Associative Algebra Approach to Irrational Conformal Field Theory

i.e. categorical methods for non-semisimple CFT

N. Read

w. H. Saleur

cond-mat/0701259
hep-th/0701117 } → Nucl. Phys. B

Rational Conformal Field Theory

- def: ① finite no of irreps of a chiral alg, (or VOA) and ② full decomposability of fusion products (semisimplicity)

- fairly well understood

In <u>irrational</u> CFT, either ① or ② (or both) fails.

Thus maybe <u>infinite</u> no of irreps of (maximal, finitely-generated) chiral algebra

<u>or</u> <u>reducible</u> but <u>indecomposable</u> reps of Virasoro or chiral algebra

- typically producing <u>logarithms</u> in some correlators (Gurarie)

Here we aim to control these effects using <u>symmetry</u> and <u>algebraic methods</u>.

Problems of interest:

Nonlinear sigma models with target supermanifold

e.g. $$\frac{SU(m+n|n)}{S(U(1)\times U(m+n-1|n))}$$

integer quantum Hall effect:

$$\frac{U(n,n|2n)}{U(n|n)\times U(n|n)} \quad \text{at} \quad \theta=\pi$$

(Pruisken, Zirnbauer)

spin quantum Hall effect:

$$\frac{OSp(2n|2n)}{U(n|n)} \quad \text{at} \quad \theta=\pi,$$

$n=1$ is iso to $\dfrac{SU(2|1)}{S(U(1)\times U(1|1))}$ above

(Gruzberg, Ludwig, NR)

Loop models inc percolation, polymers

(denNijs, Nienhuis, Saleur-Duplantier, Cardy)

Quantum spin chains related to these (Haldane, Affleck)

Antiferromagnetic SU(m) spin chain

(Affleck)

$2L$ sites, $i = 0, 1, \ldots, 2L-1$

Vector space of dim $= m$ on each site

i even $\to$ transforms in fundamental of $SU(m)$

i odd $\to$ " " dual (conjugate) fundamental

$SU(m)$-invariant nearest-neighbor interaction

Rep. using bosons: b_i^a, b_{ia}^+ (i even), $[b_i^a, b_{i'a'}^+] = \delta_{ii'}\delta^a_{a'}$

$(a = 1, \ldots, m)$ $\qquad \bar{b}_{ia}, \bar{b}_i^{a+}$ (i odd)

Constraints: $\left.\begin{array}{ll} b_{ia}^+ b_i^a = 1 & (i \text{ even}) \\ \bar{b}_i^{a+} \bar{b}_{ia} = 1 & (i \text{ odd}) \end{array}\right\}$ (Summ. over a)

Symmetry ($SU(m)$) gens, each site: $J_{ia}^b = \begin{cases} b_{ia}^+ b_i^b & (i \text{ even}) \\ -\bar{b}_i^{b+} \bar{b}_{ia} & (i \text{ odd}) \end{cases}$

Ham: $H = -\epsilon \sum_{i\,\text{even}} e_i - \epsilon^{-1} \sum_{i\,\text{odd}} e_i$

where $\quad e_i = \begin{cases} \bar{b}_{i+1}^{a+} b_{ia}^+ b_i^b \bar{b}_{i+1\,b} & (i \text{ even}) \\ \bar{b}_i^{a+} b_{i+1,a}^+ b_{i+1}^b \bar{b}_{ib} & (i \text{ odd}) \end{cases}$

— $SU(m)$-inv coupling of $i, i+1$. $e_i = (\text{const})\, J_{ia}^b J_{i+1,b}^a + \text{const}$

Phase transition at $\underline{\epsilon = 1}$. $\mathbb{CP}^{m-1}$ sigma model, $\epsilon = 1 \leftrightarrow \theta = \pi$

If represent a state of $i, i+1$ of form

$b^{\dagger}_{ia}\, \bar{b}^{a\dagger}_{i+1} |0\rangle$ by  SU(m)-singlet "valence bond"

vacuum

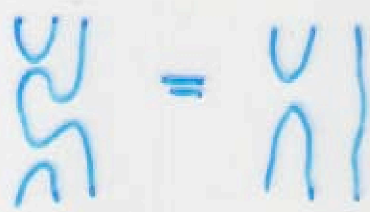
and e_i by  (i even; sim i odd)

(Also id=1 by $\uparrow\downarrow$.) — really 

Then e_i's $(i = 0, 1, \ldots, 2L-2)$ obey relations defining Temperley–Lieb algebra:

$$e_i^2 = m e_i$$

$$e_i e_{i\pm1} e_i = e_i$$



$$e_i e_{i'} = e_{i'} e_i \quad (i' \neq i, i\pm1)$$

Also have $e_i^{\dagger} = e_i$.

Note: arrows always up on even sites, down on odd (they are not degrees of freedom)

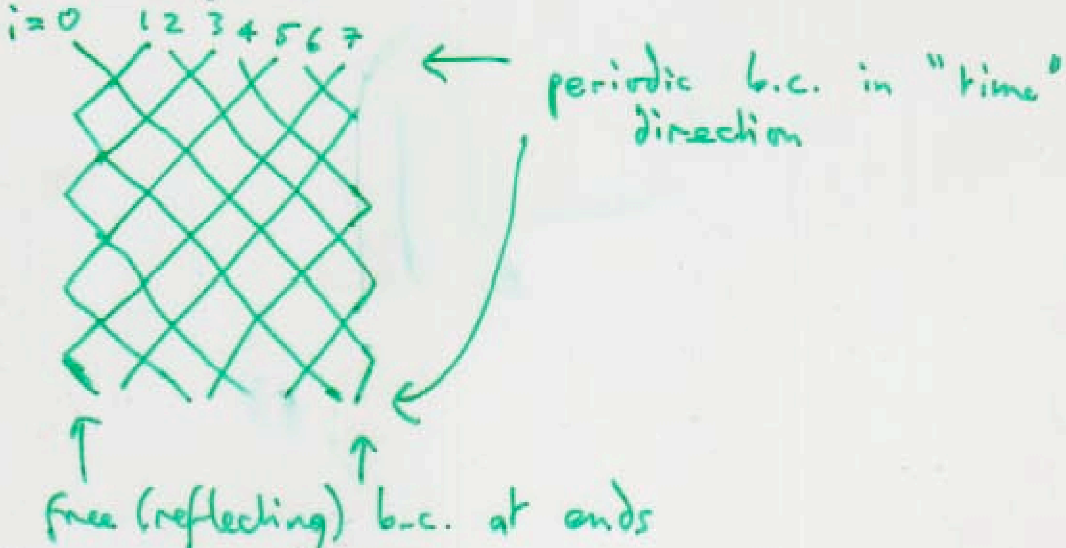
Denote algebra $TL_{2L}(q)$, where $m = q + q^{-1}$.

Spin chain space $V = V_0 \otimes V_1 \otimes \ldots \otimes V_{2L-1}$,

$V_i \cong \mathbb{C}^m$, all i

Connection with loop models:

Lattice :



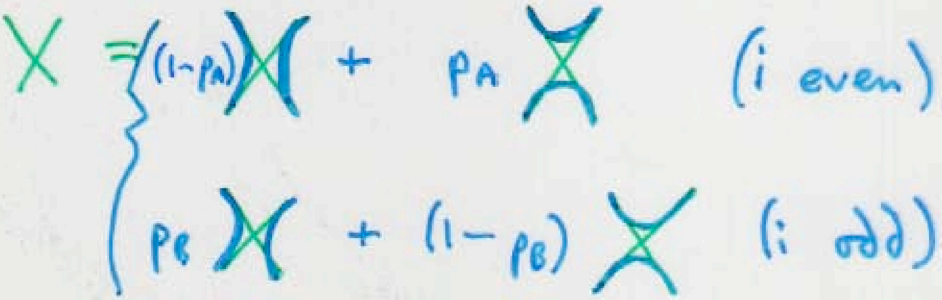
Each vertex X, write $T_i = \begin{cases} (1-p_A) + p_A e_i & (i \text{ even}) \\ p_B + (1-p_B) e_i & (i \text{ odd}) \end{cases}$

acting in same space V as spin chain.

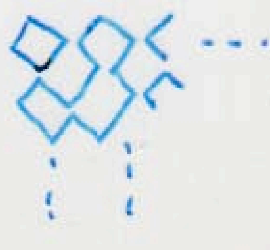
Transfer matrix: $T = T_1 T_3 \ldots T_{2L-3} T_0 T_2 \ldots T_{2L-2}$

Partition function $Z = \mathrm{Tr}\, T^{L'}$

Writing



Expand $Z = \sum_{\text{loops}} m^{\#\text{loops}} p^{\#}(1-p)^{\#}$



in loops filling edges, with avoided crossings.

$Q = m^2$- state Potts model.

Percolation $\equiv$ $m=1$

Algebraic analysis

We are interested in <u>commutant</u> algebra of TL in our chain — i.e. algebra of all linear operators on V that commute with all elements of $TL_{2L}(q)$

– sufficient " " " generators e_i.

Motivation: ① provides <u>common</u> symmetry algebra of <u>all</u> Hams of form

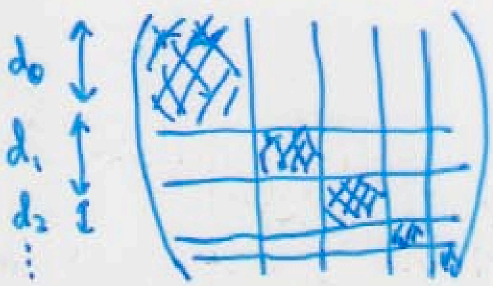
$$H = -\sum_i \lambda_i e_i \quad , \quad \lambda_i = \text{arbitrary real numbers}$$

(hence multiplicities of eigenvalues etc)

② in CFT generalization, commutes with Virasoro algebra — symmetry of CFT.

Facts: a) $TL_{2L}(q)$ is semisimple for $m \geq 2$ $(q \geq 1)$

$\Rightarrow$ it is iso to alg of block diag matrices



Sizes $d_j \times d_j$

– d_j = dim of j^{th} irreducible rep^n.

Here $$d_j = \binom{2L}{L+j} - \binom{2L}{L+j+1} , \quad j = 0, 1, \ldots, L$$

(ind. of q).

b) Provided $TL_{2L}(q)$ acts <u>faithfully</u> in V, then we have a decomposition of V as direct sum of irreps, with non-zero multiplicities $D_j \neq 0$.

$$\Rightarrow \quad \dim V = m^{2L} = \sum_{j=0}^{2L} d_j D_j$$

c) Commutant consists of all maps of TL irreps into themselves

$\Rightarrow$ semisimple, D_j = dims of irreps of commutant

– blocks of sizes $D_j \times D_j$

We find that

$$D_j = [2j+1]_q$$

where $[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}$

$$= \frac{q^n - q^{-n}}{q - q^{-1}}.$$

D_j grow <u>exp</u>ly with j for $q > 1$ $(m > 2)$.

E.g. $m=3$ $(SU(3))$ case,

$$D_j = 1,\ 8,\ \underbrace{55,\ 377, \ldots}$$

Fibonacci nos! much larger than dims of irreps of $SU(3)$. But note $8 \times 8 = 55 + 8 + 1$. Highly reducible under $SU(3)$.

Algebra structure:

TL alg = all lin combs of (isotopy classes of) diagrams of non-crossing lines

e.g.  (L = 3)

2 "throughlines"

Product:

e.g. 

$\Rightarrow$ # throughlines cannot increase, can decrease

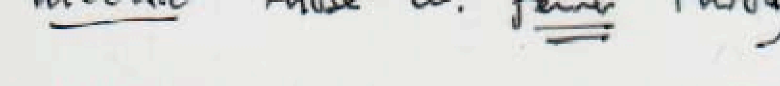
$\Rightarrow$ <u>ideals</u> $T^{(2j)}$ = span of diagrams w. $\leq 2j$ throughlines

$$0 \subset T^{(0)} \subset T^{(2)} \subset \dots \subset T^{(2L)} = TL_{2L}(q)$$

– a "cellular algebra" (Graham + Lehrer)

Half-diagrams e.g. (noncrossing ($2j$ throughlines) B.s)

<u>modulo</u> those w. fewer throughlines, so

 $= 0$ form "standard" or "cell" modules $\mathcal{W}_j$ of dim $= d_j$, $(j = 0, 1, \dots, L)$. Irreducible $m > 2$ $(q \geq 1)$

In chain (space V), map to valence bonds

$\cup$ = singlet bond

$\bullet\ \bullet\ \bullet\ \bullet$ = any state s.t. $\underset{i\ \ i+1}{\overset{\cup}{\cap}}$ acts by zero

(ie adjoint map for each pair of "neighboring" dots)

Dim of space $\mathcal{V}_j$ of latter, <u>any</u> such "valid" pattern

$$\dim \mathcal{V}_j = D_j = [2j+1]_q$$

These are irreps of commutant algebra $\mathcal{A}_m(2L)$.

Basis elements:

$$\boxed{\tilde{J}^{a_1 a_2 \cdots a_k}_{b_1 b_2 \cdots b_k} = \sum_{0 \le i_1 < i_2 < \cdots < i_k} J^{a_1}_{i_1 b_1} J^{a_2}_{i_2 b_2} \cdots J^{a_k}_{i_k b_k}}$$

($\tilde{J} = 1$ for $k=0$, $\tilde{J}^{a_1}_{b_1}$ = gens of $U(m)$)

Take subspace

$$\boxed{\begin{aligned} J^{\cdots a_\ell\, a_{\ell+1} \cdots}_{\quad b_\ell\, a_\ell \cdots} &= 0 \\ J^{\cdots a_\ell\, a_{\ell+1} \cdots}_{\cdots a_{\ell+1} b_{\ell+1} \cdots} &= 0 \end{aligned}}$$

(using Jones-Wenzl projectors)

These commute with all e_i; those with k even span $\mathcal{A}(2L)$

Further J's with $k > 2j$ annihilate jth irrep. Also <u>cellular</u>.

$L \to \infty$ limit

is <u>nice</u> for commutant $\mathcal{A}_m$:

$$\mathcal{A}_m = \lim_{\longleftarrow} \mathcal{A}_m(2L)$$

using natural <u>projections</u> of (abstract) algebras

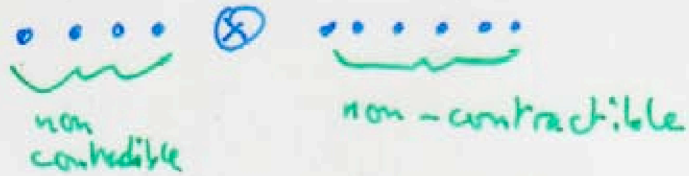
$$p_{2L} : \mathcal{A}_m(2L) \to \mathcal{A}_m(2L-2)$$

by annihilating $\mathcal{V}_{2L+2}$. "Pullback" rep^{ns} using p_{2L}.
Note D_j <u>ind</u> of L, so $\mathcal{A}_m$ has irreps $\mathcal{V}_j$, $j = 0, 1, 2, \ldots$

TL irreps corres to $\mathcal{V}_j$, each j.
Corresponding limit, in terms of representations of TL.
But <u>no</u> limit of TL algebra this way.
(Dims of "irreps" $= \infty$.)

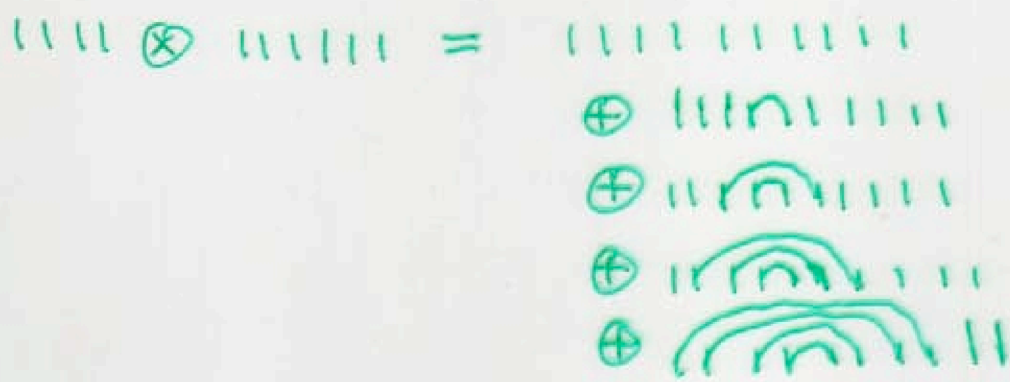
Fusion (tensor product) for commutant $\mathcal{A}_m$

In terms of irreps:



but might be able to contract in middle:

E.g.



I.e. $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2} = \mathcal{V}_{j_1+j_2} \oplus \mathcal{V}_{j_1+j_2-1} \oplus \cdots$

$\cdots \oplus \mathcal{V}_{|j_1-j_2|}$

exactly as for $SU(2)$ (or $U_q(sl_2)$).

Or construct comultiplication for $\mathcal{A}$.

Stable in $L \to \infty$ limit.

Used restriction: $\mathcal{A}_m(2L_1) \otimes \mathcal{A}_m(2L_2) \supseteq \mathcal{A}_m(2(L_1+L_2))$

Fusion (induction product) for TL algebras

Join end to end:

$$| \, | \, | \, | \, | \otimes | \, | \, | \, | \, | \, |$$

$$e_1, e_2, e_3, \quad e_5, e_6, e_7, e_8, e_9$$

Introduce e_{2L_1} to join $(= e_4)$

$$TL_{2L_1}(q) \otimes TL_{2L_2}(q) \subset TL_{2(L_1+L_2)}(q)$$

Must use <u>induction</u> to get rep of larger alg from rep of subalgebra

— does <u>not</u> obey $\dim W_1 \cdot \dim W_2 = \dim W_1 \times W_2$

Fusion rules (for reps of appropriate algs) <u>same</u> as in A_m:

$$W_{j_1} \times W_{j_2} = \mathrm{ind}_{TL_{2L_1} \otimes TL_{2L_2}(q)}^{TL_{2(L_1+L_2)}(q)} \left(W_{j_1} \otimes W_{j_2} \right)$$

$$= \bigoplus_{j = |j_1 - j_2|}^{j_1 + j_2} W_j$$

Note for $SU(2)$ (and a gen to $SU(m)$) this is classical Frobenius – Schur – Weyl duality ($\rightarrow$ rep. thy. of $SU(m)$ using Young tableaux etc).

Spin-1/2 chain and $U_q(sl_2)$

For spin-$\frac{1}{2}$ chain, $2L$ sites,

$$e_i = \frac{q+q^{-1}}{4} - \frac{1}{2}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right) - \left(\frac{q+q^{-1}}{4}\right)\sigma_i^z \sigma_{i+1}^z$$
$$- \left(\frac{q-q^{-1}}{4}\right)\left(\sigma_i^z - \sigma_{i+1}^z\right)$$

represent TL algebra faithfully (TL), all $q \in \mathbb{C}$

Commutant is $U_q(sl_2)$ — a quantum group.

For q^2 not a root of unity, both $TL_{2L}(q)$ and $U_q(sl_2)$ are semisimple, fusion as above etc.

Can define <u>antipode</u>, <u>braiding</u> and <u>twist</u> for $\mathcal{A}_m$ as well as for $U_q(sl_2)$. → ribbon Hopf algebras

E.g. $\sigma_i = i(q^{1/2} - q^{-1/2} e_i)$ (Jones, Kaufman)

$= i q^{1/2} \; - i q^{-1/2}$ — rep^n of braid gp.

Can then say

$$\boxed{\begin{array}{c}\mathcal{A}_m \text{ is Morita equivalent to } U_q(sl_2) \\ \text{as ribbon Hopf algebras}\end{array}}$$

(Here $U_q(sl_2) = \varprojlim U_q(sl_2)^{(2L)}$ using finite chains)

<u>Superalgebra generalization</u>

$m=1$: percolation
$m=1, n=1$: spin QH transiti

Replace $V_0 = \mathbb{C}^m$ by $V_0 = \mathbb{C}^{m+n|n}$, $\mathbb{Z}_2$ grade

$V = V_0 \otimes V_1 \otimes \cdots \otimes V_{2L-1}$ <u>graded</u> tensor product

Even sites $\quad b^+_{ia}$, $a = 1, \ldots, m+n$ bosonic

$\quad f^+_{ia}$, $a = m+n+1, \ldots, 2n$ fermionic

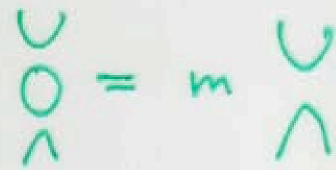
constraints $\quad b^+_{ia} b^a_i + f^+_{ia} f^a_i = 1$

Odd sites similar <u>except</u> $\{\bar{f}^{a+}_i, \bar{f}_{i'b}\} = -\delta_{ii'}\delta^a_b$

– odd states have <u>negative norm-sq.</u> $\qquad$ Note!

TL generators e_i: similar $\quad e_i^+ = e_i$

– commute with $gl(m+n|n)$



$e_i^2 = m\, e_i$

$$m = (m+n) - n = \mathrm{str}_{V_0} 1 = \mathrm{sdim}\, V_0$$

Now makes sense for <u>any integer</u> m

(but $m+n \geq 0$, $n \geq 0$). Independent of n.

For $|m| \geq 2$, theory v. similar $\to$ semisimple $A_{m+n|n}$.

$\mathrm{sdim}\, \mathcal{V}_j = [2j+1]_q$, $\dim \mathcal{V}_j = [2j+1]_{q'}$, $m+2n = q' + q'^{-1}$

Nonsemisimple cases $|m| < 2$

ie representations

Not semisimple $\Rightarrow$ $\exists$ finite dim modules
that are reducible but indecomposable
ie not decomp as direct su

E.g. TL, $|m| < 2$, and $q^{2r} = 1$, some $r > 0$
("q is a root of unity")

Standard modules not all irreducible: (P. Martin)

$$W_j \cong \begin{matrix} \circ & \\ & \searrow \\ & & \bullet \end{matrix}$$ for some j

$\bullet$ = submodule (simple) $\rightarrow$ irreducible
$\circ$ = quotient module (simple)

$\searrow$ = action of algebra connects these

Same occurs for standard (Weyl) modules of
$U_q(sl_2)$ – those of dim $2j+1$.

TL modules useful in the analysis

Projective (left) modules of algebra R ($\neq$ projective rep^{ns})

- can be defined in various ways
    - as direct sumand in a free module:

$$P \oplus P' \cong R \oplus R \oplus \cdots \oplus R$$

    -

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

← exact

    - never occur as a quotient except in a direct sum
    - if R finite dim, all indecomp projectives occur in R (viewed as module over itself)
- one for each simple module

For $q = e^{i\pi/3}$ ($m=1$), indecomp. projectives have forms

(P. Martin)

$$P_j : \begin{cases} \begin{tikzcd} 0 \arrow[dr] & \\ & 2 \end{tikzcd} & j = 0 \\ \begin{tikzcd} & j \arrow[dl] \arrow[dr] & \\ j-1 \arrow[dr] & & j+2 \arrow[dl] \\ & j & \end{tikzcd} & j \equiv 0 \pmod 3,\ j > 0 \\ j & j \equiv 1 \pmod 3 \\ \begin{tikzcd} & j \arrow[dl] \arrow[dr] & \\ j-2 \arrow[dr] & & j+1 \arrow[dl] \\ & j & \end{tikzcd} & j \equiv 2 \pmod 3 \end{cases}$$

standard

standard

"quartets" formed from two standard modules

$U_q(sl_2)$ at same q

Useful modules are "tilting" modules
— view as direct summands in spin $\frac{1}{2}$ chain under $U_q(sl_2)$

← standard (Pasquier, Saleur)

T_j :

← standard

$$\begin{tikzcd} & j-1 \arrow[dl] \arrow[dr] & \\ j-3 \arrow[dr] & & j \arrow[dl] \\ & j-1 & \end{tikzcd} \qquad j \equiv 0 \pmod 3$$

$$j \qquad j \equiv 1 \pmod 3$$

$$\begin{tikzcd} & j-2 \arrow[dl] \arrow[dr] & \\ j-3 \arrow[dr] & & j \arrow[dl] \\ & j-2 & \end{tikzcd} \qquad j \equiv 2 \pmod 3$$

$j = 0, 1, \ldots, L$. (Values outside this range dropped)

(Projectives of $U_q(sl_2)^{(2L)}$, and tilting modules of $TL_{2L}(q)$ also exist.)

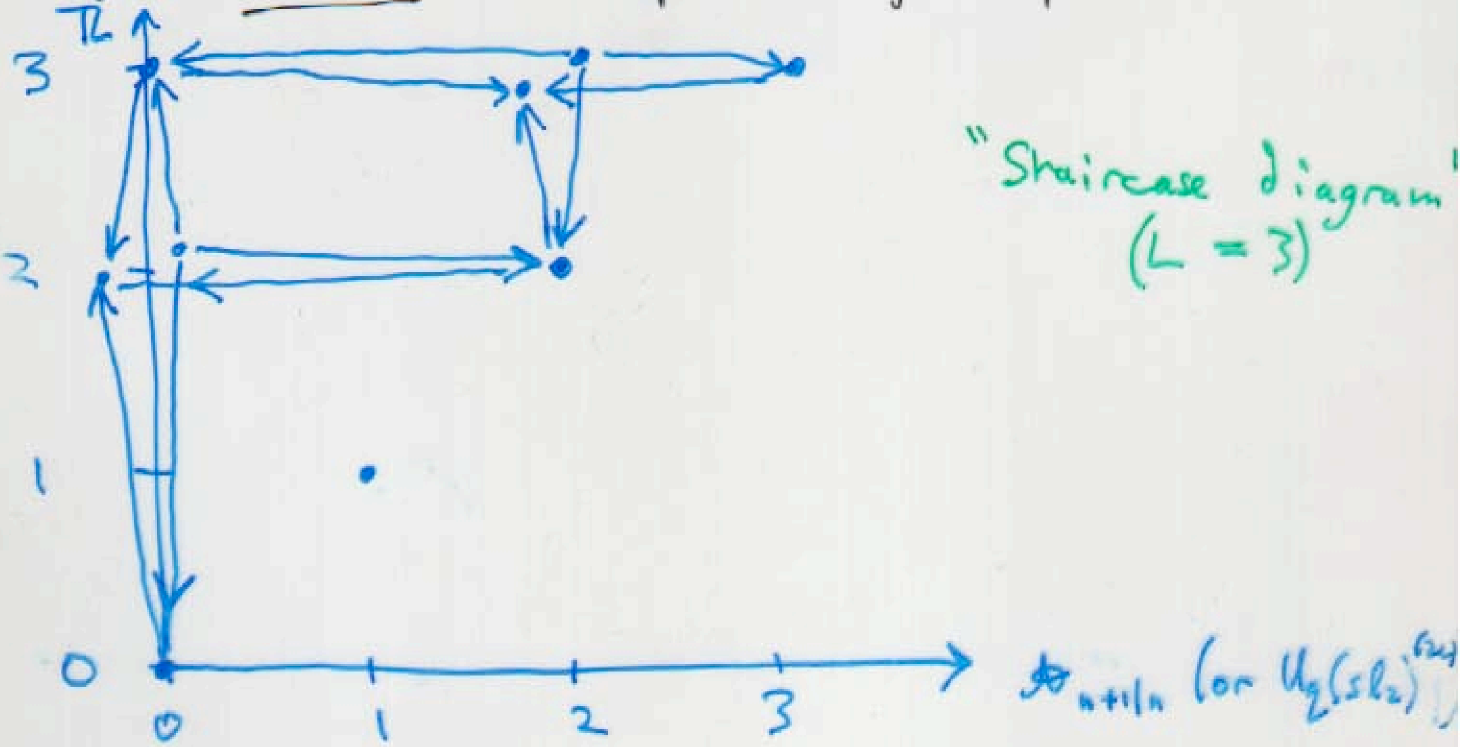
Important lemma:

Finite dim alg R acting in finite dim V, with comm alg S. Then the two categories of modules $\mathrm{Tilt}(R)$ and $\mathrm{Proj}(S)$ are equivalent, equivalence given by V.

(For $TL_{2L}(q)$ and $U_q(sl_2)^{(2L)}$, labeling agrees with this equiv.)
ie $P_j \leftrightarrow T_j$

To analyze our susy chain V, use known TL rep. thy. and our standard modules of TL and $\mathcal{A}_{m+1|n}$, plus self-duality of V.

We find <u>identical</u> decomp^{n} as for spin-$\frac{1}{2}$ chain.



Simple subquotients are dots at (j, j') under $\mathcal{A}_{m+1|n}$ (or $U_q(sl_2)$), $TL_{2L}(q)$. Those occurring twice are offset for clarity, as are the arrows:

vertical arrow $\leftrightarrow$ action of TL

horizontal arrow $\leftrightarrow$ action of $\mathcal{A}$

Exercise:

Verify consistent with commutation of these algebras!

$\mathcal{A}$ modules are tilting.

TL modules are projective, except one simple at highest j, and P_0 is absent.

Consequence: $\boxed{\text{M. equivalence of } \mathcal{A}_{m+n|n}(2L) \text{ and } U_q(sl_2)^{(2L)} \text{ (as associative algebras) for each } L, |m|.}$

Also $L \to \infty$.

Fusion rules:

def^n of fusion as before

Fusion rules for tilting modules of $\mathcal{A} = \varprojlim \mathcal{A}(2L)$ same as for $U_q(sl_2)$ (studied in Par - Sal)

$\xrightarrow[L\to\infty]{S,\sigma,\Theta}$ $\boxed{\text{M. equiv as ribbon Hopf algebras}}$

Calculation (using characters)
— 6 cases

e.g. $T_{3p_1} \otimes T_{3p_2} \cong \bigoplus_r{}' (2T_{3r+2} \oplus \underline{T_{3r}} \oplus T_{3r+3} \oplus 4T_{3r+1})$

$\bigoplus_r{}'$ = direct sum over $r = |p_1-p_2|, |p_1-p_2|+1, \dots p_1+p_2-1$

$\underline{T_{3r}}$ = omitted if $r=0$

Also $T_0 \otimes T_j \cong T_j$ all j

Fusion rules for TL projectives P_j under induction product identical.

cf important lemma above

Continuum limit

"Purely algebraic" $L \to \infty$ limit as for semisimple case

More useful is <u>continuum</u> limit, uses a choice of Ham.

Here use

$$H = -\sum_i e_i$$

As $L \to \infty$

- rescale to keep length of system fixed
- rescale H to obtain limiting (low-energy) spectrum of discrete levels

This should work when $-2 < m \leq 2$, system is critical $\Rightarrow$ cont. limit is (boundary) CFT.

$L_n \propto$ Fourier modes of e_i's

Now TL modules become Virasoro modules, but structure same as in algebraic $L \to \infty$ limit.

$\to$ **same fusion rules**

Use results from earlier work to obtain conformal weights: (Bauer + Saleur 1989)

jth standard module of TL/Virasoro becomes Kac/BPZ "standard" module with

$$h_j = h_{1,1+2j} = \frac{(2jx-1)^2-1}{4x(x+1)}$$

with $$c = 1 - \frac{6}{x(x+1)}$$

where $$q = e^{i\pi/(x+1)}.$$

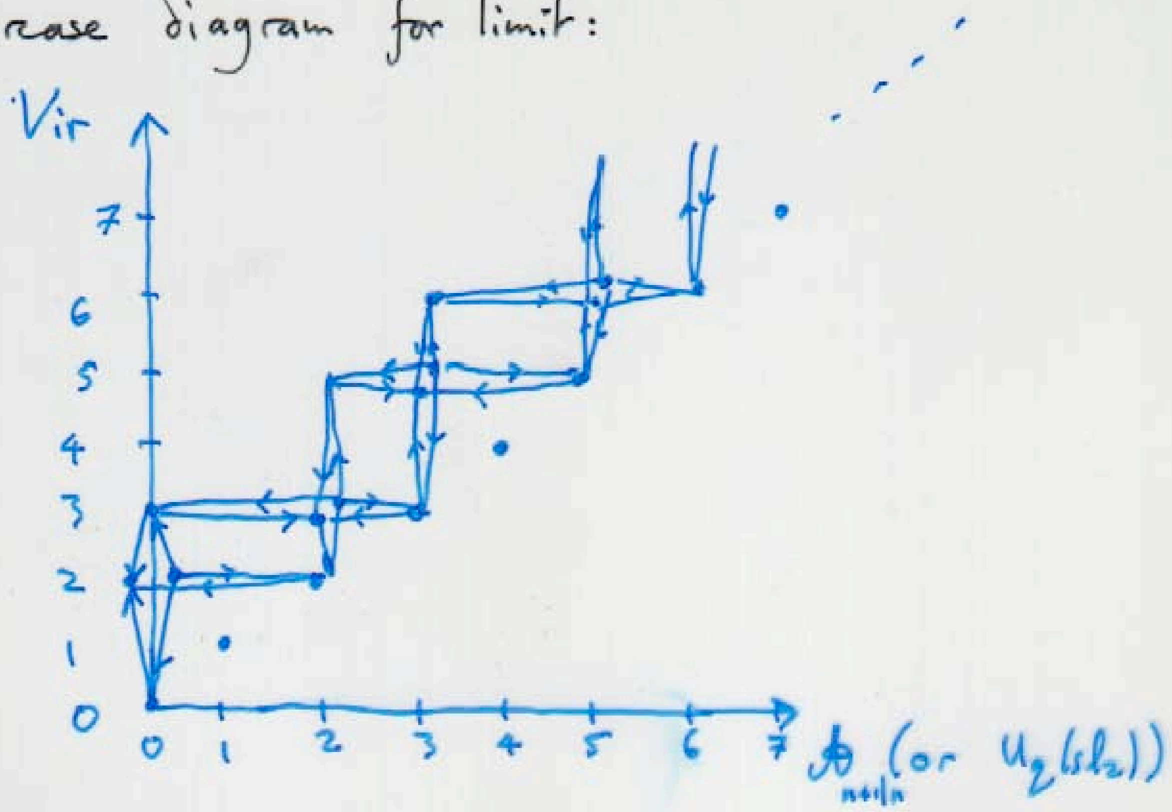
For q a root of unity (x rational), some of these are reducible, e.g. for $m=1$ ($q = e^{i\pi/3}$),

$$j \not\equiv 1 \pmod 3$$

— TL and Vir analyses agree

$$q = e^{i\pi/3}: \qquad h_j = \frac{j(2j-1)}{3}$$

Staircase diagram for limit:



TL projectives P_j $\longrightarrow$ "Vir projective modules" R_j

- <u>same</u> structure
- <u>same</u> fusion rules !! obtainable from $U_q(sl_2)$

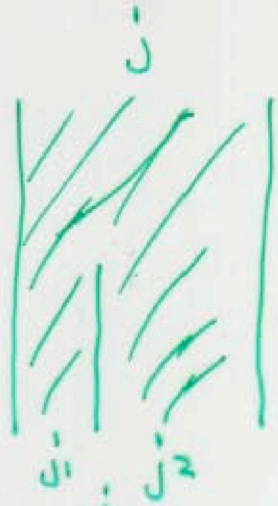
$R_{j_1} \times R_{j_2}$ = sum of R_j 's.

Correspondence of fields and states

Picture of fusion:

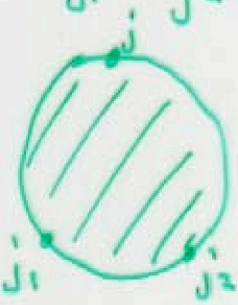
picture as

time



cf "slit strip" picture of open-string interactions

Conf mapping:



Ex of indecomp "proj" module : $\mathcal{R}_2$

$$\mathcal{R}_2 : \quad \begin{tikzcd} & R_2 \arrow[dl] \arrow[dr] & \\ \mathbb{1} = R_0 \arrow[dr] & & R_3 \; \tau \arrow[dl] \\ & T \; R_2 & \end{tikzcd}$$

R_j = simple Vir modules

Prim Fields :

Conf weights: $\quad h_0 = 0 \quad \mathbb{1}$

$h_2 = 2 \quad T(z), \; t(z)$

$h_3 = 5 \quad \tau(z)$

In CFT, alg action is ope with stress tensor:

$$T(z)\, t(0) \sim \frac{b\,\mathbb{1}}{z^4} + \frac{2t(0) + T(0)}{z^2} + \frac{t'(0)}{z} + \quad \text{(Gurarie - Ludwig)}$$

$$\ldots + z\, c^{\tau}_{Tt}\, \tau(0) + \ldots$$

$$T(z)\, \tau(0) \sim \frac{c^{T}_{T\tau}\, T(0)}{z^5} + \ldots$$

as shown in diagram above

$$\begin{tikzcd} & t \arrow[dl] \arrow[dr] & \\ \mathbb{1} \arrow[dr] & & \tau \arrow[dl] \\ & T & \end{tikzcd}$$

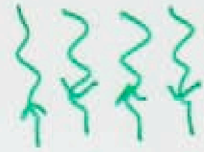
Also $\mathcal{R}_0 : \; R_0 \to R_2$ ie $\mathbb{1} \to T$

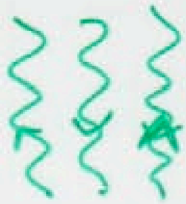
acts as identity in fusion rules $\leftrightarrow$ reps of Vir, gen. by T

What about j = half-integer, $j = \frac{1}{2}, \frac{3}{2}, \ldots$?

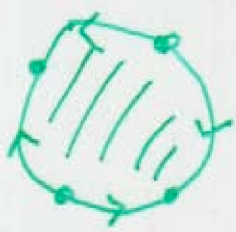
Violates rules for orientation

Odd length, $2L$ = odd





- changes boundary condition for even-length chains
- need even number:



$h_{1/2} = 0$

- Cardy's b.c. changing operator, crossing probability formulas

- can do fusion rules, etc.

Main Results (Theorems)

Construction of ribbon Hopf algebras $\mathcal{A}_{m+n|n}$

$\boxed{\mathcal{A}_{m+n|n} \text{ Morita equiv to } U_q(sl_2) \text{ (as r.H.a.)}}$

– used Ringel duality with $TL(q)$

$\boxed{Proj(TL(q)) \text{ is a ribbon category, equiv to } Tilt(U_q(sl_2))}$

$\boxed{\text{A ribbon cat of "projective" Virasoro modules, equiv to } Tilt(U_q(sl_2))}$

All come in two variants, and

$m \leftrightarrow q \leftrightarrow c$

Conclusion

1) Can also do unoriented $(O(n))$ models. $j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$

Dense and dilute phases $\to$ 4 series of theories

2) Periodic case: much richer
some results, NR - Saleur 2001
still in progress

3) Main points:
a) Morita equiv of symm alg with $U_q(sl_2)$

b) fusion rules from lattice using symmetry alg.