{1,1}\rangle \equiv L_{-2}|\phi_{1,1}\rangle \neq 0$

The linking relation

$$L_2|\phi_{1,5}\rangle = -\frac{5}{8}|\phi_{1,1}\rangle$$

implies

$$\langle\phi_{1,1}|L_2|\phi_{1,5}\rangle = -\frac{5}{8}\langle\phi_{1,1}|\phi_{1,1}\rangle = -\frac{5}{8}$$

i.e.

$$\langle\chi_{1,1}|\phi_{1,5}\rangle \equiv \beta_{1,5} = -\frac{5}{8}$$

$\beta_{1,5}$ is gauge invariant

Similarly

$$\beta_{1,5} = \langle \chi_{1,1} | \phi_{1,5} \rangle$$

is invariant under

$$|\phi_{1,5}\rangle \to |\phi'_{1,5}\rangle = |\phi_{1,5}\rangle + \alpha |\chi_{1,1}\rangle$$

Gauge transformation

General gauge transformation (module dependent)

$$|\phi_{1,s}\rangle \to |\phi'_{1,s}\rangle = |\phi_{1,s}\rangle + |\psi\rangle$$

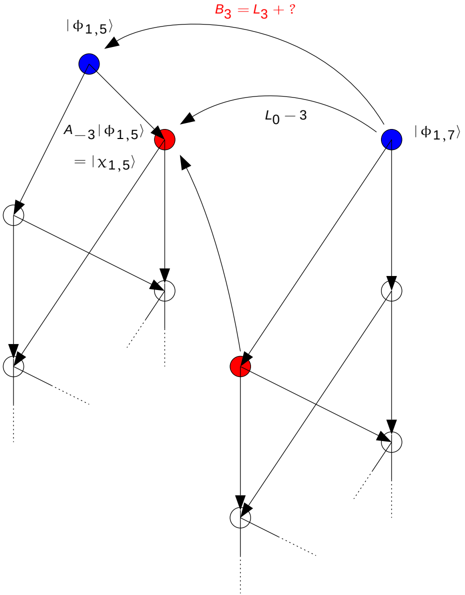
preserves the Jordan cell structure

$$L_0|\phi_{1,s}\rangle = h_{1,s}|\phi_{1,s}\rangle + |\text{Log partner}\rangle$$

$|\psi\rangle$ is a linear combination of terms of dimension $h_{1,s}$

Above cases $(s = 4,5)$: $|\psi\rangle$ is 'unique' (= $\alpha|\chi_{1,s}\rangle$)

The module $\mathscr{I}_{1,7} \supset \mathscr{M}_{1,3} \times \mathscr{I}_{1,5}$



$$L_0|\phi_{1,7}\rangle = 3|\phi_{1,7}\rangle + A_{-3}|\phi_{1,5}\rangle$$

$$A_{-3} = L_{-3} - L_{-2}L_{-1} + \frac{1}{6}L_{-1}^3$$

$\beta_{1,7}$: gauge invariant definition

Look for B_3

$$B_3|\phi_{1,7}\rangle = \beta_{1,7}|\phi_{1,5}\rangle$$

with

$$B_3 = L_3 + \gamma_1 L_1 L_2 + \gamma_2 L_1^3$$

such that $\beta_{1,7}$ is invariant under

$$|\phi_{1,7}\rangle \to |\phi'_{1,7}\rangle = |\phi_{1,7}\rangle + |\psi\rangle$$

where

$$|\psi\rangle = (L_{-3} + \alpha_1 L_{-2} L_{-1} + \alpha_2 L_{-1}^3)|\phi_{1,5}\rangle$$

$\beta_{1,7}$: gauge invariant definition

Gauge invariance forces

$$B_3|\phi_{1,7}\rangle = B_3|\phi'_{1,7}\rangle = B_3(|\phi_{1,7}\rangle + |\psi\rangle)$$

or

$$B_3|\psi\rangle = 0$$

$$|\psi\rangle = (L_{-3} + \alpha_1 L_{-2}L_{-1} + \alpha_2 L_{-1}^3)|\phi_{1,5}\rangle$$

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or

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$$|\psi\rangle = (L_{-3} + \alpha_1 L_{-2}L_{-1} + \alpha_2 L^3_{-1})|\phi_{1,5}\rangle$$

Since $B_3|\psi\rangle \in \mathscr{M}_{1,5}$:

$$\langle\phi_{1,5}|B_3|\psi\rangle = 0$$

$\beta_{1,7}$: gauge invariant definition

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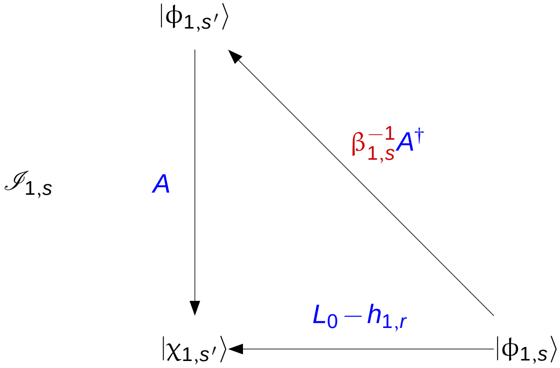
Solution:

$$\boxed{B_3 = (A_{-3})^\dagger \qquad \text{so that} \quad \langle\phi_{1,5}|B_3 = \langle\chi_{1,5}|}$$

since

$$\langle\chi_{1,5}|\psi\rangle = 0 \qquad \forall\alpha_1, \alpha_2$$

Logarithmic coupling : gauge invariant definition



$$\beta_{1,s} = \langle \chi_{1,s'} | \phi_{1,s} \rangle = \langle \phi_{1,s'} | A^\dagger | \phi_{1,s} \rangle$$

$\beta_{1,s}$ is fixed by the normalization of $|\chi_{1,s'}\rangle$ $(L_{-n} + \alpha L_{-(n-1)} L_{-1} + \cdots)$

Existence of β's: [Gaberdiel-Kausch and Eberle-Flohr]

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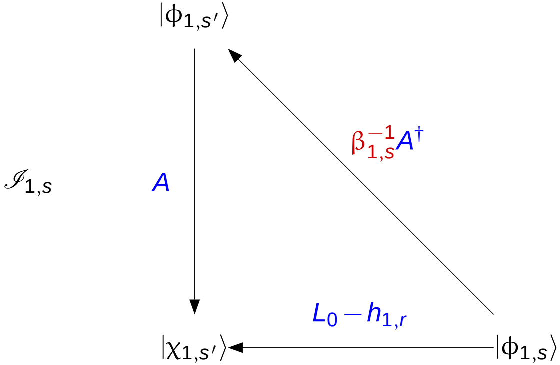

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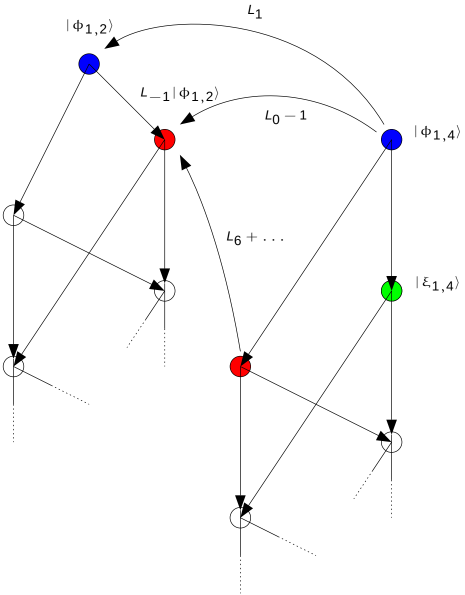
$\mathscr{I}_{1,s}$: Staggered module [Roshiepe]

Kytölä–Ridout: “On Staggered Indecomposable Virasoro Modules”

Logarithmic couplings: new definition via SV ($\mathscr{I}_{1,4}$)



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$$|\xi_{1,4}\rangle = \left(L_{-4} - L_{-3}L_{-1} - L_{-2}^{2} + \frac{5}{3}L_{-2}L_{-1}^{2} - \frac{1}{4}L_{-1}^{4}\right)|\phi_{1,4}\rangle$$

$$+ \quad \left(a_1 L_{-5} + a_2 L_{-4}L_{-1} + a_3 L_{-3}L_{-2} + a_4 L_{-2}^{2}L_{-1}\right)|\phi_{1,2}\rangle$$

and let $\beta_{1,4}$ be free:

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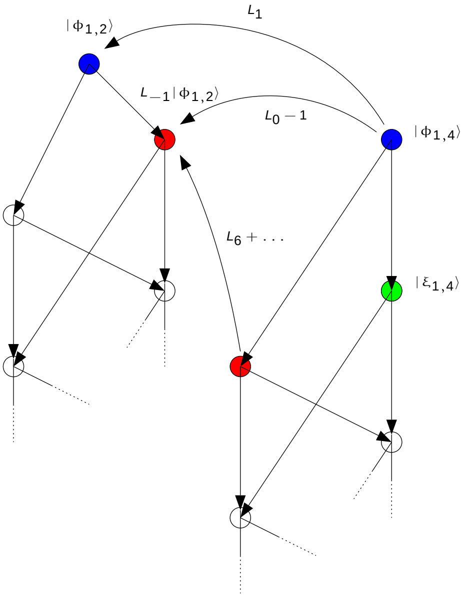

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$$\Rightarrow \quad \beta_{1,4} = -\frac{1}{2}$$

Percolation: spectrum of the theory

Spectrum of the theory

Multiple fusions of $\mathscr{M}_{1,2}$ generate all $\mathscr{M}_{1,s}$ for $s \geq 1$ as:

- $\mathscr{M}_{1,3n}$

- $\mathscr{I}_{1,s}$ with $s \neq 3n$ and

$$[\mathscr{I}_{1,s}]_{v.s.} \approx \mathscr{M}_{1,s} \oplus \mathscr{M}_{1,s'} \quad where \quad s' = \begin{cases} s-2 & \text{if } s = 1 \bmod 3 \\ s-4 & \text{if } s = 2 \bmod 3 \end{cases}$$

- All exponents $h_{1,s}$ with $s \geq 1$ appear

$$\{h_{1,r}\} = \left\{0, 0, \tfrac{1}{3}, 1, 2, \tfrac{10}{3}, 5, 7, \tfrac{28}{3}, 12, \cdots\right\}$$

- This is the minimal spectrum that fits the Cardy's formula

Spectrum extension: no-go theorem

Can we add more fields/modules in the theory?

e.g.: $\mathscr{M}_{2,1}$ with

$$\left(L_{-2} - \frac{2}{3} L_{-1}^2\right) |\phi_{2,1}\rangle = 0$$

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Fusion rules

$$\mathscr{M}_{2,1} \times \mathscr{M}_{2,1} = \mathscr{I}_{3,1}$$

where

$$[\mathscr{I}_{3,1}]_{v.s.} \approx \mathscr{M}_{1,1} \oplus \mathscr{M}_{3,1}$$

with $h_{3,1} = 2$ and

$$L_2 |\phi_{3,1}\rangle \equiv \beta_{3,1} |0\rangle = \frac{5}{6} |0\rangle$$

Evaluate the 2-point function

$$\langle \phi_{1,5}(z)\phi_{3,1}(w)\rangle$$

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L_{-1} and L_0 Ward identities (for translation and scale invariance) yield

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Since

$$\beta_{3,1} = \frac{5}{6} \neq \beta_{1,5} = -\frac{5}{8}$$

$\Rightarrow$ the addition of $\mathscr{M}_{2,1}$ in the theory violates conformal invariance

- Gurarie-Ludwig-type argument:

  Logarithmic extension of the Virasoro algebra:

  $T(z)$ and $t(z)$ s.t.

$$\langle T(z)t(w)\rangle = \frac{b}{(z-w)^4}$$

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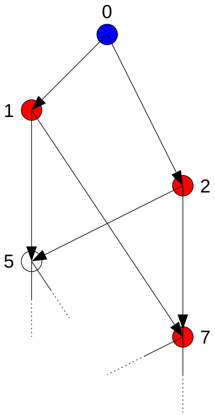
- This conclusion holds for a BCFT

Need for extension: Watts' formula for π_{hv}

Ridout's proposal: 4-pt correlation of a field of $h = 0$ and with



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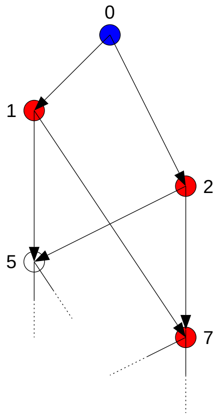

This fixes $\mathscr{M}_{2,5/2}$: fusion generates fields with $h_{2,(2k+1)/2}$

$\mathscr{M}_{2,5/2} \subset$ a rank-two staggered module containing $|0\rangle$ with $\langle 0|0\rangle = 0$

Dualities

Percolation and SLE

- Percolation (minimal) = theory generated by fusions of $\mathscr{M}_{1,2}$

  This is a logarithmic deformation of $M(2,3)$, call it $LM(2,3)$

  [$\neq LM(2,3)$ model of Pearce-Rasmussen-Zuber]

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- Cardy's formula is proved by SLE and $LM(2,3)$:

$$LM(2,3) \sim SLE(6)$$

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- Kytölä: *From SLE to the operator content of percolation*

  confirms the resulting structure from the SLE point of view

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- Recall: $\mathscr{M}_{1,2}$ and $\mathscr{M}_{2,1}$ are mutually exclusive
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$$L^*M(2,3) \sim SLE(8/3)$$

$SLE(8/3)$ = Self-avoiding walks

Recall: SLE $\leftrightarrow$ CFT is via a SV(κ) at level 2

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- Column-row duality: [Read-Saleur]

Percolation: boundary vs bulk

- Integrable perturbation of the Q-state Potts model

$$A_Q(\tau) = A_{\mathrm{CFT(Q)}} + \tau \int \phi_{2,1}(z)\, \bar{\phi}_{2,1}(\bar{z})\, d^2x$$

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- This is (expected to be) a (bulk) LCFT

$$\phi_{2,1}^{\mathrm{bulk}} \sim \mathscr{M}_{2,1}$$

– Integrability requires the SV at level 2

– $\phi_{2,1}$ is outside the Kac table: the modules cannot be irreducible

- Gurarie-Ludwig type “no-go theorem”:

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- Spectrum (percolation):

$$\text{boundary} \quad \{h_{1,s}\} \qquad \text{bulk}: \quad \{h_{r,1}\}$$

$$\beta_{1,5}^{\text{bdry}} = -\frac{5}{8} \qquad \beta_{3,1}^{\text{bulk}} = \frac{5}{6}$$

[GL orginal claim is ok]

Similar proposal [Simmons-Cardy]

- Dual version for self-avoiding walks (with $\phi_{1,3}$ perturbation)