A short review of the sandpile model as a logarithmic CFT

Philippe Ruelle

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Forward

Historically, sandpile models have been proposed by Bak, Tang & Wiesenfeld ('87) as prototypes of self-organized critical models (SOC).

Idea was: many critical behaviours (power laws) in nature, but unlikely to result from fine-tuning $\longrightarrow$ it is the dynamics that drives the system to a critical state, even if the system is prepared in a non-critical state.

Example (BTW) = Abelian sandpile model = ASM, with slow addition of sand (pile builds up, then avalanches of all sizes).

Many other sandpile models, with deterministic or stochastic toppling rules, directed or isotropic, more or less complex than ASM.

Being more tractable, 2d ASM is the most studied, though still challenging ...

[Deepak Dhar, Theoretical studies of self-organized criticality, Physica A 369 (2006) 29-70]

Important for us:

1. interesting non-equilibrium system, with stationary measure
2. lattice realization of logarithmic CFT, $c = -2$ (so it seems ...)

Plan

1. The Abelian sandpile model
   recurrent configurations – definition of invariant measure – spanning trees – boundary conditions

2. Lattice observables in ASM $\leftrightarrow$ LCFT (the pros)
   dissipation – change of boundary conditions – height variables

3. Difficulties and open problems (the cons)
   cylinder or torus partition functions – boundary conditions

4. Conclusions and developments

The sandpile model

Take a grid Λ with N sites.

Attach a random variable $h_i = 1, 2, 3, 4$ to every site (h_i is $\#$ grains).

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	2	4	3	1	2	3	4	1

$\#$ stable configs $= 4^N$

Dynamics

ASM is a dynamical system in discrete $2+1$: $\mathcal{C}_t \xrightarrow{\mathcal{T}} \mathcal{C}_{t+1}$.

Defined in two steps:

1. on **random** site i, drop one grain: $h_i \to h_i + 1$
2. relaxation: all unstable sites topple (avalanche)

   If $h_i \geq 5$, then $\begin{cases} h_i \to h_i - 4 \\ h_j \to h_j + 1, \quad j = \text{nearest neighbour of } i \end{cases}$

   Until all sites are stable again $\longleftarrow$ OK BECAUSE DISSIPATION !!
   Resulting configuration is $\mathcal{C}_{t+1}$. (on boundaries)

Potential chain reaction: one grain dropped can trigger a large avalanche.
System spanning avalanches will happen, and induce correlations of heights over long distances $\longrightarrow$ critical state.

Main properties

- stochastic dynamics $\rightarrow$ proba distribution $P_t(\mathcal{C})$ on set of configs.
- certain configs, called transient, have a zero probability to occur after the dynamics has been run for long enough.
  The image of the repeated dynamics $\mathcal{T}$ shrinks and then stabilizes.
- unique probability measure P^* invariant under dynamics

$$P^*_\Lambda(\mathcal{C}) = P_\infty(\mathcal{C}) = \lim_{t\to\infty} \mathcal{T}^t\, P_0(\mathcal{C})$$

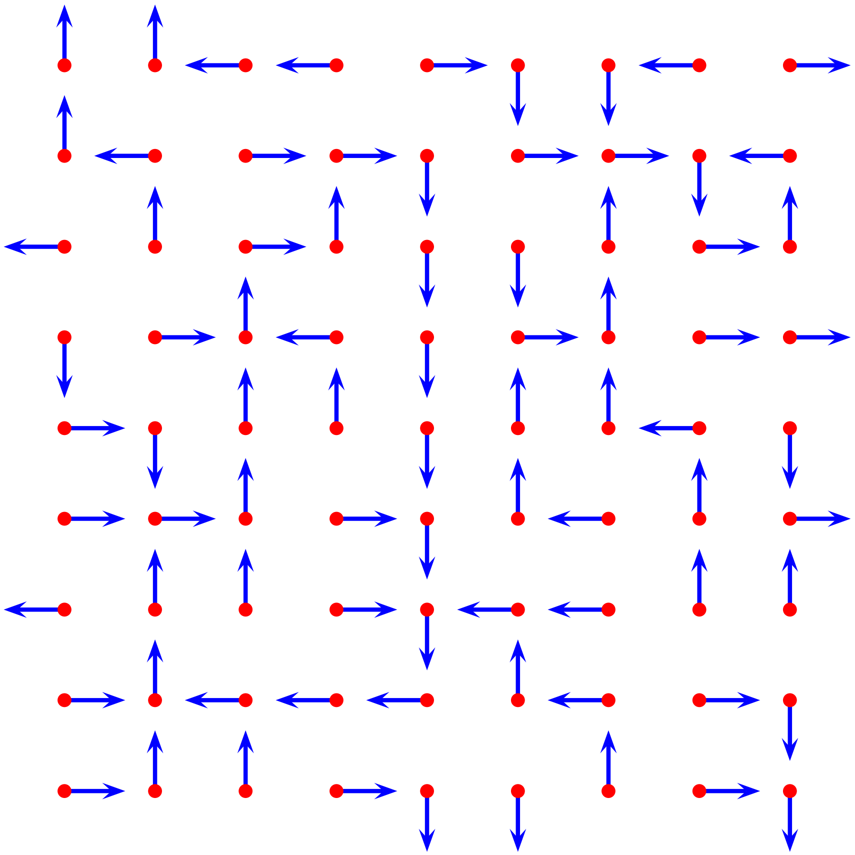
- P^*_Λ is non-zero, and uniform, on **recurrent** configs, a tiny fraction of all stable configs

$$|\mathcal{R}| \simeq 3.21^N \ll 4^N$$

  But being recurrent imposes non-local constraints ...

From heights to arrows

Height variable $h_i = 1, 2, 3, 4$ at every lattice site.
Replace h_i by an arrow pointing N, E, S, W (to one of its neighbours)



There are 4^N arrow configs.

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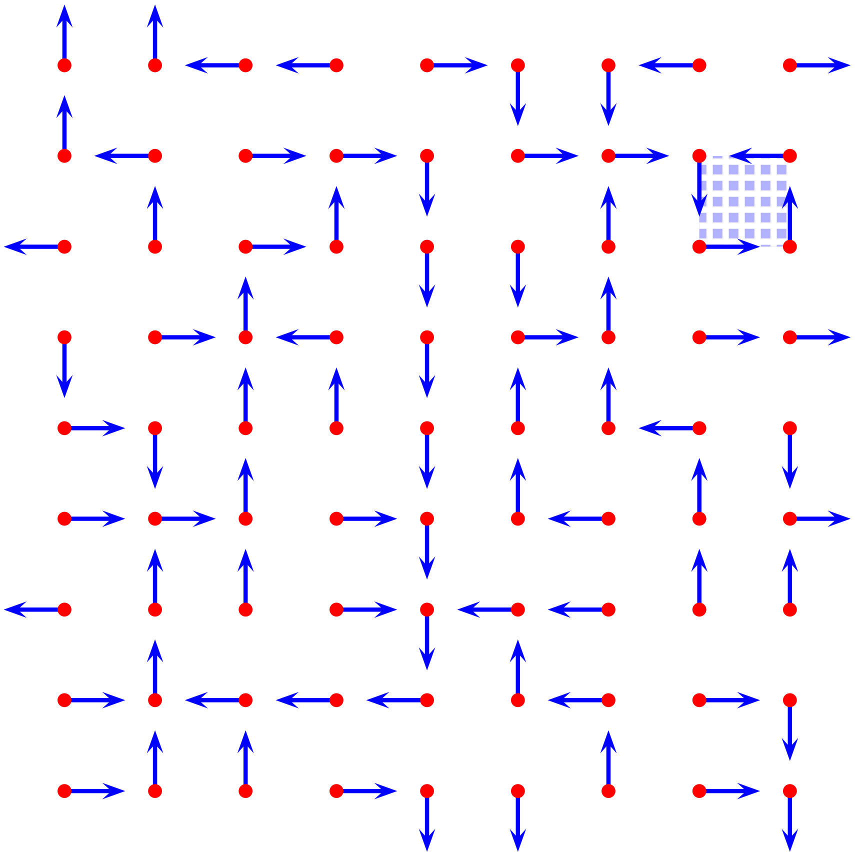

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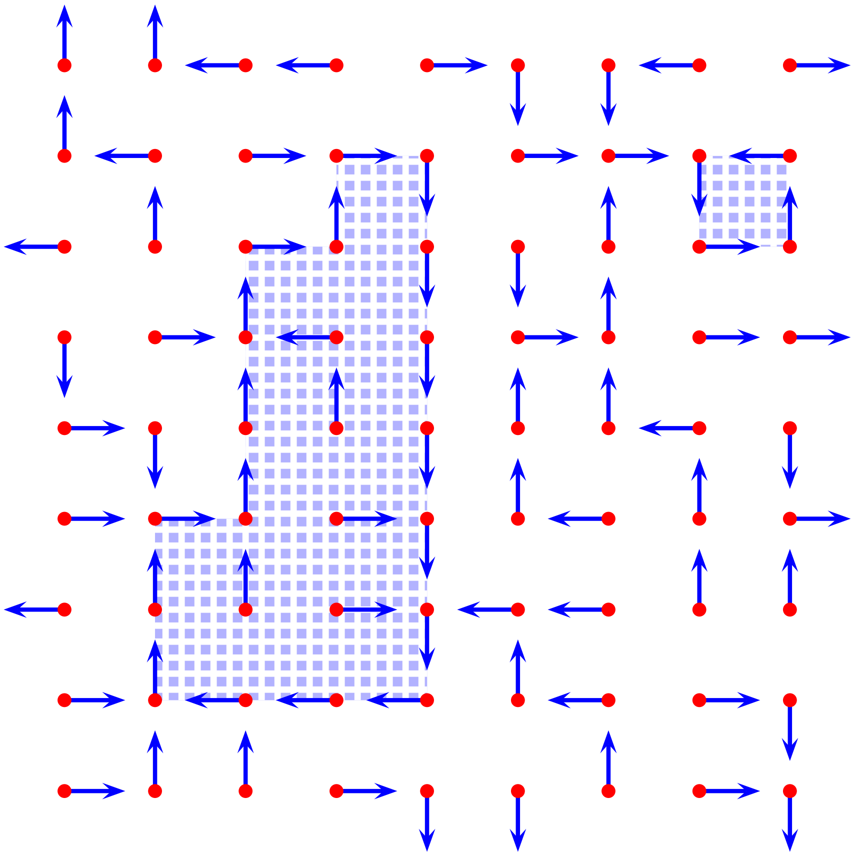

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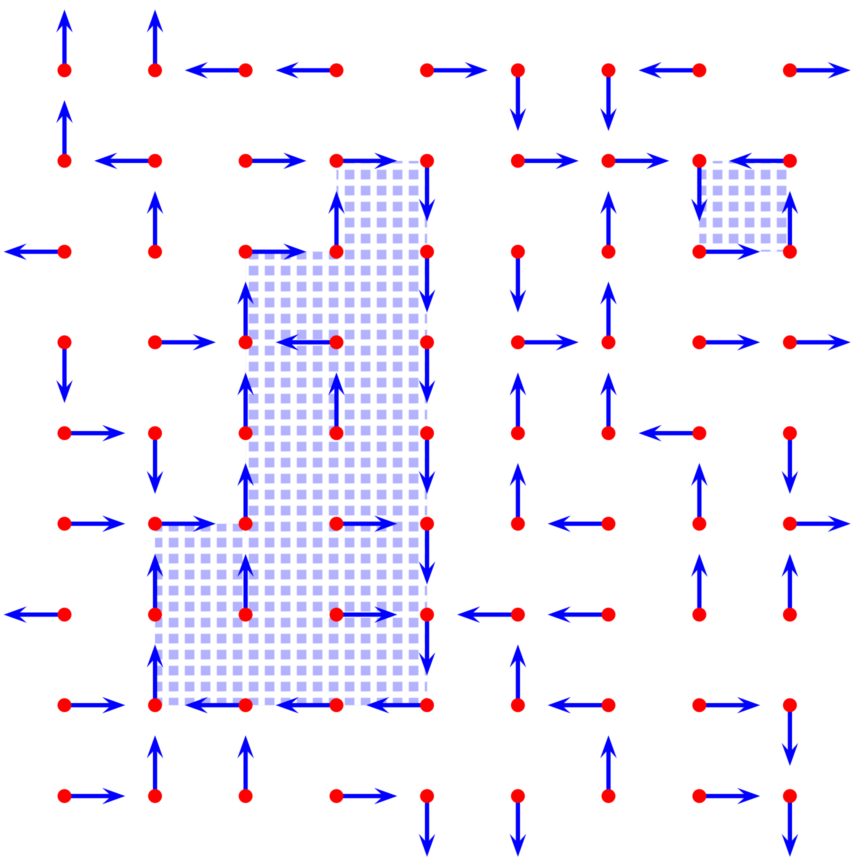


There are 4^N arrow configs.

If a subset of arrows form a loop, then simply exclude that arrow configuration !

Keep only arrows without loops

$\Downarrow$

ROOTED SPANNING TREES

(oriented towards the roots)

Spanning trees

1. There is a 1-to-1 correspondence between the recurrent configs and spanning trees.

   There is an explicit, non-local, mapping between height configs and spanning trees (burning algorithm). See later for examples.

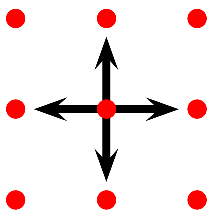
2. From Kirchhoff's theorem, partition function is

$$Z_\Lambda \equiv \#\ \text{spanning trees} = |\mathcal{R}| = \det \Delta \simeq 3.21^N$$

   with Δ the Laplacian matrix,

$$\Delta_{ij} = \begin{cases} 4 & \text{for } i = j \\ -1 & \text{for } \langle i, j \rangle \end{cases}$$

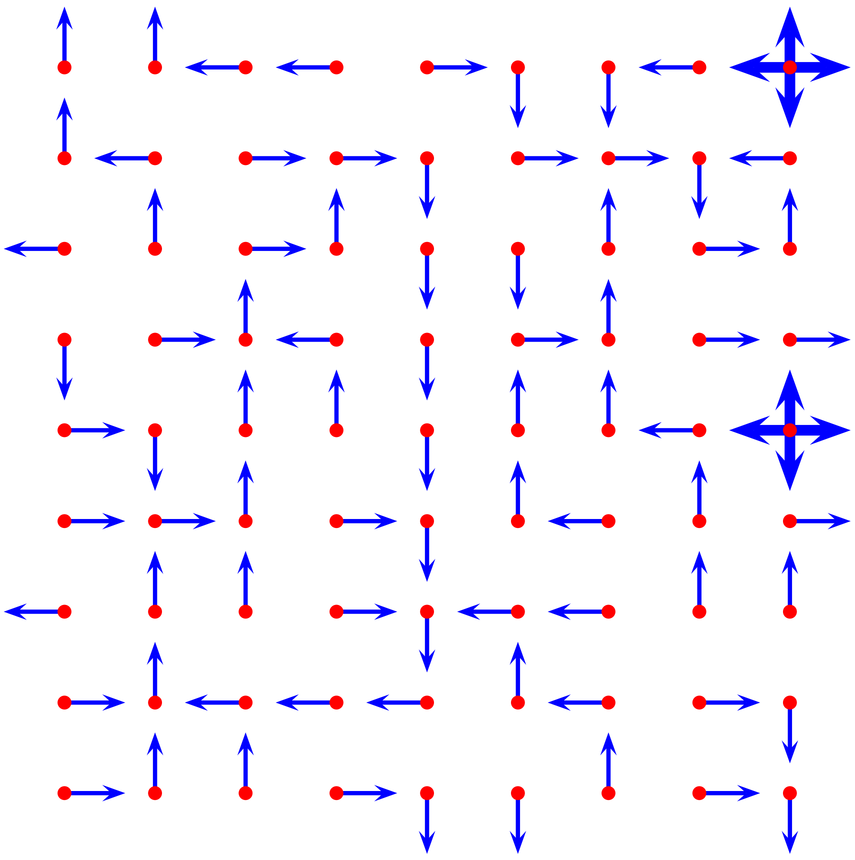
   In description by heights, Δ is the toppling matrix :



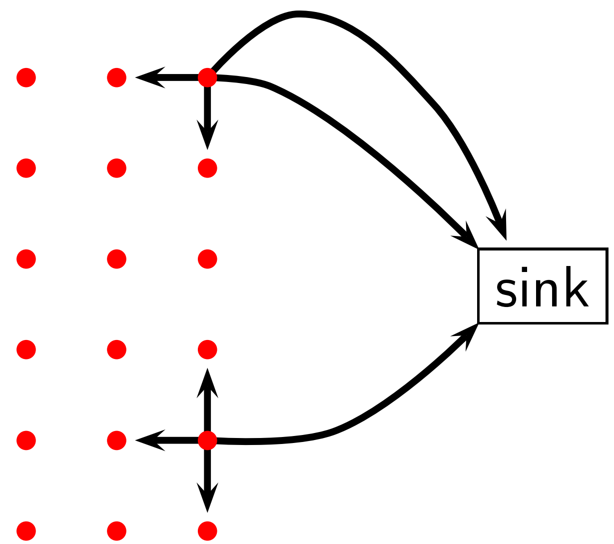
Boundaries

Need for a prescription for boundary sites.

The arrows may be allowed to have all four directions ($\#\ h$-values $= 4$)



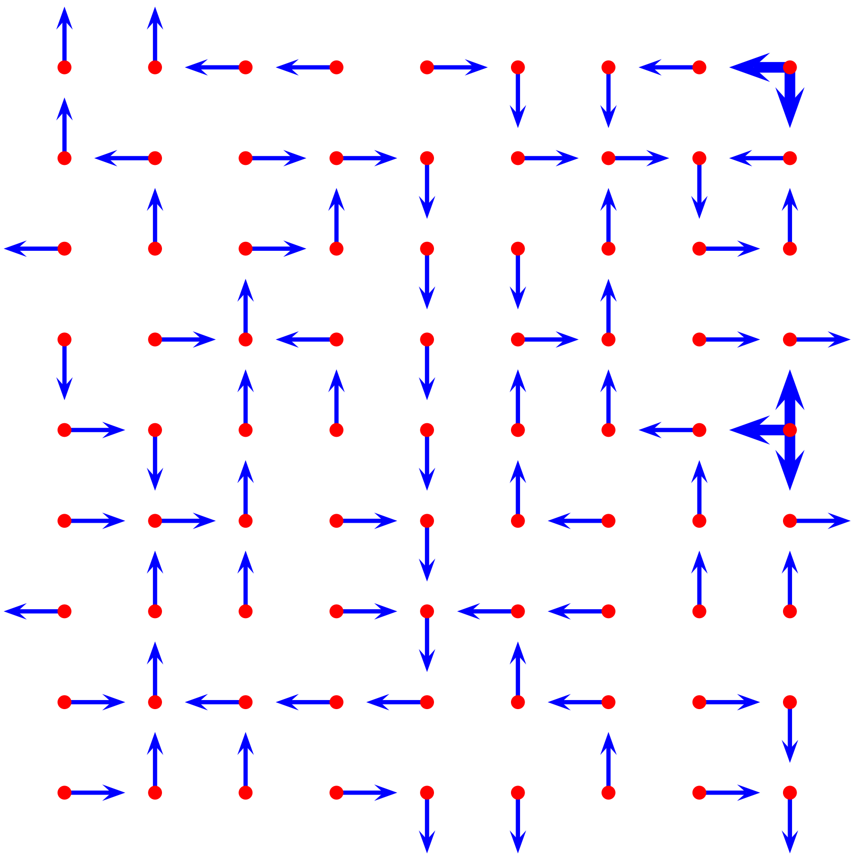
Such boundary sites are OPEN (or Dirichlet, dissipative):



Boundaries

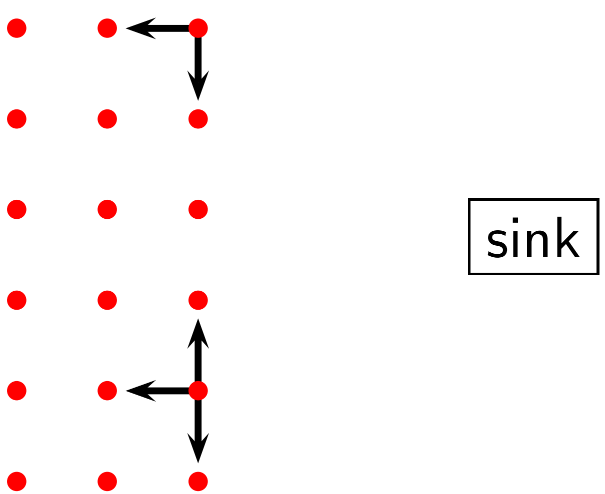
Need for a prescription for boundary sites.

Boundary arrows may be forced to point inwards only ($\#$ h-values < 4)



These are called CLOSED

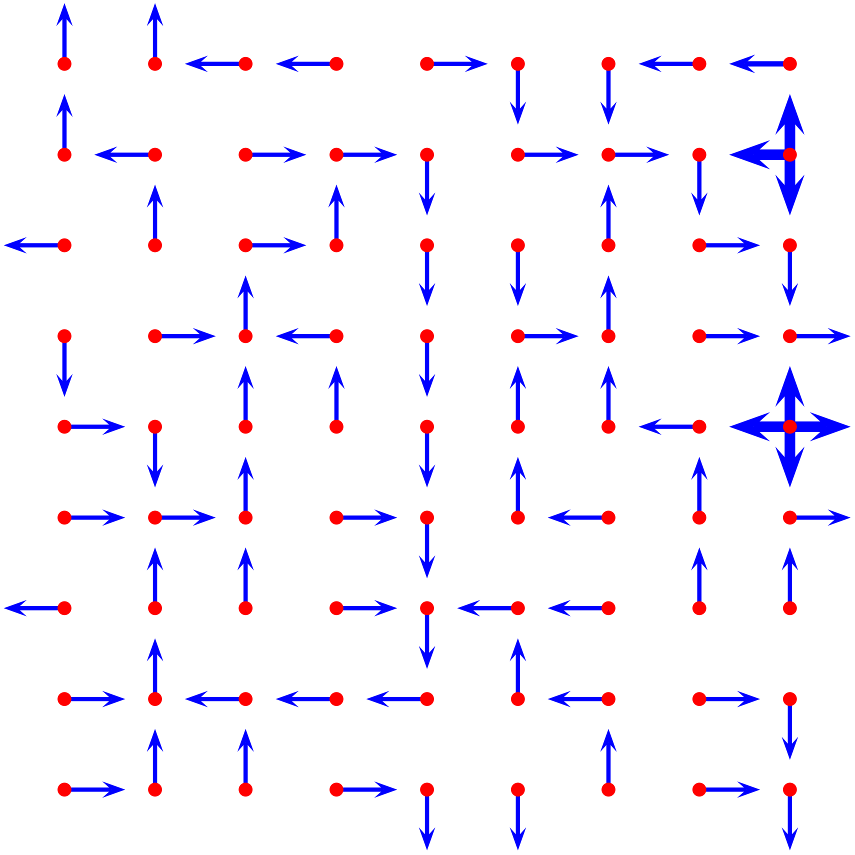
(or Neumann, conservative):



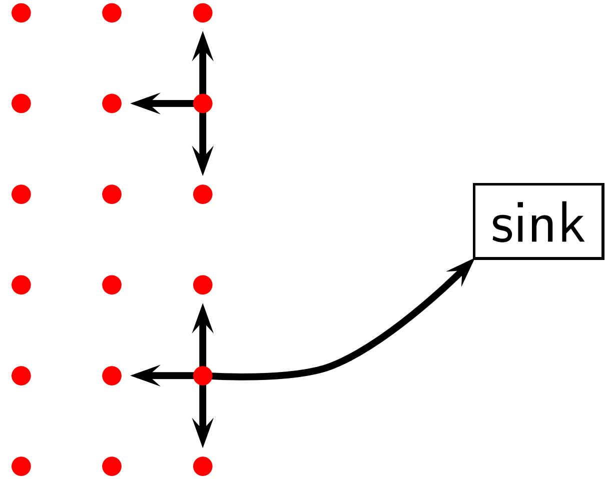
Boundaries

Need for a prescription for boundary sites.

And we can make some open and some others closed:



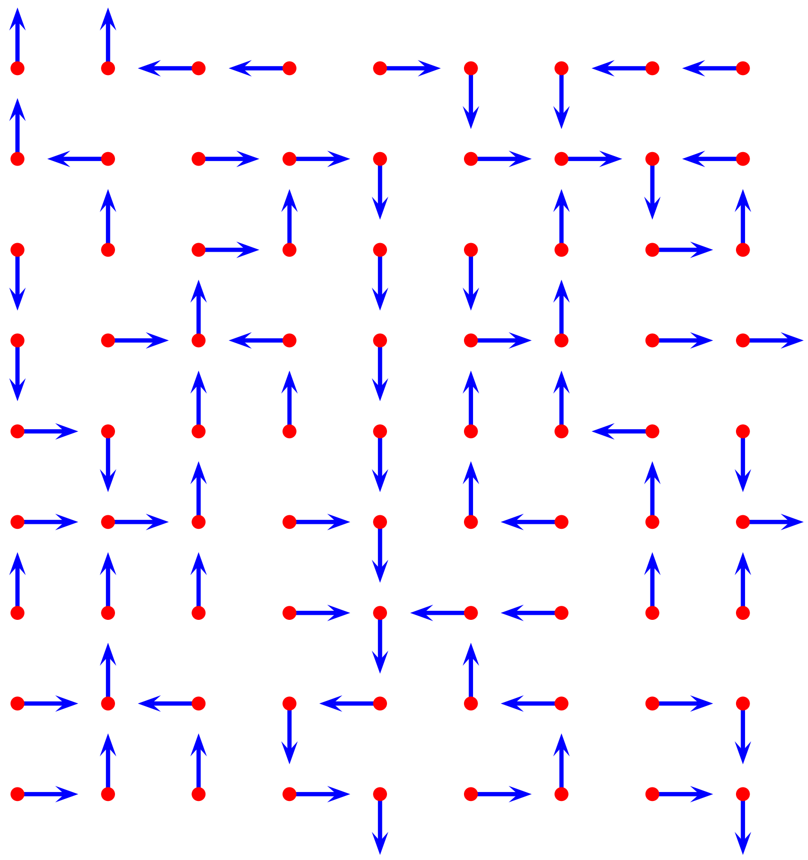
Mix of OPEN and CLOSED



Boundary conditions

Yields two boundary conditions for every boundary site: open or closed,

$\longrightarrow$ Open or closed boundary (or portion of).



Number of spanning trees depends on number of open sites.

Kirchhoff's theorem still holds ($\#$ spanning trees $= \det \Delta$) with appropriate Laplacian matrix.

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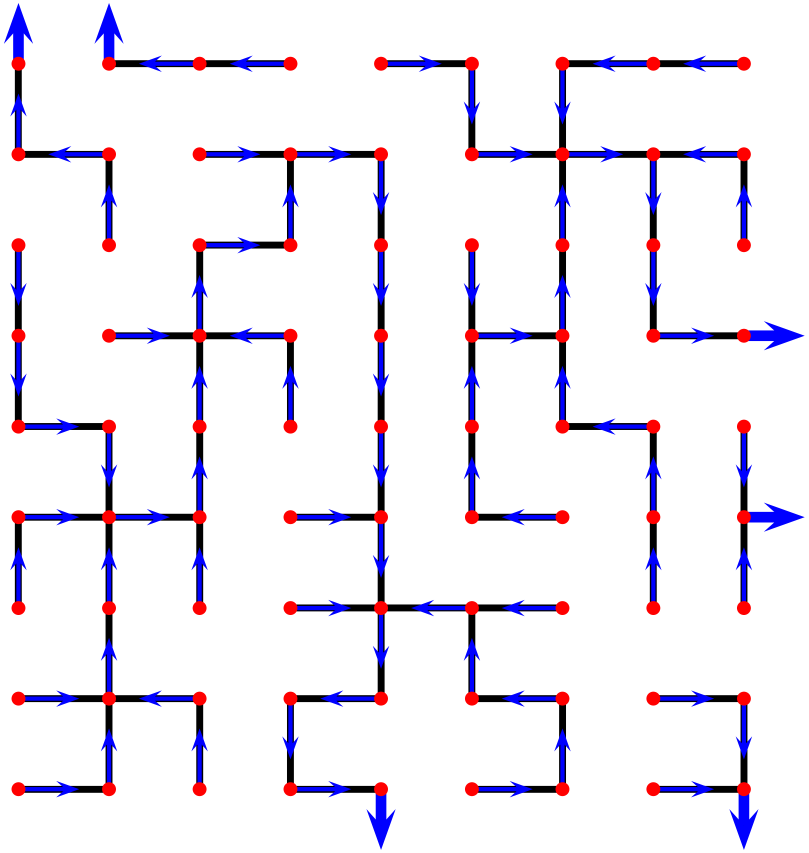
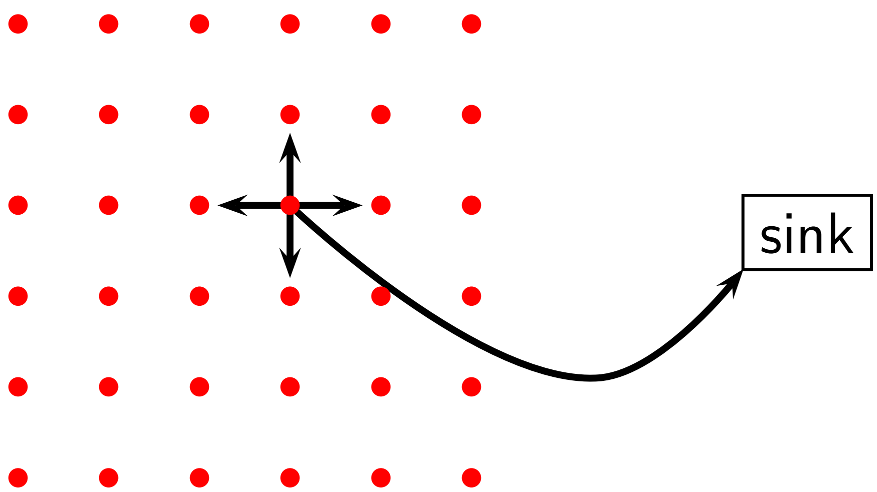


In the spanning tree description, all arrows point towards a root: the tree grows from the roots.

As many potential roots as open boundary sites.

Open bulk sites

Note that bulk sites can also be made open/dissipative by allowing their arrow to point towards the sink (when toppling, one grain is lost)
$\longrightarrow \#\ h$-values $= 5$

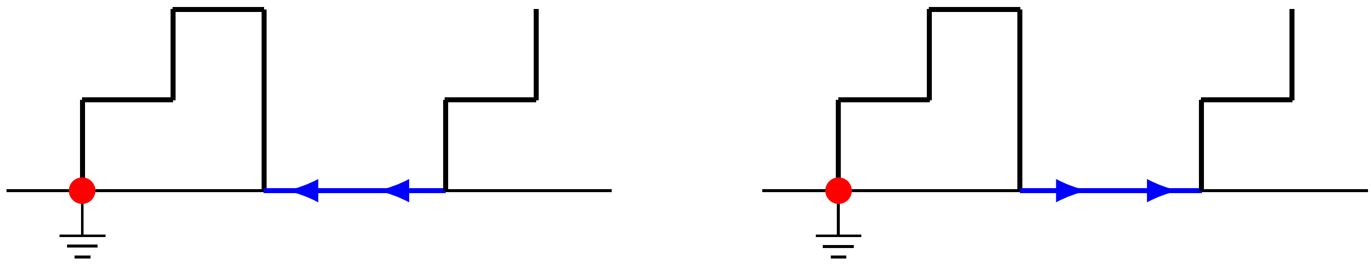


if all sites are open/dissipative, the model ceases to be critical.

Note that at least one open site (boundary or bulk) is required (one needs dissipation !)

Other b.c.'s

- force boundary arrows (in spanning tree variables).
  Trees are constrained to contain certain boundary bonds.
  Direction matters $\longrightarrow$ directional boundary conditions !!



- periodic boundary condition
  Cylindrical or toric geometry can be considered provided open sites are present (must be bulk sites for torus).

- others ???

ASM summary

1. Steady state behaviour of sandpile is controlled by invariant measure $P^*_\Lambda = \lim_{t\to\infty} P_t$.

2. For fixed lattice shape and size, and fixed number of open sites, the invariant measure P^*_Λ is unique, and uniform on the set of rooted spanning trees (= recurrent configs).
   Non-local degrees of freedom !

3. P^*_Λ explicitly depends on type of lattice, size of lattice, boundary conditions, number of dissipative sites, dissipation rates, ...

ASM summary

1. Steady state behaviour of sandpile is controlled by invariant measure $P_\Lambda^* = \lim_{t\to\infty} P_t$.

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3. P_Λ^* explicitly depends on type of lattice, size of lattice, boundary conditions, number of dissipative sites, dissipation rates, ...

Is the scaling limit $\lim_{|\Lambda|\to\infty} P_\Lambda^*$ of the invariant measure the quantum field theoretic measure of a (logarithmic) conformal field theory ???

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Among testable issues

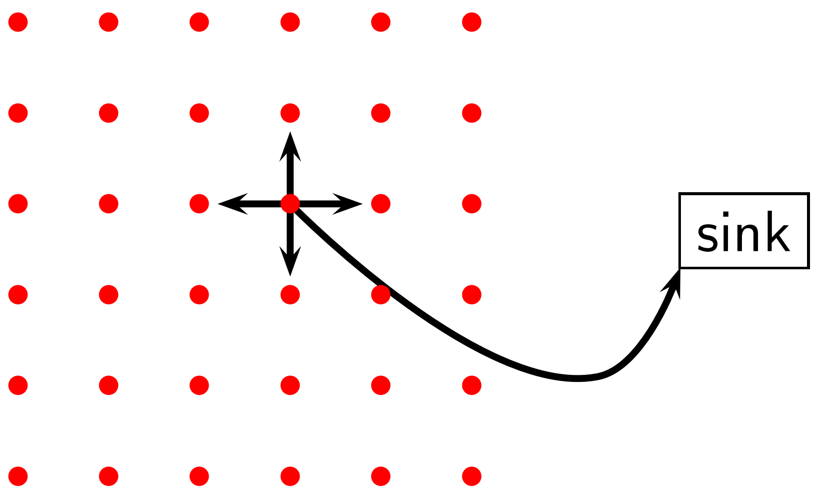
To confirm the relevance of conformal description, ask questions that have an answer in CFT:

1. Effect of introducing additional dissipation/roots (*)
2. Effect of changing the boundary conditions (**)
3. Correlations of height variables (***)
4. Combine previous three (*******)

Note that we need lattice correlators in infinite volume.

<u>Here</u> : we take the infinite volume limit of finite volume results.

Isolated dissipation = adding roots



Change from usual Laplacian

$$\Delta_{ii} = 4, \quad \Delta_{\langle ij \rangle} = -1,$$

to new one with only change at z:

$$\Delta'_{zz} = 5, \quad \Delta'_{\langle zj \rangle} = -1.$$

New Laplacian:

$$\Delta' = \Delta + B, \qquad B_{ij} = \delta_{i,z}\, \delta_{j,z}.$$

The effect of introducing dissipation can be measured by the fraction by which the number of recurrent configurations increases:

$$\frac{Z(\text{with dissip. at } z)}{Z} = \frac{\det \Delta'}{\det \Delta} = \frac{\#\text{ recurrent configs in new model}}{\#\text{ recurrent configs in original model}}$$

$$\overset{\text{scalim}}{\longleftrightarrow} \langle \omega(z, \bar{z}) \rangle \quad ??$$

Same on closed boundary (open boundary is already dissipative), and at more than one site, $\Delta' = \Delta + B_1 + B_2 + \ldots$,

$$\frac{Z(\text{with dissip. at } z_1, z_2, \ldots)}{Z} = \frac{\det \Delta'}{\det \Delta} \quad \overset{\text{scalim}}{\longleftrightarrow} \quad \langle \omega_1 \omega_2 \ldots \rangle \quad ??$$

with ω_i bulk or boundary (chiral) field.

Not difficult to see that above ratio contains logarithms (+ lower).

Since $\Delta' - \Delta = B$ has finite rank,

$$\frac{\det \Delta'}{\det \Delta} = \frac{\det[\Delta + B]}{\det \Delta} = \frac{\det \Delta[\mathbb{I} + \Delta^{-1} B]}{\det \Delta} = \det[\mathbb{I} + \Delta^{-1}]_{z_1, z_2, \ldots}$$

dominated by logarithms ...

In the scaling limit, it was found to be consistent to identify the insertion of isolated dissipation at a closed site with the insertion of a dimension 0 logarithmic field ω, partner of the identity.

[Dissipation at open sites leads to scale dimension 2 field, less relevant.]

Checked :

✓ insertion of dissipation at different points, both bulk and boundary

✓ bulk to boundary OPE ($\omega_{\rm bulk} \longrightarrow \omega_{\rm boundary}$)

✓ insertion of boundary and change of b.c. (see later)

✓ dissipation at all sites : system no longer critical (expon. decays)

Pertubation of CFT by $m^2 \int \omega(z, \bar{z}) \sim m^2 \int \tilde{\theta}\theta$ (mass term)

(Realized by fermions)

The ω's have a realization in terms of symplectic fermions.

All calculations are exactly compatible with following identifications :

$$\omega_{\rm bulk}(z,\bar{z}) \equiv \text{(insertion of dissipation at bulk } z) = \frac{1}{2\pi}\theta\tilde{\theta} + \gamma_0\,\mathbb{I}$$

$$\omega_{\rm cl}(x) \equiv \text{(insertion of dissipation at closed } x) = \frac{1}{2\pi}\theta\tilde{\theta} + (2\gamma_0 - \frac{5}{4})\,\mathbb{I}$$

computed from Wick contractions.

On open boundary, dissipation is compatible with

$$\text{(insertion of dissipation at open } x) = \frac{2}{\pi}\partial\theta\partial\tilde{\theta}$$

Change of boundary conditions

- set $\mathcal{B} = \{\alpha\}$ of conformally invariant b.c.'s.

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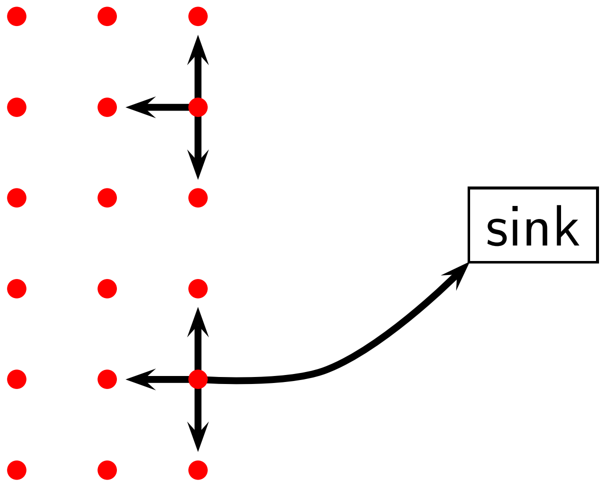
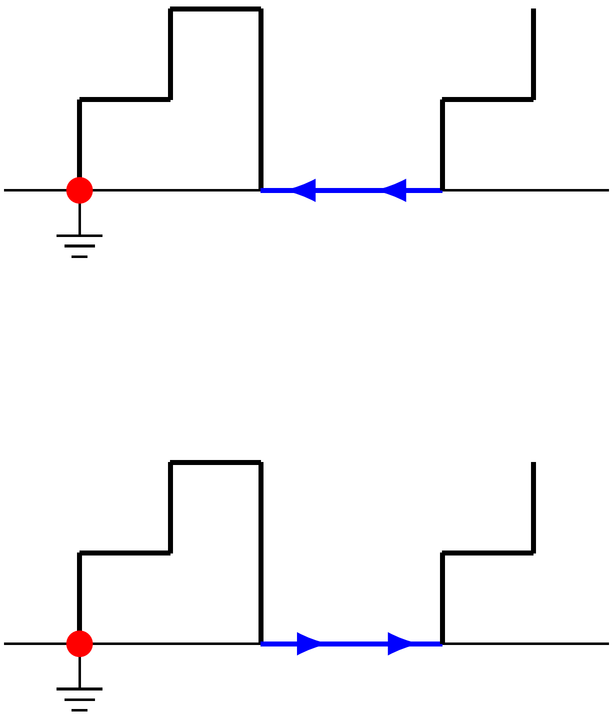
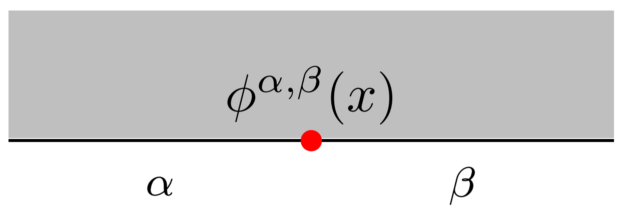
  Open, closed, left arrows, right arrows:





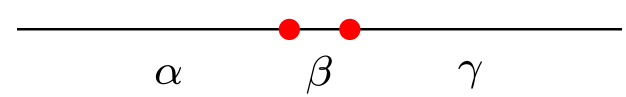
Change of boundary conditions

- set $\mathcal{B} = \{\alpha\}$ of conformally invariant b.c.'s. For now, only four.
- $\mathcal{B}$ can be finite or infinite (supposedly our case).
- a change of boundary condition at a point x, from α to β is realized by the insertion of a (chiral) boundary field $\phi^{\alpha,\beta}$



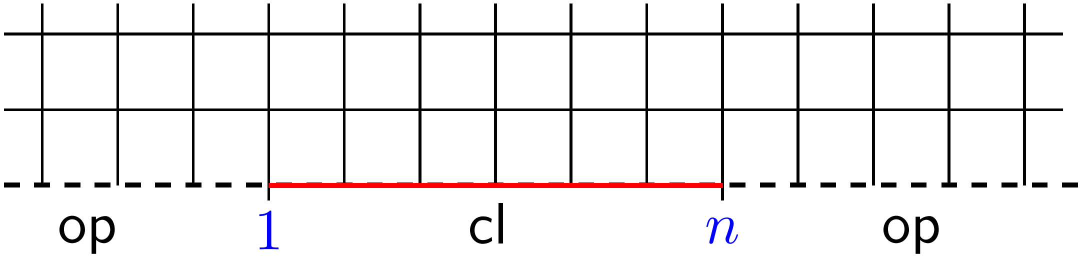
Consistency : b.c.c.f. $\phi^{\alpha,\beta}$ are primary fields satisfying a boundary fusion algebra (composition law) with identity $\phi^{\alpha,\alpha} = \mathbb{I}$:

$$\lim_{x \to y} \phi^{\alpha,\beta}(x) \star \phi^{\beta,\gamma}(y) \simeq \phi^{\alpha,\gamma}(y)$$



Open $\leftrightarrow$ closed

To turn open into closed, remove dissipation: $\Delta' = \Delta - B_1 - B_2 - \ldots$



Similar to previous,

$$\frac{\det \Delta'}{\det \Delta} = \frac{\det[\Delta - B]}{\det \Delta} = \det[\mathbb{I} - \Delta^{-1}]_{1 \le i,j, \le n} \simeq A n^{1/4} \leftrightarrow \langle \mu(0) \mu(n) \rangle$$

plus many others checks lead to:

The change of boundary condition from open to closed, and vice-versa, is effected, in the scaling limit, by the insertion of a chiral, boundary primary field $\phi^{\mathrm{op,cl}} = \phi^{\mathrm{cl,op}} \equiv \mu$ with conformal dimension $-\frac{1}{8}$.

Fixed arrows

Same idea as before: insert a string of n consecutive arrows ($\longrightarrow$ perturbed Laplacian $\Delta' = \Delta + B$) and measure the effect by the ratio:

$$\frac{\#\{\text{spanning trees with } n \text{ prescribed arrows}\}}{\#\{\text{spanning trees}\}} = \det[\mathbb{I} + \Delta^{-1}B].$$

<u>Remember</u> : left and right arrows are <u>not</u> identical $\longrightarrow$ oriented b.c.'s

Appropriate B allows to pick the spanning trees with the prescribed arrows; always of finite rank, leads to determinant of size $n \gg 1$.
In good cases, asymptotic value can be computed exactly (Szegö).

B.c. changing fields: summary

Present understanding leads to following table (from 2pt, 3pt, 4pt, fusion)

$\phi^{\alpha,\beta}$	open	closed	$\rightarrow$	$\leftarrow$
open	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[0] \in \mathcal{V}_{1,3}$	$[0] \in \mathcal{R}_{2,1}$
closed	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[\frac{3}{8}] \in \mathcal{V}_{2,2}$
$\rightarrow$	$[0] \in \mathcal{R}_{2,1}$	$[\frac{3}{8}] \in \mathcal{V}_{2,2}$	id.	$[0] \in \mathcal{R}_{2,1}$ (op) $[1] \in \mathcal{R}_{3,1}$ (cl)
$\leftarrow$	$[0] \in \mathcal{V}_{1,3}$	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[0] \in \mathcal{V}_{1,3}$	id.

- $\mathcal{V}_{r,s}$ is highest weight (null at rs), and $\mathcal{R}_{r,1}$ are rank 2 staggered modules
- all b.c.c.f. are primary (highest weight of subrepresentation if in a $\mathcal{R}$)
- note $\phi^{\rightarrow \overset{\mathrm{cl}}{,} \leftarrow} \neq \phi^{\rightarrow \overset{\mathrm{op}}{,} \leftarrow}$ (not so for outgoing arrows)

Height variables

Most natural but hardest !

Idea = compute joint probas $P^*[h_{z_1} = a,\, h_{z_2} = b,\, \ldots]$ on lattice and make the correspondence

$$\delta(h_z - a) - P^*(a) \quad \longleftrightarrow \quad \text{field } \; h_a(z) \in \text{LCFT}$$

so that in the scaling limit,

$$\Big\{ P^*[h_{z_1} = a,\, h_{z_2} = b] - P^*(a)\, P^*(b) \Big\}$$
$$= \Big\langle [\delta(h_{z_1} - a) - P^*(a)][\delta(h_{z_2} - b) - P^*(b)] \Big\rangle = \langle h_a(z_1)\, h_b(z_2) \rangle$$

and for all higher point functions.

Number of orientations of arrow at z = number of height values :
4 for bulk or open boundary sites, 3 for closed boundary sites.

Height variables

The identification of scaling fields h_a requires computing lattice correlation functions of height variables ...

Fine for heights 1 (boundary or bulk)

More difficult for heights 2,3,4 on boundary (open or closed)

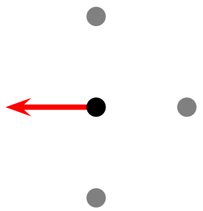
Still harder for heights 2,3,4 in bulk !

Why ??

Trees, branches, leaves

Need to translate arrows into heights ...

Height at z determined by number of neighbours pointing **eventually** to z !



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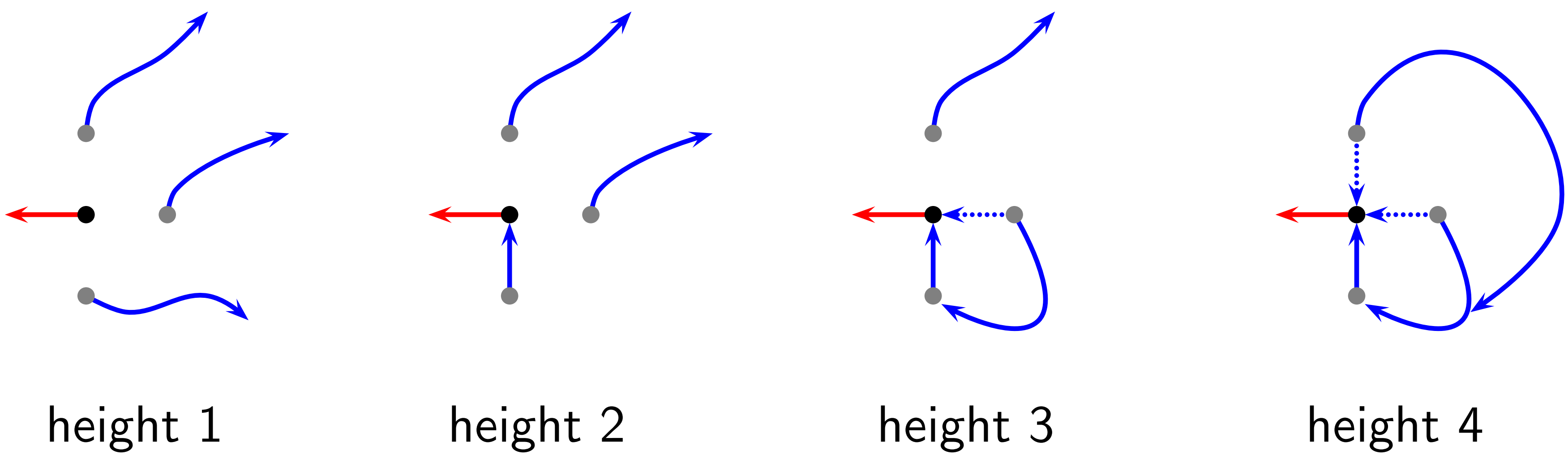

Height 1 : reference site is a leaf $=$ local constraint

Height ≥ 2 : have to control long paths across whole lattice $\longrightarrow$ non-local !!

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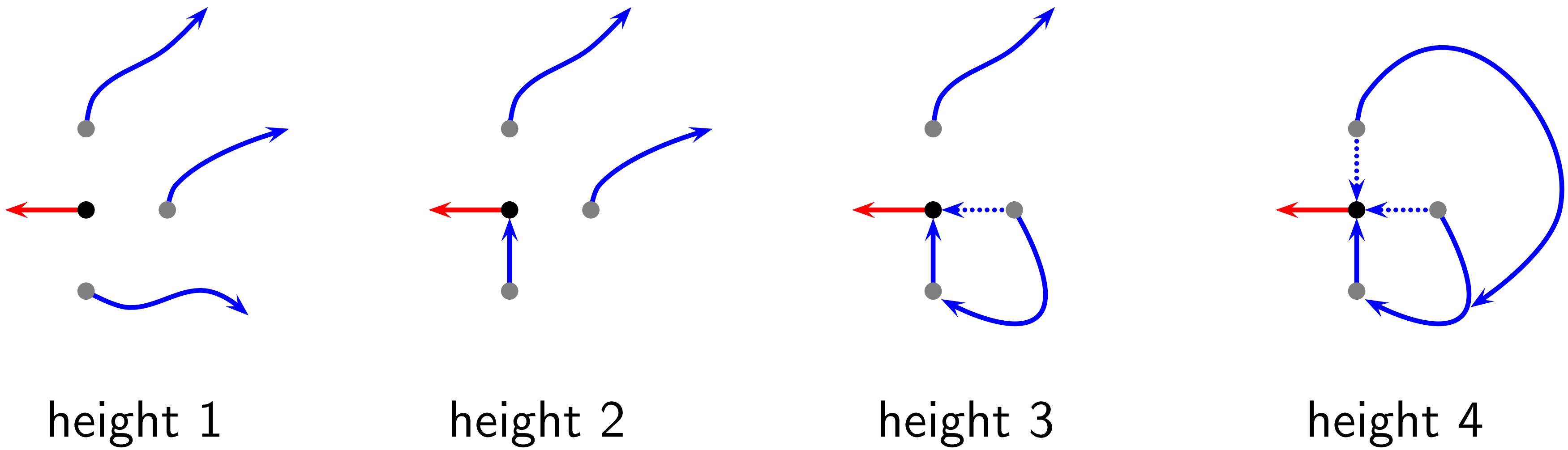

Height 1 : reference site is a leaf $=$ local constraint

Height ≥ 2 : have to control long paths across whole lattice $\longrightarrow$ non-local !!

Heights 1 are easier, while heights 2, 3, 4 are much harder !!

Height variables

On UHP, compute 1-site probability to have height 1,2,3,4 at a distance m from boundary, open or closed.

Asymptotic analysis for m large yields dominant contributions in SL :

$$\begin{aligned} P_i^{\rm op}(m) &= P_i + \frac{1}{m^2}\Big(a_i + \frac{b_i}{2} + b_i\,\log m\Big) + \ldots, \\ P_i^{\rm cl}(m) &= P_i - \frac{1}{m^2}\big(a_i + b_i\,\log m\big) + \ldots, \end{aligned}$$

with coefficients a_i, b_i known exactly : $b_1 = 0$, but $b_2, b_3, b_4 \neq 0$.

$$\begin{aligned} a_1 &= \frac{\pi-2}{2\pi^3}, \qquad b_1 = 0 \\ a_2 &= \frac{\pi-2}{2\pi^3}\Big(\gamma + \frac{5}{2}\log 2\Big) - \frac{11\pi-34}{8\pi^3}, \qquad b_2 = \frac{\pi-2}{2\pi^3} \\ a_3 &= \frac{8-\pi}{4\pi^3}\Big(\gamma + \frac{5}{2}\log 2\Big) + \frac{2\pi^2+5\pi-88}{16\pi^3}, \qquad b_3 = \frac{8-\pi}{4\pi^3} \end{aligned}$$

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with coefficients a_i, b_i known exactly : $b_1 = 0$, but $b_2, b_3, b_4 \neq 0$. These results allow to conclude :

• The height 1 field h_1 is a primary field with weights (1,1).

• Up to normalization, the others three h_2, h_3, h_4 are equal to the logarithmic partner of h_1. They belong to a non-chiral indecomposable $\mathcal{R}_{2,1}$.

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Asymptotic analysis for m large yields dominant contributions in SL :

$$P_i^{\rm op}(m) = P_i + \frac{1}{m^2}(a_i + \frac{b_i}{2} + b_i \log m) + \ldots,$$
$$P_i^{\rm cl}(m) = P_i - \frac{1}{m^2}(a_i + b_i \log m) + \ldots,$$

with coefficients a_i, b_i known exactly : $b_1 = 0$, but $b_2, b_3, b_4 \neq 0$. These results allow to conclude :

• The height 1 field h_1 is a primary field with weights (1,1).

• Up to normalization, the others three h_2, h_3, h_4 are equal to the logarithmic partner of h_1. They belong to a non-chiral indecomposable $\mathcal{R}_{2,1}$.

!!!!! These log fields do not seem to belong to the triplet theory !!!!!

Plan

1. The Abelian sandpile model
   recurrent configurations – definition of invariant measure – spanning trees – boundary conditions

2. Lattice observables in ASM $\leftrightarrow$ LCFT (the pros)
   dissipation – change of boundary conditions – height variables

3. Difficulties and open problems (the cons)
   cylinder or torus partition functions – boundary conditions

4. Conclusions and developments

Partition functions

Most natural definition of partition function is

$$Z_\Lambda = \# \text{ of spanning trees} = \det \Delta.$$

First finite-size corrections give correct central charge, $c = -2$.
But leads to trouble on cylinder and torus ...

Cylinder with open-open b.c.

$$Z_{\text{op,op}} = \det \Delta_{\text{op,op}} = \eta^2(q) = \sum_{k\geq 1} (-1)^{k+1} k \, \chi_{(1,2k+1)}(q)$$
$$= \chi_0 - 2\chi_1 + 3\chi_3 - 4\chi_6 + 5\chi_{10} - \dots$$

in terms of irreducible Virasoro characters.

Cylinder with open-closed b.c.

$$Z_{\mathrm{op,cl}} = \det \Delta_{\mathrm{op,cl}} = \frac{\theta_4(q)}{\eta(q)} = \sum_{k \geq 1} (-1)^{k+1} k \, \chi_{(1,2k)}(q)$$
$$= \chi_{-1/8} - 2\chi_{3/8} + 3\chi_{15/8} - 4\chi_{35/8} + 5\chi_{63/8} - \dots$$

Cylinder with closed-closed b.c.

$$Z_{\mathrm{cl,cl}} = \det \Delta_{\mathrm{cl,cl}} = 0 \; !$$

unless we introduce dissipation by hand (on one boundary f.i.), then

$$Z^*_{\mathrm{cl,cl}} = \det{}^* \Delta_{\mathrm{cl,cl}} = 2(\mathrm{Im}\,\tau)\, \eta^2(q)$$

Same on torus ...

Boundary conditions

Closed b.c. is peculiar :

it may be imposed on whole boundary if dissipation somewhere else, i.e. it cannot be imposed on all boundaries !

Left or right arrow b.c.'s are worse :

they cannot be imposed on the whole of a boundary component !

(because the so-constrained spanning trees would contain a loop)

so they may be imposed on portions of boundaries only ...

Conclusions

Good number of features well understood :

- 4 boundary conditions identified, leading to b.c. changing fields with conformal weights $0, -\frac{1}{8}, \frac{3}{8}, 1$ (some belong to indecomposables)
- isolated dissipation, on boundary or in bulk, with and without change of b.c.; bulk, boundary and bulk-boundary fusions checked
- boundary height variables on closed and open boundaries (not log)
- bulk height variables properly identified (log fields), with and without change of b.c.
- fully dissipative model, no longer critical, described by massive perturbation of $c = -2$

Also strange issues :

- proper interpretation of partition functions in terms of chars ?
- peculiar boundary conditions ? indecomposable ?

Open problems :

- look for other boundary conditions and new bulk observables
- relevant LCFT likely to be non-rational (and not triplet) : which one ?
- description in terms of and relation with SLE ?