$\mathcal{W}$-Extended Logarithmic Minimal Models

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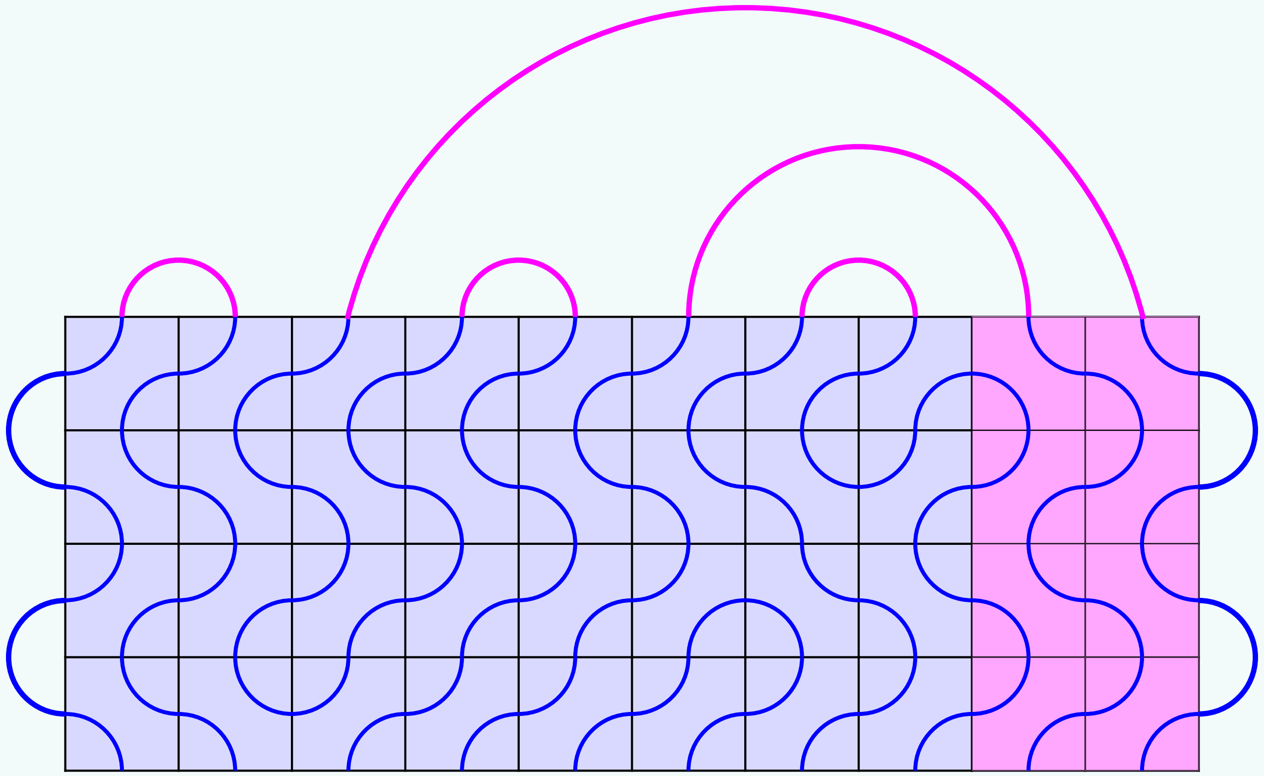
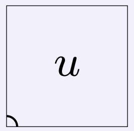
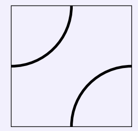
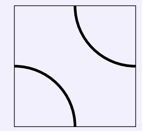


Collaborators: Paul A. Pearce, Philippe Ruelle, Yvan Saint-Aubin, Jean-Bernard Zuber

Logarithmic Minimal Models $\mathcal{LM}(p, p')$

- **Face Operators Defined in Planar Temperley-Lieb Algebra (Jones 1999)**

$X(u) =$  $= \dfrac{\sin(\lambda - u)}{\sin\lambda}$  $+ \dfrac{\sin u}{\sin\lambda}$  ; $\quad X_j(u) = \dfrac{\sin(\lambda - u)}{\sin\lambda} I + \dfrac{\sin u}{\sin\lambda} e_j$

$1 \le p < p'$ coprime integers, $\quad \lambda = \dfrac{(p'-p)\pi}{p'} =$ crossing parameter

$u =$ spectral parameter, $\quad \beta = 2\cos\lambda =$ fugacity of loops

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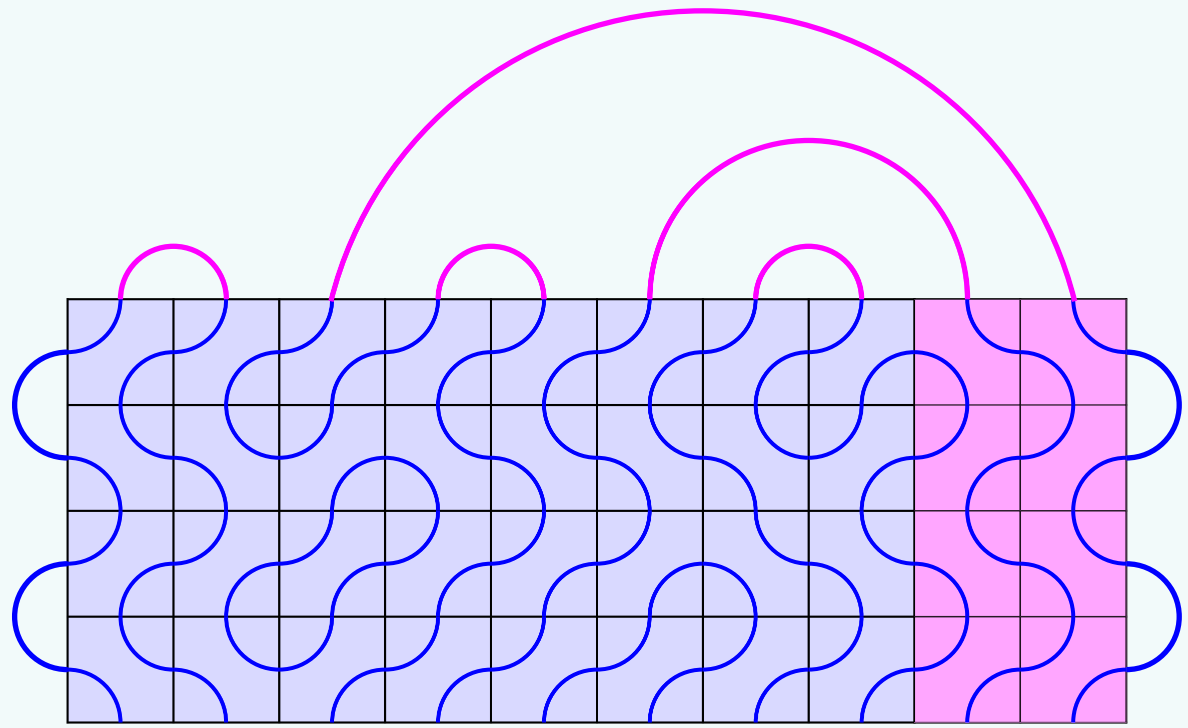
Planar Algebra
(Temperley-Lieb Algebra)

↓ YBE

Nonlocal Statistical Mechanics
(Yang-Baxter Integrable Link Models)

continuum limit ↓ ↑ lattice realization

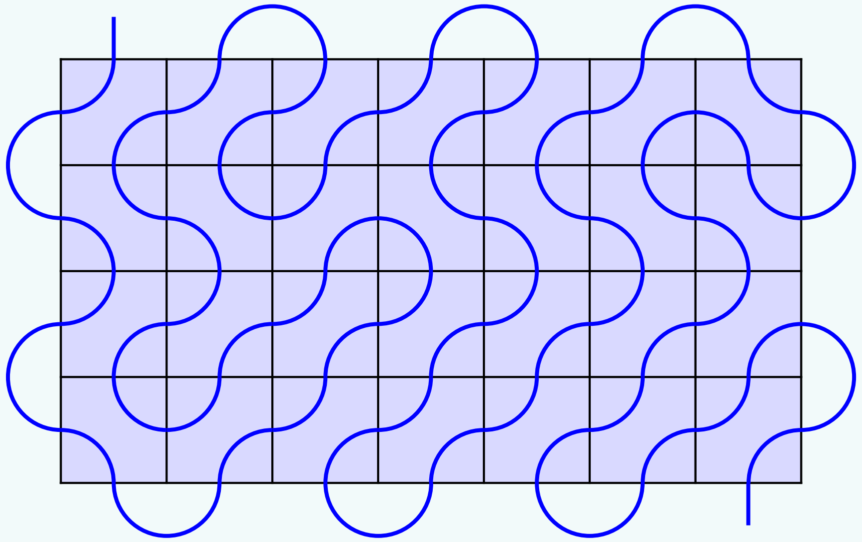
Logarithmic CFTs
(Logarithmic Minimal Models)



Nonlocal Degrees of Freedom

Polymers and Percolation on the Lattice

- **Critical Dense Polymers:** $(p,p')=(1,2)$, $\lambda=\frac{\pi}{2}$

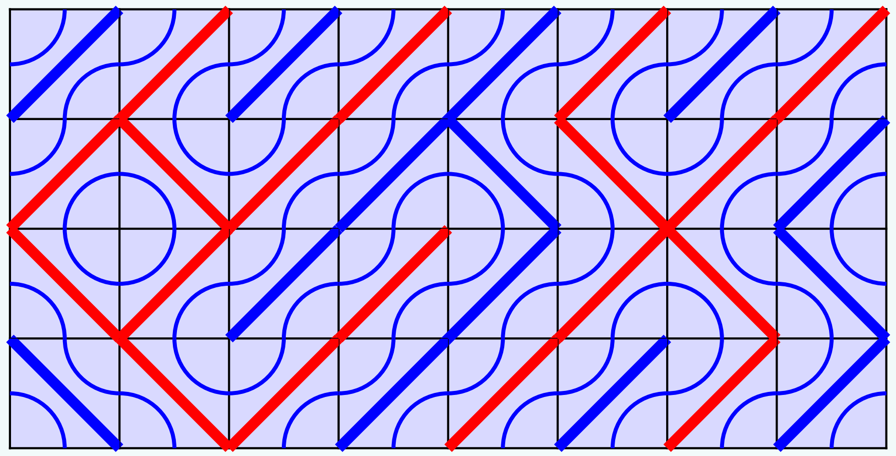


$$d_{path}^{SLE}=2-2\Delta_{p,p'-1}=2, \qquad \kappa=\frac{4p'}{p}=8$$

$\Delta_{1,1}=0$ lies outside rational $\mathcal{M}(1,2)$ Kac table

$\beta=0 \;\Rightarrow\;$ no loops $\;\Rightarrow\;$ space-filling dense polymer

- **Critical Percolation:** $(p,p')=(2,3)$, $\lambda=\frac{\pi}{3}$, $u=\frac{\lambda}{2}=\frac{\pi}{6}$ (isotropic)



$$d_{path}^{SLE}=2-2\Delta_{p,p'-1}=\frac{7}{4}, \qquad \kappa=\frac{4p'}{p}=6$$

$\Delta_{2,2}=\frac{1}{8}$ lies outside rational $\mathcal{M}(2,3)$ Kac table

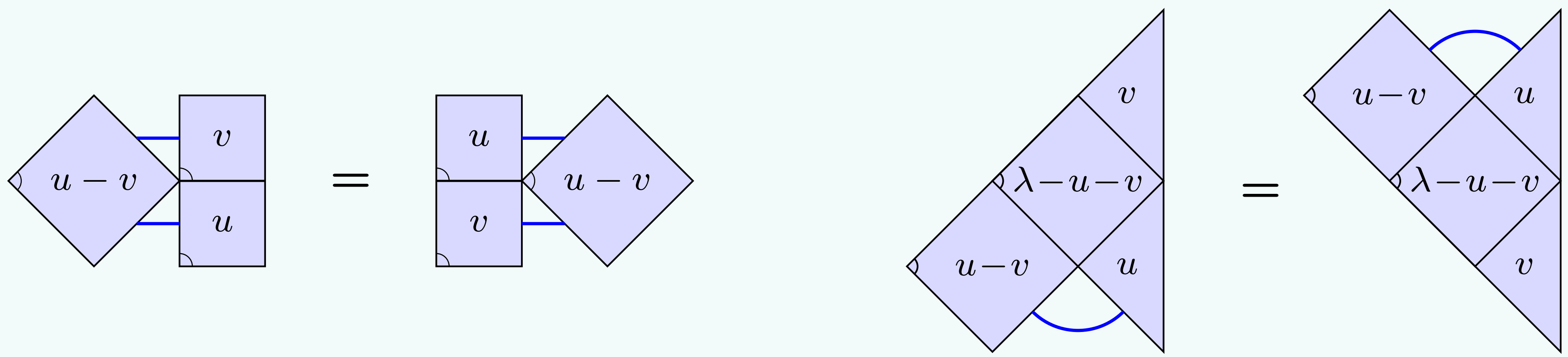
Bond percolation on the blue square lattice:

Critical probability $=p_c=\sin(\lambda-u)=\sin u=\frac{1}{2}$

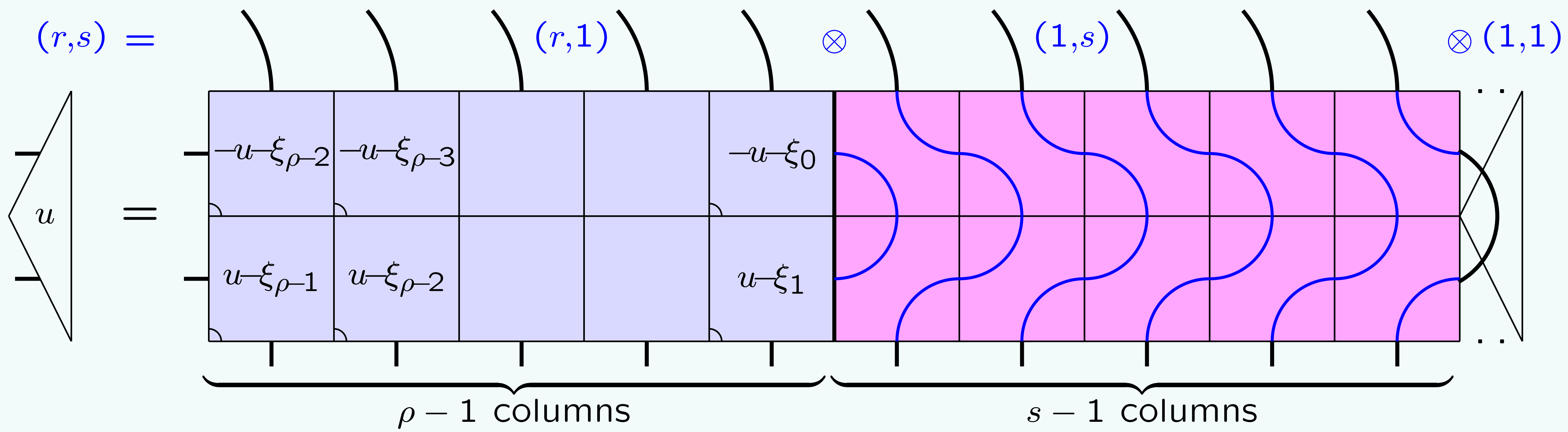
$\beta=1 \quad\Rightarrow\quad$ local stochastic process

Yang-Baxter Equations and Boundary Conditions

- **Yang-Baxter Equations**



- Equality is the equality of N-tangles.
- (r, s) **Solution** $r, s \in \mathbb{N}$, ρ is related to r, and ξ_k is linear in λ.

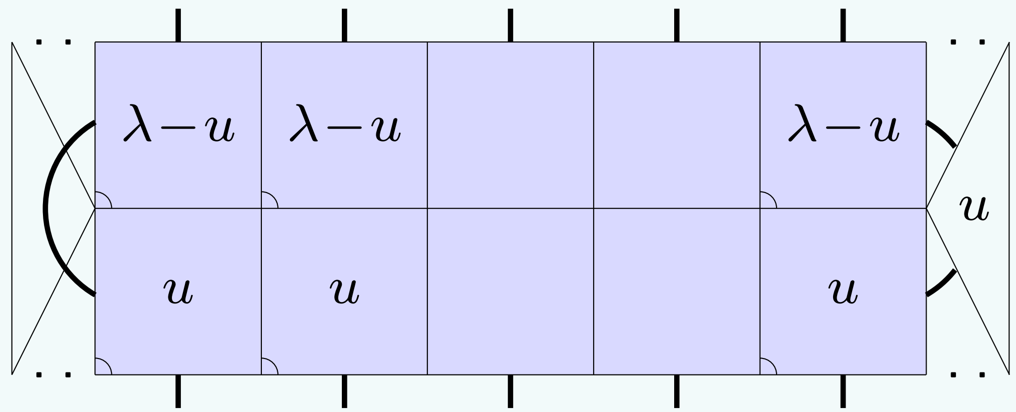


- Left boundary conditions are constructed similarly.

Double-Row Transfer Matrix

- For a strip with N columns, the double-row transfer "matrix" is the N-tangle

$\boldsymbol{D}(u) =$



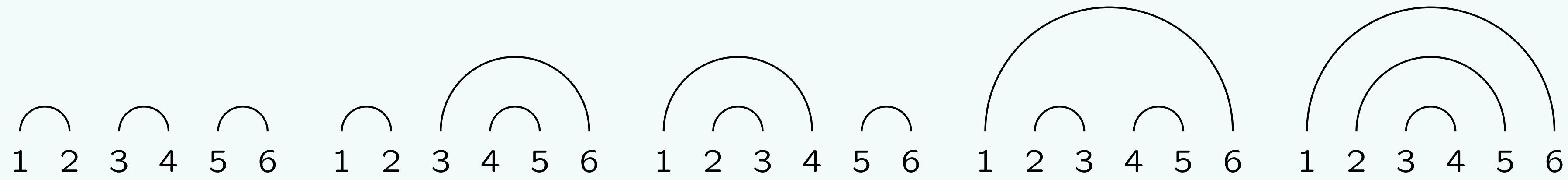
- Using the Yang-Baxter and Boundary Yang-Baxter Equations in the planar Temperley-Lieb algebra, it can be shown that, for any (r, s), the double-row transfer tangles **commute** and are **crossing symmetric**

$$\boldsymbol{D}(u)\boldsymbol{D}(v) = \boldsymbol{D}(v)\boldsymbol{D}(u), \qquad \boldsymbol{D}(u) = \boldsymbol{D}(\lambda - u)$$

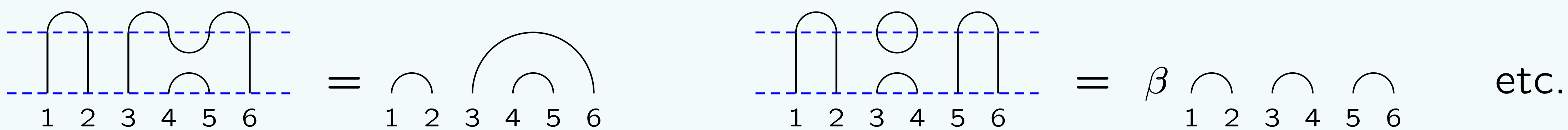
- Multiplication is vertical concatenation of diagrams.
- Act on **vector spaces** of states to obtain **matrix realizations** and spectra.

Planar Link Diagrams

- The planar N-tangles act on a vector space $\mathcal{V}_N$ of *planar link diagrams*. The dimension of $\mathcal{V}_N$ is given by Catalan numbers. For $N = 6$, there is a basis of 5 link diagrams:



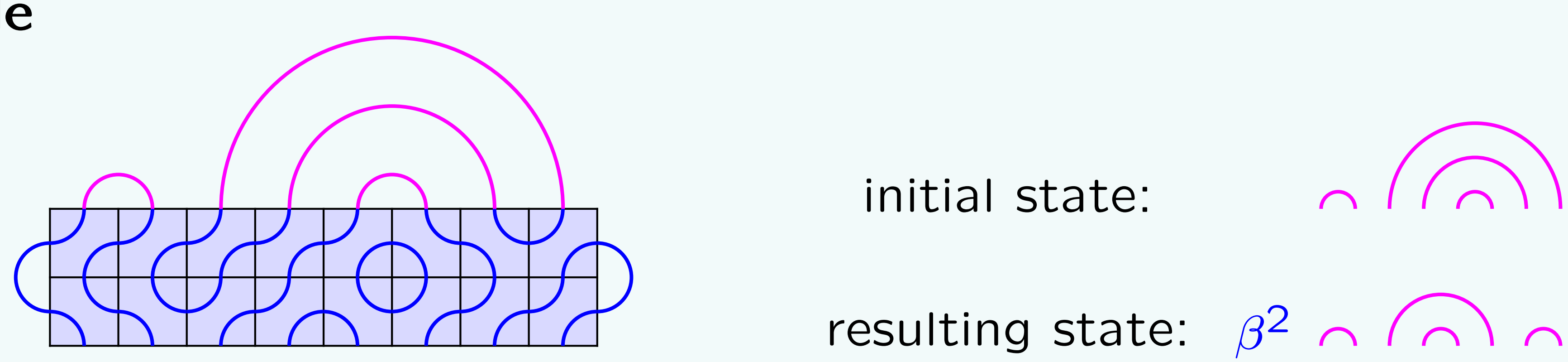
- The first link diagram is the reference state. Other states are generated by the action of the TL generators by concatenation from below



- The action of the TL generators on the states is non-local. It leads to matrices with entries $0, 1, \beta$ that represent the TL generators. For $N = 6$, the action of e_1 and e_2 on $\mathcal{V}_6$ is

$$e_1 = \begin{pmatrix} \beta & 0 & 1 & 0 & 1 \\ 0 & \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \beta & 0 & 0 \\ 0 & 1 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{etc.}$$

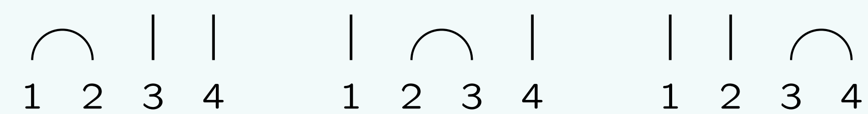
- **Example**



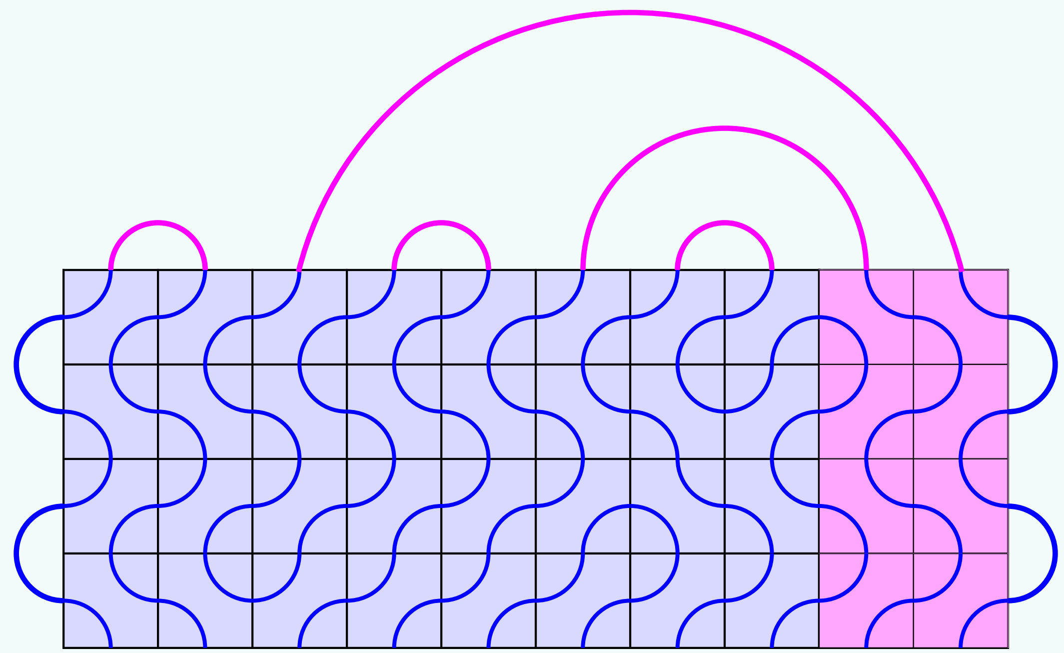
Defects

- More generally, the vector space of states $\mathcal{V}_N^{(\ell)}$ can contain ℓ **defects**

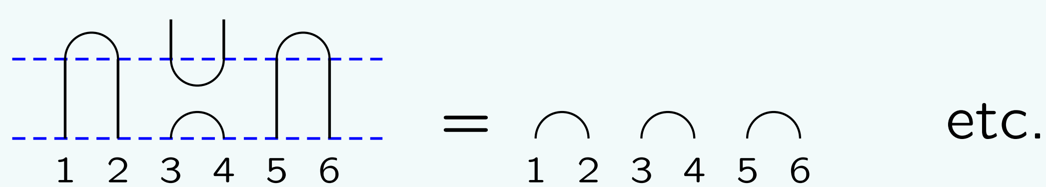
$N = 4,\ \ell = 2:$



- The ℓ defects can be closed on the right or the left. In this way, the number of defects propagating in the bulk is controlled by the boundary conditions. In particular, for $(1, s)$ boundary conditions, the $\ell = s - 1$ defects simply propagate along a boundary



- Defects in the bulk can be annihilated in pairs but not created under the action of TL



- The transfer matrices are thus **block-triangular** with respect to the number of defects.

Conformal Field Theory and Kac Representations

- With only one non-trivial (r,s)-type boundary condition, the double-row transfer matrix is found to be **diagonalizable**.
- In the continuum scaling limit, each logarithmic minimal model gives rise to a CFT

$$\boldsymbol{D}(u) \sim e^{-u\mathcal{H}}, \qquad -\mathcal{H} \to L_0 - \frac{c}{24}, \qquad Z_{r,s}(q) = \mathrm{Tr}\, \boldsymbol{D}(u)^P \to q^{-c/24}\, \mathrm{Tr}\, q^{L_0} = \chi_{r,s}(q)$$

where q is the modular parameter.

- Associated to the boundary condition (r,s) is the so-called **Kac representation** (r,s).
- As representations of the Virasoro algebra, the Kac representations fall in three groups:

(i) irreducible representations,
(ii) reducible yet indecomposable representations,
(iii) fully reducible representations.

- Two irreducible representations with the same conformal weight are identified: $(kp,p') \equiv (p,kp'),\ \ k \in \mathbb{N}$.
- There are **infinitely** many distinct Kac representations.
- This infinite set of representations is associated to an infinitely extended Kac table.
- The Kac representations are the building blocks for fusion.
- The **identity** representation is $(1,1)$. It is $\begin{cases} \text{irreducible}, & p = 1 \\ \text{reducible yet indecomposable}, & p \geq 2 \end{cases}$

Critical Dense Polymer Kac Table

- **Central Charge:** $(p,p')=(1,2)$

$$c \;=\; 1-\frac{6(p-p')^2}{pp'} \;=\; -2$$

- **Conformal Weights:**

$$\Delta_{r,s} \;=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'} \;=\; \frac{(2r-s)^2-1}{8}, \qquad r,s\in\mathbb{N}$$

- **Kac Representation Characters:**

$$\chi_{r,s}(q)=q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- **Irreducible Representations:**

There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

Critical Percolation Kac Table

- **Central Charge:** $(p, p') = (2, 3)$

$$c = 1 - \frac{6(p-p')^2}{pp'} = 0$$

- **Conformal Weights:**

$$\Delta_{r,s} = \frac{(p'r - ps)^2 - (p-p')^2}{4pp'} = \frac{(3r-2s)^2 - 1}{24}, \qquad r, s \in \mathbb{N}$$

- **Kac Representation Characters:**

$$\chi_{r,s}(q) = q^{-c/24} \frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- **Irreducible Representations:**

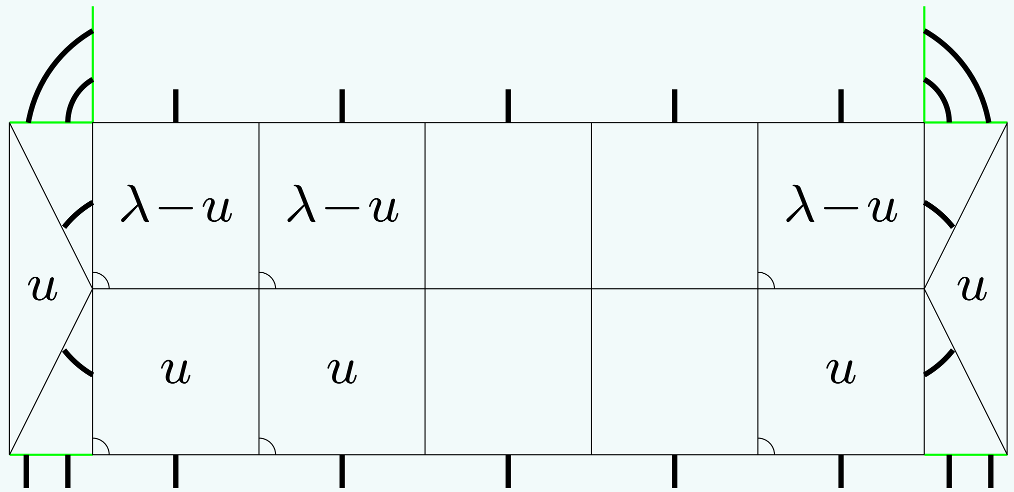
There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Lattice Implementation of Fusion

- Fusion is implemented on the lattice by taking non-trivial boundary conditions on the left and right $(r',s')\otimes(r,s)$

$\boldsymbol{D}(u) =$



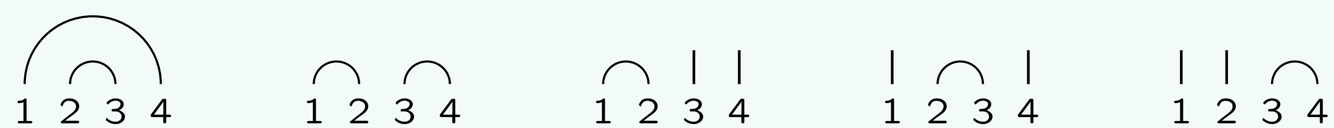
- In general, these fusion transfer matrices are **non-diagonalizable** as they can exhibit **non-trivial Jordan blocks**.
- In terms of representations, such examples correspond to reducible representations $\mathcal{R}$ of **rank greater than 1** $\Rightarrow$ **Logarithmic CFT**. There are infinitely many of these reps; all of rank 2 or 3 and all associated to the infinitely extended Kac table.

An Indecomposable Representation of Rank 2

- For $\mathcal{LM}(1,2)$, the fusion "$(1,2)\otimes(1,2) = \left(-\frac{1}{8}\right)\otimes\left(-\frac{1}{8}\right) = 0\;+\;0 = (1,1)+(1,3)$" yields a reducible yet indecomposable representation of rank 2.
- For $N=4$, the Hamiltonian

$$\boldsymbol{D}(u)\sim e^{-u\mathcal{H}} \qquad -\mathcal{H} \;=\; \sum_j e_j \;\sim\; \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) + \sqrt{2}\,I \qquad -\mathcal{H}\;\mapsto L_0-\tfrac{c}{24}$$

acts on the five states with $\ell=0$ or $\ell=2$ defects



- The Jordan canonical form of $\mathcal{H}$ has rank-2 Jordan blocks

$$-\mathcal{H}\sim\left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{8} & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{array}\right) \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{8} & 1 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = L_0^{(4)}$$

- As $N\to\infty$, the eigenvalues of $-\mathcal{H}$ approach the integer energies indicated in $L_0^{(4)}$.
- For $N=4$, the finitized partition function is ($q=$ modular parameter)

$$Z^{(N)}_{(1,2)|(1,2)}(q) = \underbrace{\chi^{(N)}_{(1,1)}(q)}_{0\text{ defects}} + \underbrace{\chi^{(N)}_{(1,3)}(q)}_{2\text{ defects}} = q^{-c/24}[(1+q^2)+(1+q+q^2)] = q^{-c/24}(2+q+2q^2)$$

Dense Polymer Virasoro Fusion Algebra

- The fundamental Virasoro fusion algebra of critical dense polymers $\mathcal{LM}(1,2)$ is

$$\big\langle (2,1),(1,2)\big\rangle \;=\; \big\langle (r,1),(1,2k),\mathcal{R}_k;\;\; r,k\in\mathbb{N}\big\rangle$$

- With the identifications $(k,2k')\equiv(k',2k)$, the fusion rules obtained empirically from the lattice are **commutative**, **associative** and agree with Gaberdiel & Kausch (1996)

$$(r,1)\otimes(r',1) = \bigoplus_{j=|r-r'|+1,\text{ by }2}^{r+r'-1} (j,1)$$

$$(1,2k)\otimes(1,2k') = \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} \mathcal{R}_j$$

$$(1,2k)\otimes\mathcal{R}_{k'} = \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}(1,2j)$$

$$\mathcal{R}_k\otimes\mathcal{R}_{k'} = \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}\mathcal{R}_j$$

$$(r,1)\otimes(1,2k) = \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} (1,2j) = (r,2k)$$

$$(r,1)\otimes\mathcal{R}_k = \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} \mathcal{R}_j$$

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

$\mathcal{R}_k = (1,2k-1)\oplus_i(1,2k+1)$, (indecomposable of rank 2)

$$\delta^{(2)}_{j,\{k,k'\}} = 2-\delta_{j,|k-k'|}-\delta_{j,k+k'}$$

$\mathcal{W}$-Extended Vacuum of $\mathcal{WLM}(1,2)$

- Critical dense polymers $\mathcal{LM}(1,2)$ in the $\mathcal{W}$-extended picture is identified with the so-called **symplectic fermions**.
- The $\mathcal{W}$-extended vacuum character of symplectic fermions is known to be

$$\hat{\chi}_{1,1}(q) = \sum_{n=1}^{\infty} (2n-1)\,\chi_{2n-1,1}(q)$$

- The BYBE is *not* linear and sums of solutions do *not* usually give new solutions. Rather, the **BYBE is closed under fusions**. We thus consider the triple fusion

$$(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = (1,1)\oplus 3(3,1)\oplus 5(5,1)\oplus\ldots\oplus(2n-1)(2n-1,1)\oplus\ldots$$

For large n, the coefficients stabilize and reproduce the extended vacuum character $\hat{\chi}_{1,1}(q)$.

- The **$\mathcal{W}$-Extended Vacuum** is thus defined by

$$(1,1)_{\mathcal{W}} := \lim_{n\to\infty}(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = \bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,1)$$

- In general, we denote by $\mathcal{WLM}(p,p')$ the logarithmic minimal model $\mathcal{LM}(p,p)$ viewed in the $\mathcal{W}$-extended picture.

$\mathcal{W}$-Extended Boundary Conditions and Fusion

- The $\mathcal{W}$-extended vacuum $(1,1)_{\mathcal{W}}$ of $\mathcal{WLM}(1,2)$ must act as the identity. In particular

$$(1,1)_{\mathcal{W}} \,\hat{\otimes}\, (1,1)_{\mathcal{W}} = (1,1)_{\mathcal{W}}$$

where $\hat{\otimes}$ denotes the fusion multiplication in the extended picture.

- The $\mathcal{W}$-extended vacuum has the **stability property**

$$(2n-1,1) \otimes (1,1)_{\mathcal{W}} = (2n-1)\,(1,1)_{\mathcal{W}}$$

- The $\mathcal{W}$-extended fusion $\hat{\otimes}$ is therefore defined by

$$(1,1)_{\mathcal{W}} \,\hat{\otimes}\, (1,1)_{\mathcal{W}} := \lim_{n\to\infty} \left(\frac{1}{(2n-1)^3} (2n-1,1) \otimes (2n-1,1) \otimes (2n-1,1) \otimes (1,1)_{\mathcal{W}} \right) = (1,1)_{\mathcal{W}}$$

- Additional stability properties enable us to define

$$
\begin{aligned}
(1,s)_{\mathcal{W}} &:= (1,s) \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,s), \qquad s=1,2 \\
(2,s)_{\mathcal{W}} &:= \tfrac{1}{2}(2,s) \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n\,(2n,s), \qquad s=1,2 \\
\hat{\mathcal{R}}_1 \equiv (\mathcal{R}_1)_{\mathcal{W}} &:= \mathcal{R}_1 \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,\mathcal{R}_{2n-1} \\
\hat{\mathcal{R}}_0 \equiv (\mathcal{R}_2)_{\mathcal{W}} &:= \tfrac{1}{2}\mathcal{R}_2 \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n\,\mathcal{R}_{2n}
\end{aligned}
$$

- The ensuing representation content: 4 $\mathcal{W}$-irreducible representations and 2 $\mathcal{W}$-reducible yet $\mathcal{W}$-indecomposable representations of rank 2.

Fusion Rules for $\mathcal{WLM}(1,2)$

- The $\mathcal{W}$-extended fusion rules follow from the Virasoro fusion rules combined with stability

$\hat{\otimes}$	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
0	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
1	1	$\mathbf{0}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$
$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$
$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$

where the 4 $\mathcal{W}$-irreducible representations are represented by their conformal weights.

Example: Consider the $\mathcal{W}$-extended fusion rule $1\,\hat{\otimes}\,1=\mathbf{0}$:

$$\begin{aligned}(2,1)_{\mathcal{W}}\,\hat{\otimes}\,(2,1)_{\mathcal{W}} &= \Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big)\,\hat{\otimes}\,\Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big)\\ &= \tfrac{1}{4}\Big((2,1)\otimes(2,1)\Big)\otimes\Big((1,1)_{\mathcal{W}}\,\hat{\otimes}\,(1,1)_{\mathcal{W}}\Big)\\ &= \tfrac{1}{4}\Big((1,1)\oplus(3,1)\Big)\otimes(1,1)_{\mathcal{W}}\\ &= \tfrac{1}{4}(1+3)(1,1)_{\mathcal{W}}\\ &= (1,1)_{\mathcal{W}}\end{aligned}$$

- **For general $\mathcal{WLM}(1,p')$**, the $\mathcal{W}$-extended fusion rules and characters agree with Gaberdiel & Kausch (1996) and Gaberdiel & Runkel (2008).

Representation Content of $\mathcal{WLM}(p,p')$

	Number	Symplectic Fermions	Critical Percolation
$\mathcal{W}$-indec reps	$6pp' - 2p - 2p'$	6	26
Rank 1	$2p + 2p' - 2$	4	8
Rank 2	$4pp' - 2p - 2p'$	2	14
Rank 3	$2(p-1)(p'-1)$	0	4
$\mathcal{W}$-irred chars	$2pp' + \frac{1}{2}(p-1)(p'-1)$	4	13
$\mathcal{W}$-proj reps	$2pp'$	4	12
$\mathcal{W}$-proj chars	$\frac{1}{2}(p+1)(p'+1)$	3	6

- The **finitely** many $\mathcal{W}$-indecomposable reps close under fusion with respect to $\hat{\otimes}$.
- For $p \geq 2$, this fusion algebra has **no identity**. A canonical algebraic extension exists.
- A "disentangling procedure" is employed when identifying the various representations.
- The $\mathcal{W}$-*projective* representations form a fusion sub-algebra. Here, a $\mathcal{W}$-projective representation is a "maximal" $\mathcal{W}$-indecomposable representation in the sense that it does not appear as a subfactor of any other $\mathcal{W}$-indecomposable representation.

Virasoro Decompositions

- In terms of Virasoro-indecomposable representations, the $\mathcal{W}$-indecomposable representations decompose into infinite direct sums.

rank-1: $(\kappa \in \mathbb{Z}_{1,2},\ r \in \mathbb{Z}_{1,p},\ s \in \mathbb{Z}_{1,p'})$

$$(\kappa p, s)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)((2k-2+\kappa)p, s)$$
$$(r, \kappa p')_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)(r,(2k-2+\kappa)p')$$

where the two expressions for $(p,p')_{\mathcal{W}}$ agree while $(p,2p')_{\mathcal{W}} \equiv (2p,p')_{\mathcal{W}}$.

rank-2: $(\kappa \in \mathbb{Z}_{1,2},\ a \in \mathbb{Z}_{1,p-1},\ b \in \mathbb{Z}_{1,p'-1},\ r \in \mathbb{Z}_{1,p},\ s \in \mathbb{Z}_{1,p'})$

$$(\mathcal{R}^{a,0}_{\kappa p,s})_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)\mathcal{R}^{a,0}_{(2k-2+\kappa)p,s}, \qquad (\mathcal{R}^{0,b}_{r,\kappa p'})_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)\mathcal{R}^{0,b}_{r,(2k-2+\kappa)p'}$$

rank-3: $(\kappa \in \mathbb{Z}_{1,2},\ a \in \mathbb{Z}_{1,p-1},\ b \in \mathbb{Z}_{1,p'-1})$

$$(\mathcal{R}^{a,b}_{\kappa p,p'})_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)\mathcal{R}^{a,b}_{p,(2k-2+\kappa)p'} = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)\mathcal{R}^{a,b}_{(2k-2+\kappa)p,p'}$$

- The embedding diagrams are partially understood.
- The set of $\mathcal{W}$-projective representations is

$$\left\{(\mathcal{R}^{\alpha,\beta}_{\kappa p,p'})_{\mathcal{W}}; \kappa \in \mathbb{Z}_{1,2}, \alpha \in \mathbb{Z}_{0,p-1}, \beta \in \mathbb{Z}_{0,p'-1}\right\}, \qquad (\mathcal{R}^{0,0}_{\kappa p,p'})_{\mathcal{W}} \equiv (\kappa p, p')_{\mathcal{W}}$$

$\mathcal{W}$-Irreducible Characters of Critical Percolation

- $\mathcal{W}$-irreducible representations:

$$\hat{\chi}_{\frac{1}{3}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)q^{3(4k-3)^2/8} \qquad \hat{\chi}_{\frac{21}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{(6k-2)^2/6}$$

$$\hat{\chi}_{\frac{10}{3}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{3(4k-1)^2/8} \qquad \hat{\chi}_{\frac{33}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{(6k-1)^2/6}$$

$$\hat{\chi}_{\frac{1}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)\,q^{(6k-5)^2/6} \qquad \hat{\chi}_{-\frac{1}{24}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)\,q^{(6k-6)^2/6}$$

$$\hat{\chi}_{\frac{5}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)\,q^{(6k-4)^2/6} \qquad \hat{\chi}_{\frac{35}{24}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{(6k-3)^2/6}$$

- Subfactors of $\mathcal{W}$-reducible yet $\mathcal{W}$-indecomposable representations:

$$\hat{\chi}_0(q) = 1$$

$$\hat{\chi}_1(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k^2\Big[q^{(12k-7)^2/24} - q^{(12k+1)^2/24}\Big]$$

$$\hat{\chi}_2(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k^2\Big[q^{(12k-5)^2/24} - q^{(12k-1)^2/24}\Big] \qquad \eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

$$\hat{\chi}_5(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k(k+1)\Big[q^{(12k-1)^2/24} - q^{(12k+7)^2/24}\Big]$$

$$\hat{\chi}_7(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k(k+1)\Big[q^{(12k+1)^2/24} - q^{(12k+5)^2/24}\Big]$$

- **For general** $\mathcal{WLM}(p,p')$, the $\mathcal{W}$-characters agree with Feigin, Gainutdinov, Semikhatov & Tipunin (2006).

Polynomial Fusion Ring of $\mathcal{WLM}(p,p')$

- Notation: $\kappa\in\mathbb{Z}_{1,2},\quad a\in\mathbb{Z}_{1,p-1},\quad b\in\mathbb{Z}_{1,p'-1},\quad \alpha\in\mathbb{Z}_{0,p-1},\quad \beta\in\mathbb{Z}_{0,p'-1}$
- Rank-1 representations: $\{(\mathcal{R}_{a,\kappa p'}^{0,0})_{\mathcal{W}},(\mathcal{R}_{\kappa p,b}^{0,0})_{\mathcal{W}},(\mathcal{R}_{\kappa p,p'}^{0,0})_{\mathcal{W}}\}$ $\quad\sharp=2p+2p'-2$

  Rank-2 representations: $\{(\mathcal{R}_{\kappa p,b}^{a,0})_{\mathcal{W}},(\mathcal{R}_{\kappa p,p'}^{a,0})_{\mathcal{W}},(\mathcal{R}_{a,\kappa p'}^{0,b})_{\mathcal{W}},(\mathcal{R}_{\kappa p,p'}^{0,b})_{\mathcal{W}}\}$ $\quad\sharp=4pp'-2p-2p'$

  Rank-3 representations: $\{(\mathcal{R}_{\kappa p,p'}^{a,b})_{\mathcal{W}}\}$ $\quad\sharp=2(p-1)(p'-1)$
- Polynomials:

$$P_n(x)\;=\;U_{3n-1}\big(\tfrac{x}{2}\big)-3U_{n-1}\big(\tfrac{x}{2}\big)\;=\;2\Big(T_{2n}\big(\tfrac{x}{2}\big)-1\Big)U_{n-1}\big(\tfrac{x}{2}\big)\;=\;(x^2-4)U_{n-1}^3\big(\tfrac{x}{2}\big)$$
$$P_{n,n'}(x,y)\;=\;\Big(T_n\big(\tfrac{x}{2}\big)-T_{n'}\big(\tfrac{y}{2}\big)\Big)U_{n-1}\big(\tfrac{x}{2}\big)U_{n'-1}\big(\tfrac{y}{2}\big)$$

where T_n and U_n are Chebyshev polynomials of the first and second kind, respectively.

- The polynomials

$$\begin{aligned}
\mathrm{pol}_{(\mathcal{R}_{\kappa p,b}^{\alpha,0})_{\mathcal{W}}}(X,Y)&=\tfrac{2-\delta_{\alpha,0}}{\kappa}T_\alpha\big(\tfrac{X}{2}\big)U_{\kappa p-1}\big(\tfrac{X}{2}\big)U_{b-1}\big(\tfrac{Y}{2}\big)\\
\mathrm{pol}_{(\mathcal{R}_{a,\kappa p'}^{0,\beta})_{\mathcal{W}}}(X,Y)&=U_{a-1}\big(\tfrac{X}{2}\big)\tfrac{2-\delta_{\beta,0}}{\kappa}T_\beta\big(\tfrac{Y}{2}\big)U_{\kappa p'-1}\big(\tfrac{Y}{2}\big)\\
\mathrm{pol}_{(\mathcal{R}_{\kappa p,p'}^{\alpha,\beta})_{\mathcal{W}}}(X,Y)&=\tfrac{2-\delta_{\alpha,0}}{\kappa}T_\alpha\big(\tfrac{X}{2}\big)U_{\kappa p-1}\big(\tfrac{X}{2}\big)\big(2-\delta_{\beta,0}\big)T_\beta\big(\tfrac{Y}{2}\big)U_{p'-1}\big(\tfrac{Y}{2}\big)
\end{aligned}$$

generate an ideal of the quotient polynomial ring

$$\mathbb{C}[X,Y]\Big/\Big(P_p(X),P_{p'}(Y),P_{p,p'}(X,Y)\Big)$$

- The $\mathcal{W}$-extended fusion algebra of $\mathcal{WLM}(p,p')$ is **isomorphic** to this ideal.

Summary and Outlook

- Infinite series of Yang-Baxter integrable lattice models of non-local statistical mechanics.
- Logarithmic CFT with infinitely many (higher-rank) indecomposable representations.
- Empirical Virasoro fusion rules for $\mathcal{LM}(p,p')$

Checks:
1. $\mathcal{LM}(p,p')$ fusion rules agree with level-by-level fusion rules of Eberle & Flohr (2006) using the Nahm-Gaberdiel-Kausch algorithm (1994-96).
2. Vertical fusion subalgebras agree with Read & Saleur (2007) and Mathieu & Ridout (2008).
3. Associativity.

- $\mathcal{W}$-extended picture with finitely many (higher-rank) indecomposable representations.
- Inferred $\mathcal{W}$-algebra fusion rules for $\mathcal{WLM}(p,p')$

Checks:
1. $\mathcal{WLM}(1,p')$ fusion rules agree with Gaberdiel & Kausch (1996) and Gaberdiel & Runkel (2008).
2. $\mathcal{WLM}(p,p')$ characters agree with Feigin et al (2006).
3. Associativity.

- Links to SLE.
- Verlinde formulas from spectral decompositions:
  - Projective representations (with Pearce).
  - Grothendieck ring (with Pearce & Ruelle).
  - Fusion algebra.
- From strip to cylinder (with Pearce & Villani) and torus $\rightarrow$ modular invariance.
- Open boundary conditions (with Pearce & Tipunin) $\rightarrow$ half-integer Kac labels.