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Fusion rules and boundary conditions in logarithmic CFT

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joint work with

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Outline

- From boundary to bulk in non-logarithmic rational CFT
- The $W_{2,3}$ model with c=0
- A boundary theory for the $W_{2,3}$ model
- (A few words on the bulk theory)

Boundary CFT - the Cardy case (non-log., rational)

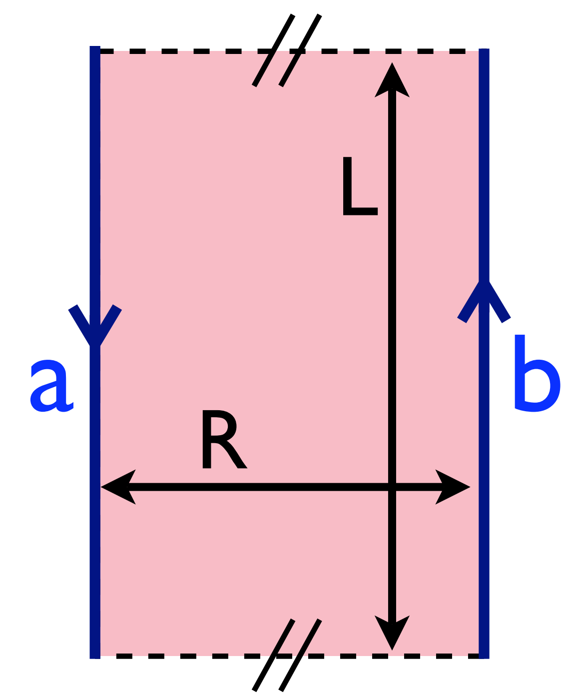
e.g. Virasoro minimal models V(p,q)

Cardy '89

irreducible representations $h_{r,s}$
elementary boundary conditions } Kac label (r,s)

Cylinder partition function

$$A_{a,b}(q) = \sum_c N_{ab}^{\ c}\, \chi_c(q) \qquad q = e^{-\pi L/R}$$



$$U_a \otimes U_b \cong \bigoplus_c N_{ab}^{\ c}\, U_c$$

Rewrite: $A_{U,V}(q) = \chi_{V\otimes U^*}(q)$

Boundary is simpler than bulk (non-log., rational)

Bulk partition function :

$$Z(q) = \sum_i |\chi_i(q)|^2$$

Cylinder partition function for a=b=(1,1) :

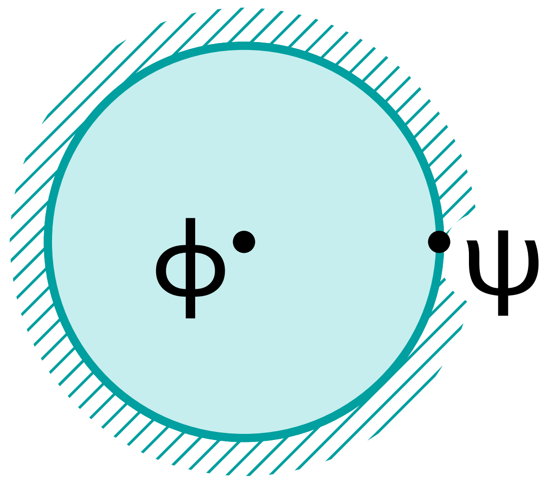
$$A_{a,a}(q) = \chi_{(1,1)}(q)$$

The space of states on the (1,1)-boundary is the irreducible Virasoro vacuum representation (h=0)

Fuchs, Schweigert, IR ‘02
Gaberdiel, IR ‘07

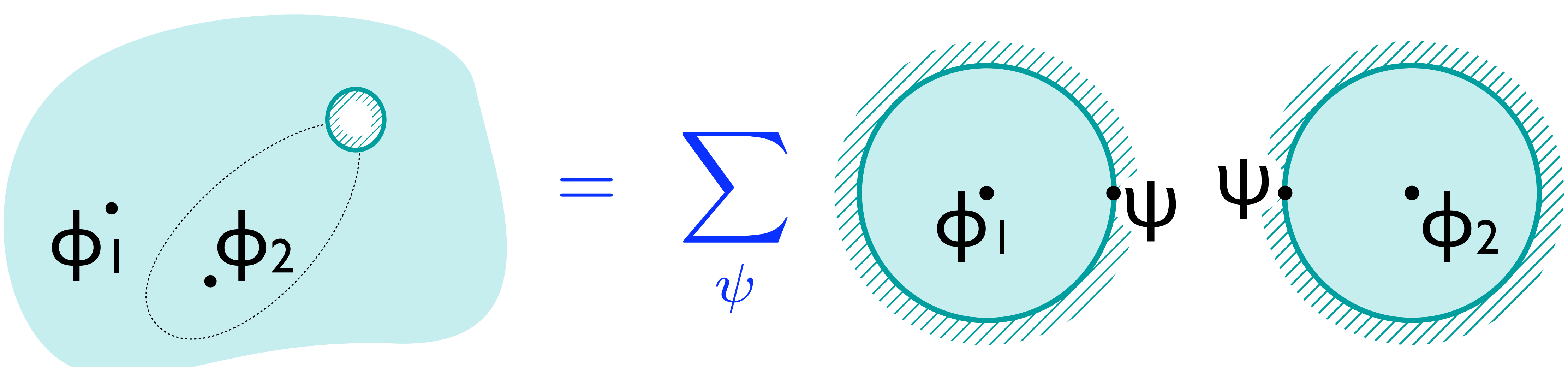
From boundary to bulk

1) Disc correlator of one bulk and one boundary field non-degenerate:



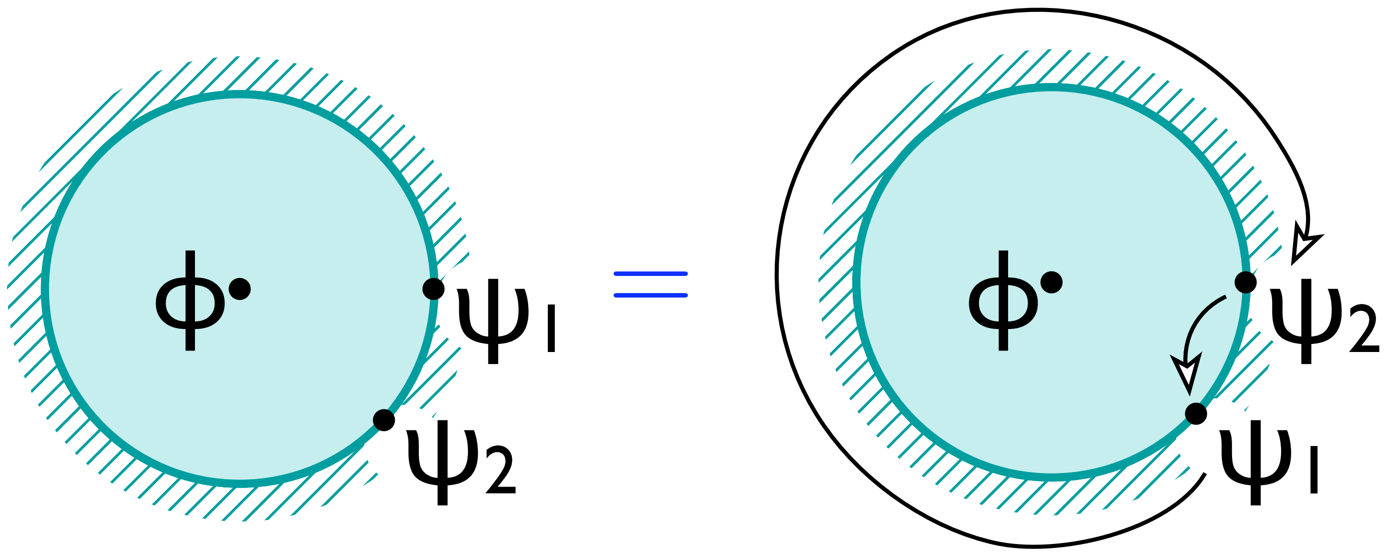
= 0 for all ψ then ϕ = 0

Want this, because:



. . . from boundary to bulk

2) Disc correlator symmetric:



. . . from boundary to bulk

3) Take biggest $\mathcal{H}_{\text{bulk}}$ satisfying 1) and 2)

(non-log., rational)

e.g. boundary condition labelled by vacuum rep. (1,1)

- space of boundary fields is the VOA itself
  $$\mathcal{H}_{(1,1)\to(1,1)} = \mathcal{V}$$

- ansatz for space of bulk fields
  $$\mathcal{H}_{\text{bulk}} = \bigoplus_{i,j} Z_{ij}\ U_i \otimes_{\mathbb{C}} \bar{U}_j^*$$

- constrain Z_{ij} . . .

. . . from boundary to bulk (non-log., rational)

$$\mathcal{H}_{\mathrm{bulk}} = \bigoplus_{i,j} Z_{ij}\; U_i \otimes_{\mathbb{C}} \bar{U}_j^*$$

- $Z_{ij} = 0$ for $i \neq j$:

  If $\phi \in U_i \otimes_{\mathbb{C}} \bar{U}_j^*$ and $\psi \in \mathcal{H}_{(1,1)\to(1,1)} = \mathcal{V}$

  then $\langle\, \phi\, \psi \,\rangle = 0$

  (3-point conformal block on the sphere vanishes for insertions of U_i, U_j^* and V if $i \neq j$.)

. . . from boundary to bulk (non-log., rational)

$$\mathcal{H}_{\rm bulk} = \bigoplus_i Z_{ii}\; U_i \otimes_{\mathbb{C}} \bar{U}_i^*$$

- $Z_{ii} \leq 1$: Suppose

$$\mathcal{H}_{\rm bulk} = \cdots \oplus (U_i \otimes_{\mathbb{C}} \bar{U}_i^*)_1 \oplus (U_i \otimes_{\mathbb{C}} \bar{U}_i^*)_2 \oplus \cdots$$

Take $\phi_1 \in (U_i \otimes_{\mathbb{C}} \bar{U}_i^*)_1$, $\phi_2 \in (U_i \otimes_{\mathbb{C}} \bar{U}_i^*)_2$, then

$$\langle\, \phi_1\, \psi \,\rangle = \lambda_1 \cdot b(\phi_1, \psi) \qquad \langle\, \phi_2\, \psi \,\rangle = \lambda_2 \cdot b(\phi_2, \psi)$$

for the <u>same</u> $b : (U_i \otimes_{\mathbb{C}} \bar{U}_i^*) \times \mathcal{V} \to \mathbb{C}$.

Hence, $\lambda_2 \phi_1 - \lambda_1 \phi_2$ vanishes in all disc correlators.

Results from rational CFT (non-log., rational)

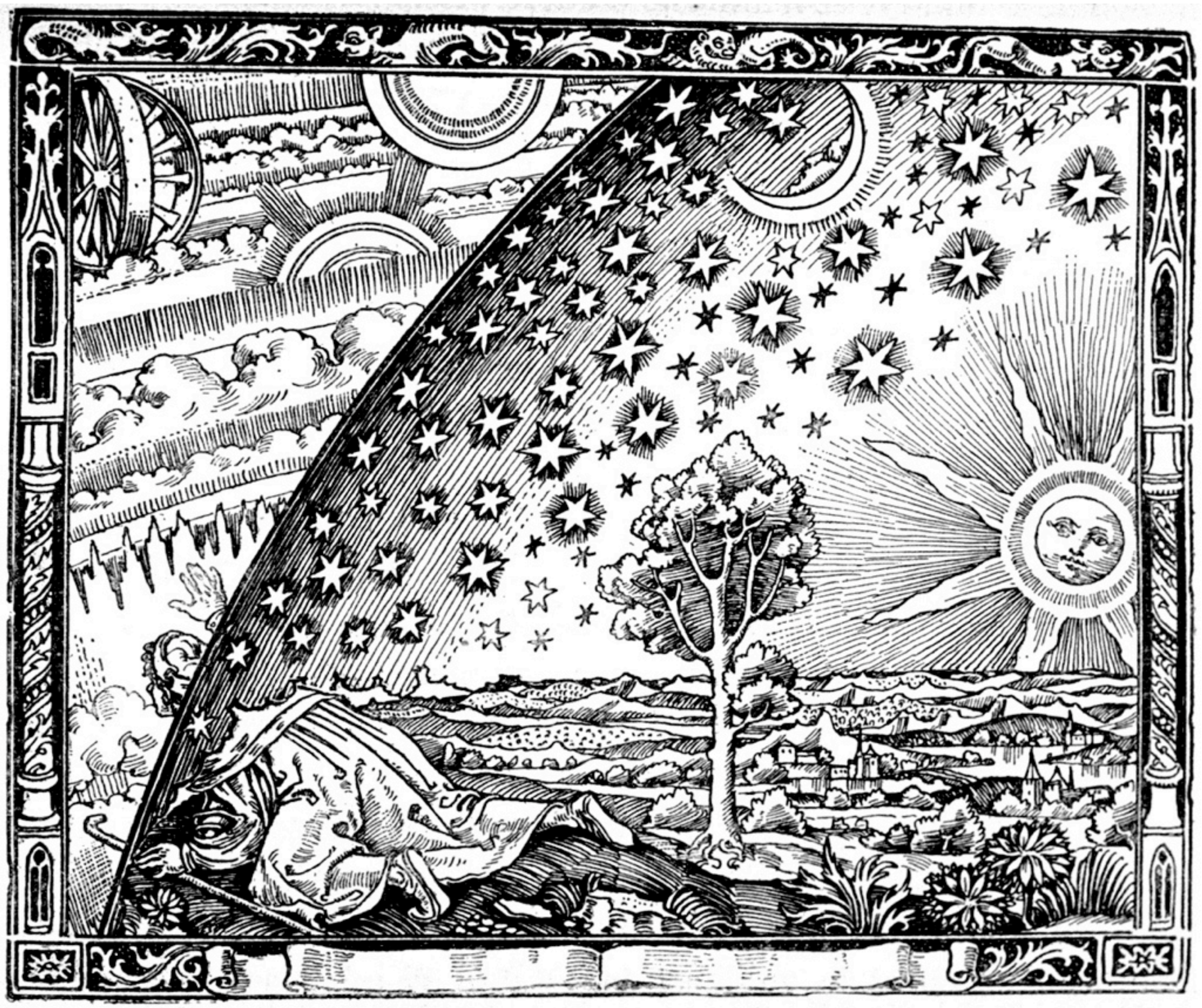
- Given a boundary theory (boundary labels, spaces of boundary fields, boundary OPEs) there exists a unique bulk theory that fits to this boundary theory.

- The bulk theory is determined by 1) - 3) above.

- Every bulk theory with the same holomorphic and anti-holomorphic rational chiral algebra can be obtained in this way.

Fjelstad, Fuchs, Schweigert, IR '06
Kong, IR '08

Rational logarithmic CFT



Study the $W_{2,3}$ triplet model with central charge $c=0$.

(because the $W_{1,p}$ models are "too simple")

The W_{23} model

Feigin, Gainutdinov, Semikhatov, Tipunin '06

- Virasoro Verma module for h=0 and c=0: two independent null vectors

$$N_1 = L_{-1}\Omega \qquad\qquad N_2 = (L_{-2} - \tfrac{3}{2}L_{-1}L_{-1})\Omega$$

- Divide by N_1 and N_2 : get

$$\mathcal{V}(0) = \mathbb{C}\cdot\Omega$$

- Divide by N_1 but not by N_2 : get V with character

$$\chi_{\mathcal{V}}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} + 14q^{11} + 21q^{12} + 24q^{13} + 34q^{14} + 41q^{15} + 55q^{16} + \cdots$$

(quasi-rational, but not rational)

. . . the W_{23} model

- Extend by three fields with h=15, get W with character

$$\chi_{\mathcal{W}}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} + 14q^{11} + 21q^{12} + 24q^{13} + 34q^{14} + 44q^{15} + 58q^{16} + \cdots$$

$$\left(\chi_{\mathcal{V}}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} + 14q^{11} + 21q^{12} + 24q^{13} + 34q^{14} + 41q^{15} + 55q^{16} + \cdots \right)$$

- W is indecomposable but not irreducible

$$0 \longrightarrow \mathcal{W}(2) \longrightarrow \mathcal{W} \longrightarrow \mathcal{W}(0) \longrightarrow 0$$

$\mathcal{W}(2)$: irreducible sub-representation

$\mathcal{W}(0)$: irreducible quotient

(Does not happen for the $W_{1,p}$ models.)

. . . the W_{23} model

- 13 irreducibles:

Feigin, Gainutdinov, Semikhatov, Tipunin '06

$\mathcal{W}(h)$ with h from

(believed to be all)

	$s=1$	$s=2$	$s=3$
$r=1$	$0,\ 2,\ 7$	$0,\ 1,\ 5$	$\frac{1}{3},\ \frac{10}{3}$
$r=2$	$\frac{5}{8},\ \frac{33}{8}$	$\frac{1}{8},\ \frac{21}{8}$	$-\frac{1}{24},\ \frac{35}{24}$

- These are all self-conjugate, but W is not
  → new representation W*

$$0 \longrightarrow \mathcal{W}(2) \longrightarrow \mathcal{W} \longrightarrow \mathcal{W}(0) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{W}(0) \longrightarrow \mathcal{W}^* \longrightarrow \mathcal{W}(2) \longrightarrow 0$$

(Does not happen for the $W_{1,p}$ models.)

Fusion of W_{23} representations

- Not known if logarithmic tensor product theory of Huang, Lepowsky, Zhang '07 applies.

  (Does not happen for the $W_{1,p}$ models.)

- Compute fusion rules
  - start from representations of V
  - compute fusion via Nahm '94, Gaberdiel, Kausch '96

    → done in Eberle, Flohr '06
  - use induced W-representations and associativity
  - compare subset to Rasmussen, Pearce '08

- 13 irreducibles do not close under fusion, need to add 22 indecomposables to close under fusion + conjugates.

. . . fusion of W_{23} representations

Rasmussen, Pearce '08
Gaberdiel, Wood, IR '09

- Some fusion rules: $\mathcal{W} \otimes \mathcal{R} = \mathcal{R}$

$$\mathcal{W}(0) \otimes \mathcal{W}(h) = \begin{cases} \mathcal{W}(0) & : h = 0 \\ 0 & : \text{else} \end{cases}$$

$$\mathcal{W}(2) \otimes \mathcal{W}(2) = \mathcal{W}^*$$

Gaberdiel, Wood, IR '09

- Resolves associativity puzzle in Eberle, Flohr '06

$$\mathcal{W}(0) \otimes \big(\mathcal{W}(2) \otimes \mathcal{W}(2)\big) = \big(\mathcal{W}(0) \otimes \mathcal{W}(2)\big) \otimes \mathcal{W}(2)$$

$$= \mathcal{W}(0) \otimes \mathcal{W}^* \qquad\qquad = 0$$

and $\mathcal{W}(0) \otimes \mathcal{W}^* = 0$, while $\mathcal{W}(0) \otimes \mathcal{W} = \mathcal{W}(0)$.

Fusion rules and Grothendieck group K_0

Here: K_0 = equivalence classes [U] of representations where [U]=[V] iff $\chi_U(q) = \chi_V(q)$

Product on K_0 via $[U] \cdot [V] = [U \otimes V]$?

No: Have $\chi_{\mathcal{W}}(q) = \chi_{\mathcal{W}^*}(q)$ but

$$[\mathcal{W}(0)] \cdot [\mathcal{W}] = [\mathcal{W}(0)]$$
$$[\mathcal{W}(0)] \cdot [\mathcal{W}^*] = 0$$

In fact: Tensor product not exact.

(Does not happen for the $W_{1,p}$ models.)

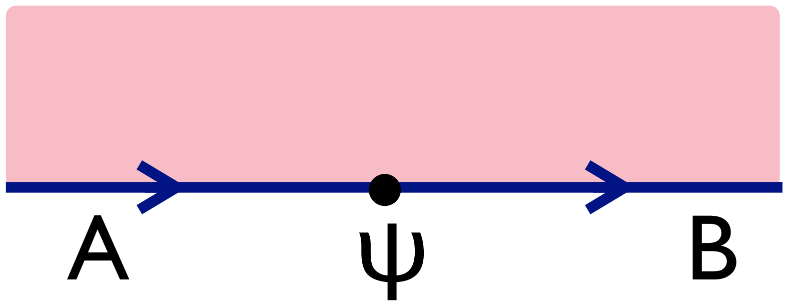
Boundary theory for the $W_{2,3}$ model

Gaberdiel, Wood, IR '09

In non-log., rational CFT (Cardy case):

representations $\leftrightarrow$ boundary conditions

boundary changing fields A→B : $\mathcal{H}_{A\to B} = B \otimes A^*$



$W_{2,3}$ model will be the same but only on <u>subset</u> of reps.

(Does not happen for the $W_{1,p}$ models.)

. . . boundary theory for the $W_{2,3}$ model

Problems with $\mathcal{H}_{A\to B} = B \otimes A^*$:

- Boundary condition for the irreducible **W(0)**?

  $\mathcal{H}_{\mathcal{W}(0)\to\mathcal{W}(h)} = 0$ for **h≠0**.

- Boundary condition for the vacuum representation **W**?

  $\mathcal{H}_{\mathcal{W}\to\mathcal{W}} = \mathcal{W}^* \ncong \mathcal{W}$

  - no non-degenerate 2-point correlator
  - no embedding of vacuum sector

The rules of the game

Data:

- Labels for boundary conditions $\mathcal{B} = \{A, B, \dots\}$
- spaces of boundary changing fields $\mathcal{H}_{A\to B}$
- OPE of boundary fields
- boundary one-point correlator on disc $\langle\psi\rangle_A$

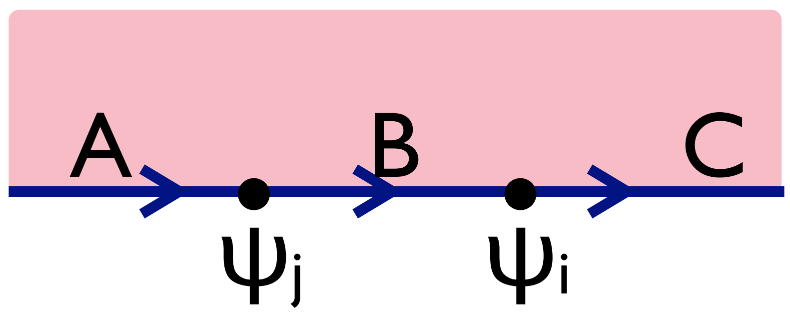
Conditions:

- boundary OPE associative
- $\mathcal{H}_{A\to B}$ is non-zero
- $\mathcal{W}$ is a sub-repn of $\mathcal{H}_{A\to A}$ containing vacuum Ω
- boundary two-point correlator $\langle\psi_1\,\psi_2\rangle_A$ is non-deg.

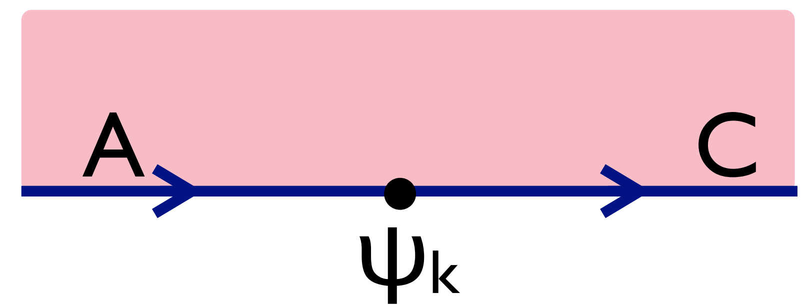
The associativity condition

In non-logarithmic, rational CFT:

boundary OPE coefficients (left out position dependence)



$= \sum_k C_{ijk}^{ABC}$



associativity condition

$$C_{jkq}^{BCD}\, C_{iql}^{ABD} = \sum_p C_{ijp}^{ABC}\, C_{pkl}^{ACD} \cdot \mathsf{F}_{pq}$$

For $W_{2,3}$ model:

difficult as conformal 4-point block not known.

Luckily: abstract nonsense construction exists . . .

Interlude: Internal Homs

In a tensor category, an internal Hom [A,B] from A to B is a representing object for the functor Hom(−⊗A , B) .

[A,B] is an object such that for all U :

$$\mathrm{Hom}(U \otimes A, B) \cong \mathrm{Hom}(U, [A,B])$$

There is an associative composition

$$m_{C,B,A} : [B,C] \otimes [A,B] \to [A,C]$$

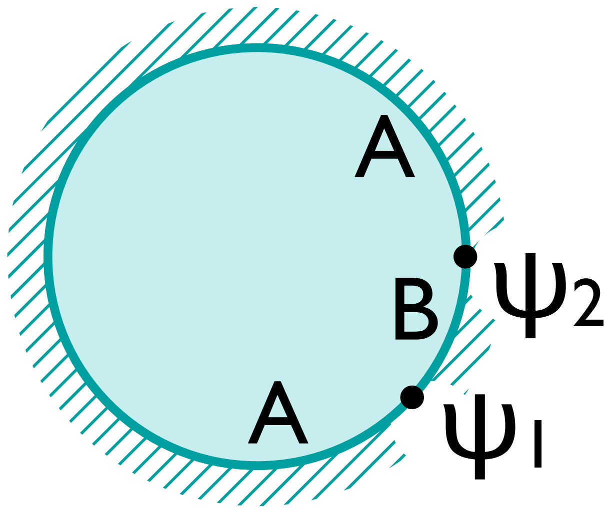
Conjecture: For the $W_{2,3}$ model, $[A,B] = (A \otimes B^*)^*$

(True if $\mathrm{Hom}(U, V^*) \cong \mathrm{Hom}(U \otimes V, \mathcal{W}^*)$)

. . . the associativity condition

If we set $\mathcal{H}_{A\to B} = (A \otimes B^*)^*$ then we have an associative boundary OPE for all representations A, B, ...

The non-degeneracy condition



$: \ \mathcal{H}_{A\to B} \times \mathcal{H}_{B\to A} \longrightarrow \mathbb{C}$

For non-degeneracy need:

$\mathcal{H}_{A\to B} \cong (\mathcal{H}_{B\to A})^*$ i.e. $(A \otimes B^*)^* \cong B \otimes A^*$

Not true e.g. for A=B=W as $W \otimes W^* = W^*$.

A little more abstract nonsense . . .

Interlude: From duals and conjugates

A representation R of W has a (right) dual if there is a representation $R^\vee$ and intertwiners

$$d_R : R^\vee \otimes R \to W \quad \text{and} \quad b_R : W \to R \otimes R^\vee$$

such that . . .

Question: Is the conjugate (contragredient) representation R^* a dual of R?

Answer: Not always, e.g. W is its own dual, but $W^* \neq W$.

(Does not happen for the $W_{1,p}$ models.)

... boundary theory for the $W_{2,3}$ model

$\mathcal{B}$: collection of all $\mathcal{W}$ representations R for which

- the conjugate R^* is a dual of R
- $b_R : \mathcal{W} \to R \otimes R^*$ is injective

For A,B $\in \mathcal{B}$ have $\mathcal{H}_{A \to B} = \left(A \otimes B^*\right)^* \cong B \otimes A^*$
and one shows:

- boundary OPE associative
- $\mathcal{H}_{A \to B}$ is non-zero
- $\mathcal{W}$ is a sub-repn of $\mathcal{H}_{A \to A}$ containing vacuum Ω
- boundary two-point correlator $\langle \psi_1 \, \psi_2 \rangle_A$ is non-deg.

. . . boundary theory for the $W_{2,3}$ model

- Only 8 of the 13 irreducibles remain:

$\mathcal{W}(h)$ with h from

	$s=1$	$s=2$	$s=3$
$r=1$	~~$0,\ 2,\ 7$~~	~~$0,\ 1,\ 5$~~	$\frac{1}{3},\ \frac{10}{3}$
$r=2$	$\frac{5}{8},\ \frac{33}{8}$	$\frac{1}{8},\ \frac{21}{8}$	$-\frac{1}{24},\ \frac{35}{24}$

- Only 18 of the 22 indecomposables generated by fusing irreducibles and taking conjugates remain.

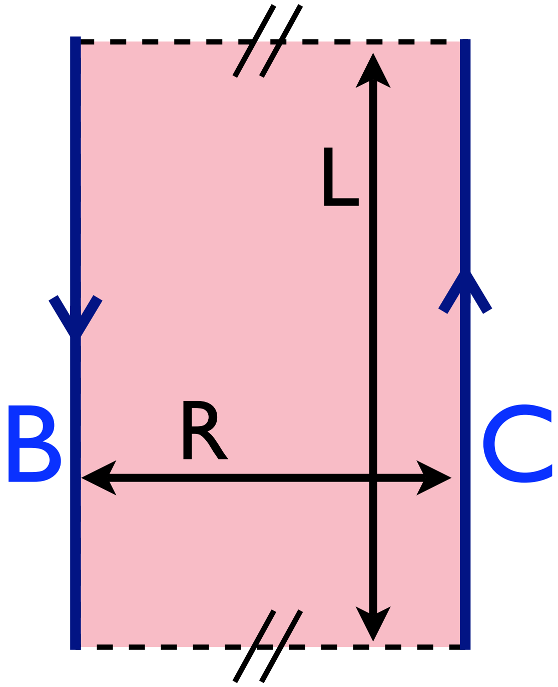
- These are the 26 boundary conditions found in the lattice realisation of Rasmussen, Pearce '08

- NO boundary condition with self-spectrum W

Cylinder partition functions

Have $A_{B,C}(q) = \chi_{\mathcal{H}}(q)$ with $\mathcal{H} = \mathcal{H}_{B\to C}$

$$= C \otimes B^*$$



Can show:

Multiplication on K_0 well defined

when restricted to $\mathcal{B}$:

$$[B]\cdot[C^*] = [B\otimes C^*]$$

So $A_{B,C}(q)$ only depends on [B] and [C].

Torus partition function

In non-logarithmic, rational CFT:

space of bulk states

$$\mathcal{H}_{\text{bulk}} = \bigoplus_i U_i \otimes_{\mathbb{C}} \bar{U}_i^*$$

torus partition function

$$Z(q) = \sum_i \chi_{U_i}(q)\, \chi_{U_i^*}(\bar{q})$$

Denote by $P_i \to U_i$ the projective cover of U_i.

Quella, Schomerus '07
Gaberdiel, IR '07

In the logarithmic $W_{1,p}$ models:

space of bulk states

$$\mathcal{H}_{\text{bulk}} = \Big(\bigoplus_i P_i \otimes_{\mathbb{C}} \bar{P}_i^* \Big) / N$$

torus partition function

$$Z(q) = \sum_i \chi_{P_i}(q)\, \chi_{U_i^*}(\bar{q})$$

… torus partition function

Gaberdiel, Wood, IR '09

In the $W_{2,3}$ model:

- $\mathcal{W}(0)$ seems to have no projective cover
- The sum $\sum_i \chi_{P_i}(q)\,\chi_{U_i^*}(\bar{q})$ over the remaining 12 irreducibles is not modular invariant
- Instead

$$Z(q) = \sum_i \dim\big(\mathrm{Hom}(P_i, P_i)\big)^{-2} \cdot |\chi_{P_i}(q)|^2$$

is modular invariant.

- proportional to Z in Feigin, Gainutdinov, Semikhatov, Tipunin '06)
- also correct answer for non-log. Cardy case and $W_{1,p}$ models

Summary

- Idea: build boundary theory first, then obtain bulk from boundary.

- Study $W_{2,3}$ model

- Representation theory:
  (W reducible , $W \neq W^*$, $\otimes$ not exact , conjugate R^* vs dual $R^\vee$)

- Boundary theory: W-repn $\leftrightarrow$ bnd condition ?
  - associative OPE from internal Hom
  - non-deg 2-point correlator guaranteed on subset of reps
  - no boundary condition with self-spectrum W