Exremal Black Holes and their Microstates

Most references for the first two lectures can be found in citations to:

Dijkgraaf, Verlinde, Verlinde, hep-th/9607026

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One of the successes of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric black holes in terms of microscopic quantum states.

$$S_{BH} = S_{micro}$$

Strominger, Vafa

$S_{BH} = A/4G_N$, $A =$ Area of event horizon

$$S_{micro} = \ln(\text{degeneracy})$$

Originally the comparison between black hole and statistical entropy was carried out in the limit of large charges.

Can we go beyond this limit?

In order to study this problem we need to address two separate issues.

1. We need to learn how to take into account the effect of the higher derivative terms / quantum corrections on the computation of black hole entropy.

2. We also need to know how to calculate the statistical entropy to greater accuracy.

In the first two lectures we shall focus on the second problem in the special case of quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories.

In the last lecture we shall try to address the first issue, namely inclusion of higher derivative and quantum corrections to black hole entropy.

The simplest $\mathcal{N} = 4$ SUSY string theory comes from

$$\text{heterotic on } T^6 \quad \leftrightarrow \quad \text{type IIA on } K3 \times T^2$$

However now we know a variety of $\mathcal{N} = 4$ supersymmetric string theories which can be obtained as orbifolds of

$$\text{heterotic on } T^6 \text{ or type II on } T^6$$

In many of these theories the exact spectrum of quarter BPS dyons has been found / guessed.

Instead of describing these exact results, we shall first focus on the general features which emerge out of these computations.

A generic $\mathcal{N}=4$ supersymmetric string theory in $D=4$ has R U(1) gauge fields. $(R\geq 6)$

6 graviphotons + $(R-6)$ matter multiplets

Thus a generic state will be characterized by an R dimensional electric charge vector Q and an R dimensional magnetic charge vector P.

This theory has a

$$6(R-6)+2$$

dimensional moduli space characterized by the vev of the massless scalar fields at infinity.

Under T-duality transformation

$$Q \to \Omega Q, \qquad P \to \Omega P$$

$\Omega \in$ a discrete subgroup of $O(6, R-6)$

This allows us to define T-duality invariant inner products Q^2, P^2 and $Q \cdot P$ using $O(6, R-6)$ invariant metric.

Under S-duality transformation

$$Q \to aQ + bP, \quad P \to cQ + dP$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ a discrete subgroup of SL(2,R).

Consider a quarter BPS dyon carrying (electric,magnetic) charges (Q, P)

A quarter BPS supermultiplet has 64 states with helicities ranging from $(h - \frac{3}{2})$ to $(h + \frac{3}{2})$ for some fixed $h \in \mathbb{Z} + \frac{1}{2}$.

$d(Q, P) \equiv$ number of quarter BPS states supermultiplets with charge (Q, P) weighted by

$$(-1)^{2h}$$

$d(Q,P)$ is a protected index

$\longrightarrow$ it does not change under continuous variation of the coupling constant and the other moduli.

For generic values of the coupling constant one expects that all states which are not protected by index will become non-BPS and massive.

$\longrightarrow$ only an index worth of states will remain BPS.

Thus $d(Q,P)$ is the number that should be used to define the statistical entropy.

$d(Q, P)$ does not change continuously as we vary the coupling constant and other moduli.

However $d(Q, P)$ can jump across walls of marginal stability in the moduli space on which the dyon can decay into a pair of half-BPS states:

$$(Q, P) \to (\alpha Q + \beta P, \gamma Q + \delta P) + (\delta Q - \beta P, -\gamma Q + \alpha P)$$

$$\alpha\beta = \gamma\delta, \qquad \alpha + \delta = 1$$

On this wall

$$m(Q, P) = m(\alpha Q + \beta P, \gamma Q + \delta P) + m(\delta Q - \beta P, -\gamma Q + \alpha P)$$

Note: Charge quantization forces $(\alpha, \beta, \gamma, \delta)$ to take only discrete values.

$d(Q,P)$ depends not only on (Q,P) but also on the domain in which the moduli lie.

Consider the ith wall bordering a domain:

$(Q,P) \to (\alpha_i Q + \beta_i P, \gamma_i Q + \delta_i P) + (\delta_i Q - \beta_i P, -\gamma_i Q + \alpha_i P)$

We shall label a domain by the collection of the parameters labelling the walls bordering the domain:

$$\vec{c} : \{(\alpha_i, \beta_i, \gamma_i, \delta_i)\}$$

T-duality transformation:

$$Q \to \Omega Q, \qquad P \to \Omega P$$

$\Omega \in$ a discrete subgroup of $O(6, R-6)$

$(\alpha_i, \beta_i, \gamma_i, \delta_i)$: T-duality invariant

S-duality transformation:

$$Q \to aQ + bP, \quad P \to cQ + dP$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ a discrete subgroup of SL(2,R).

$$\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

T-duality invariance $\rightarrow d(Q,P;\vec{c}) = d(\Omega Q, \Omega P;\vec{c})$

Thus $d(Q,P;\vec{c})$ should depend on Q and P only through the T-duality invariant combinations:

$$d(Q,P;\vec{c}) = f(Q^2, P^2, Q\cdot P; \{u\}; \vec{c})$$

$$Q^2 \equiv Q^T L Q, \quad P^2 \equiv P^T L P, \quad Q\cdot P \equiv Q^T L P$$

$L \equiv O(6, R-6)$ invariant metric

$\{u\}$: collection of other T-duality invariants

Define dyon partition function as

$$\Psi(\rho,\sigma,v;\{u\},\vec{c}) = \sum_{Q^2,P^2,Q\cdot P} (-1)^{Q\cdot P+1} f(Q^2,P^2,Q\cdot P;\{u\};\vec{c}) \exp\left[i\pi(\sigma Q^2+\rho P^2+2vQ\cdot P)\right].$$

This sum typically converges in some domain in the complex (ρ,σ,v) space.

The domain of convergence depends on $\vec{c}$.

$\rightarrow$ constraints on $\Im(\rho)$, $\Im(\sigma)$, $\Im(v)$

In all known examples, Ψ is independent of $\vec{c}$ after analytic continuation into the full (ρ,σ,v) plane.

Inverse Fourier transform:

$$\begin{aligned} & f(Q^2, P^2, Q\cdot P; \{u\}, \vec{c}) \\ \propto\ & (-1)^{Q\cdot P+1} \int_{\mathcal{C}(\vec{c})} d\rho d\sigma dv\, \Psi(\rho,\sigma,v;\{u\}) \\ & \qquad \exp\left[-i\pi(\sigma Q^2 + \rho P^2 + 2vQ\cdot P)\right] . \end{aligned}$$

$\mathcal{C}(\vec{c})$: a three dimensional subspace (contour) at fixed $\Im(\rho)$, $\Im(\sigma)$, $\Im(v)$ where the original sum converges.

Note: The dependence of f on $\vec{c}$ comes only through the choice of the contour.

Thus the jump in the index across a wall of marginal stability is given by the residue at the pole picked up during contour deformation.

Q. What is the correlation between the parameters $(\alpha, \beta, \gamma, \delta)$ labelling a wall of marginal stability and the location of the pole in the (ρ, σ, v) plane that controls the jump across this wall?

Recall: On $(\alpha, \beta, \gamma, \delta)$

$(Q, P) \to (\alpha Q+\beta P, \gamma Q+\delta P)+(\delta Q-\beta P, -\gamma Q+\alpha P)$

In all known examples the jump in the index across this wall is controlled by the pole of $\Psi(\rho, \sigma, v; \{u\})$ at

$$\rho\gamma - \sigma\beta + v(\alpha - \delta) = 0\,.$$

Constraints from S-duality

$$(Q,P) \to (Q',P') = (aQ+bP, cQ+dP)$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ a discrete subgroup of $SL(2,R)$.

Under this $\{u\} \to \{u'\}$, $\vec{c} \to \vec{c}'$.

$$f(Q^2, P^2, Q{\cdot}P; \{u\}; \vec{c}) = f(Q'^2, P'^2, Q'{\cdot}P'; \{u'\}; \vec{c}')$$

$\vec{c}$ independence of Ψ

$$\to \quad \Psi(\rho,\sigma,v;\{u\}) = \Psi(\rho',\sigma',v';\{u'\})$$

$$\rho' \equiv d^2\rho + b^2\sigma + 2bdv, \quad \sigma' \equiv c^2\rho + a^2\sigma + 2acv,$$
$$v' \equiv cd\rho + ab\sigma + (ad+bc)v$$

A surprise

In all known examples the partition function $\Psi(\rho, \sigma, v, \{u\})$ transforms as a modular function of certain weight under a subgroup of $Sp(2, \mathbb{Z})$.

$$\psi((A\Omega+B)(C\Omega+D)^{-1}) = (\det(C\Omega+D))^k \, \Psi(\Omega)$$

$$\Omega \equiv \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z})$$

The $C = 0$ subgroup of this is responsible for S-duality but the significance of the full symmetry has so far only been partially understood.

Special case: Heterotic on T^6 ($R = 28$)

Consider a pair of charge vectors (Q, P)

$(Q, P) \in$ Narain lattice

Continuous T-duality invariants:

$$Q^2, \quad P^2, \quad Q \cdot P$$

What are the discrete T-duality invariants $\{u\}$?

$\{u\}$ is a collection of four positive integers

$$(r_1, r_2, r_3, u_1)$$

Definition of r_1, r_2, r_3, u_1:

Consider the intersection of the Narain lattice with the plane spanned by Q and P.

Let (e_1, e_2) be a pair of basis vectors of this 2D lattice with e_1 being parallel to Q.

Then

$$Q = r_1 e_1, \qquad P = r_2(u_1 e_1 + r_3 e_2)$$

$$\gcd(r_1, r_2) = 1, \ \gcd(u_1, r_3) = 1, \ 1 \le u_1 \le r_3$$

S-duality transformation acts non-trivially on (r_1, r_2, r_3, u_1).

Using an S-duality transformation we can map (r_1, r_2, r_3, u_1) to $(r_1 r_2 r_3, 1, 1, 1)$.

Thus once we know $d(Q, P)$ for $(r, 1, 1, 1)$ for all r we can get $d(Q, P)$ for (r_1, r_2, r_3, u_1) using S-duality invariance of the dyon spectrum.

Note: $r_1 r_2 r_3 = \text{gcd}(Q \wedge P)$

Result for $r = 1$

$$\Psi(\rho, \sigma, v; r = 1) = 1/\Phi_{10}(\rho, \sigma, v)$$

Φ_{10}: weight 10 Igusa cusp form of $Sp(2, \mathbb{Z})$.

Explicit analysis also determines the prescription for the contour $\mathcal{C}(\vec{c})$ in different domains.

$r > 1$:

$$\psi = \sum_{\substack{s \in \mathbb{Z},\, s|r \\ \bar{s} \equiv r/s}} s \frac{1}{\bar{s}^3} \sum_{k=0}^{\bar{s}^2-1} \sum_{l=0}^{\bar{s}-1} \Phi_{10}\left(\rho, s^2\sigma + \frac{k}{\bar{s}^2}, sv + \frac{l}{\bar{s}}\right)^{-1}$$

Heterotic string theory on T^6

A generic dyon is described by a pair of 28 dimensional charge vectors (Q, P) belonging to the Narain lattice.

T-duality invariants:

Q^2, P^2, $Q \cdot P$, $\{u\} \equiv \{r_1, r_2, r_3, u_1\}$

S-duality relates (r_1, r_2, r_3, u_1) to $(r_1 r_2 r_3, 1, 1, 1)$

$\rightarrow$ is is enough to determine the spectrum for $(r, 1, 1, 1)$ for all r.

We shall now describe the computation of dyon spectrum for the $r = 1$ case in heterotic string theory on T^6.

We shall do so by using the equivalence of this theory to a type IIB string theory on $K3 \times T^2$ and working in the weakly coupled type IIB string theory.

Note: S-duality leaves the $\{u\} = (1, 1, 1, 1)$ configuration invariant.

From now on we shall drop the argument $\{u\}$ in the partition function.

The configuration:

1) Q_5 D5-brane wrapped on $K3 \times S^1$

2) Q_1 D1-branes wrapped on S^1

3) $-k$ units of momentum along S^1

4) J units of momentum along $\widetilde{S}^1$

5) One Kaluza-Klein monopole along $\widetilde{S}^1$

– BMPV black hole at the center of Taub-NUT

After translated to the heterotic description, this gives

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2k, \quad Q \cdot P = J$$

$$r = \mathrm{gcd}(Q_1, Q_5, J)$$

We take

$$Q_5 = 1$$

Thus

$$r = 1$$

We calculate the partition function in weakly coupled IIB theory and then extend it to other domains using S-duality invariance.

In the weakly coupled type IIB description the low energy dynamics of the system is described by three weakly interacting pieces:

1) The closed string excitations around the Kaluza-Klein monopole

2) The dynamics of the D1-D5 center of mass coordinate in the Kaluza-Klein monopole background

3) The relative motion between the D1 and the D5-brane

The dyon partition function is obtained as the product of the partition function of these three subsystems.

Low energy dynamics of KK monopole:

$$e^{-2\pi i\sigma}\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma})^{-24}\right\}$$

D1-D5 center of mass motion in KK monopole background:

$$\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma})^{4}\,(1-e^{2\pi in\sigma+2\pi iv})^{-2}\,(1-e^{2\pi in\sigma-2\pi iv})^{-2}\right\}$$
$$\times\, e^{-2\pi iv}\,(1-e^{-2\pi iv})^{-2}$$

Relative motion between the D1 and D5 branes:

$$e^{-2\pi i\rho}\prod_{\substack{l,b,k\in Z\\ k\geq 0,l>0}}\left\{1-\exp(2\pi i(k\sigma+l\rho+bv))\right\}^{-c(4lk-b^2)}$$

Dijkgraaf, Moore, Verlinde, Verlinde

Definition of $c(n)$:

$$F(\tau,z) \equiv 8\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]$$

$$F(\tau,z) = \sum_{b\in\mathbb{Z},n} c(4n-b^2)\, q^n\, e^{2\pi i z b}$$

After taking the product we get

$$\Psi = e^{-2\pi i\rho} \prod_{\substack{l,b,k\in Z \\ k\geq 0, l\geq 0, b<0 \text{ for } k=l=0}} \left\{1-\exp(2\pi i(k\sigma+l\rho+bv))\right\}^{-c(4lk-b^2)}$$

$$\Psi(\rho,\sigma,v;r=1) = 1/\Phi_{10}(\rho,\sigma,v)$$

Φ_{10}: weight 10 Igusa cusp form of $Sp(2,\mathbb{Z})$.

The counting that leads to the partition function also tells us how we should expand it to extract $d(Q,P)$.

First expand in powers of $e^{2\pi i\rho}$ and $e^{2\pi i\sigma}$.

Then expand in powers of $e^{\pm 2\pi i v}$.

Corresponds to the contour choice

$$\Im(\rho), \Im(\sigma) >> |\Im(v)| > 0$$

This prescription suffers from a 2-fold ambiguity.

$\Im(v) > 0$ and $\Im(v) < 0$.

Consider the factor

$$e^{-2\pi iv}(1 - e^{-2\pi iv})^{-2}$$

from the D1-D5 center of mass dynamics.

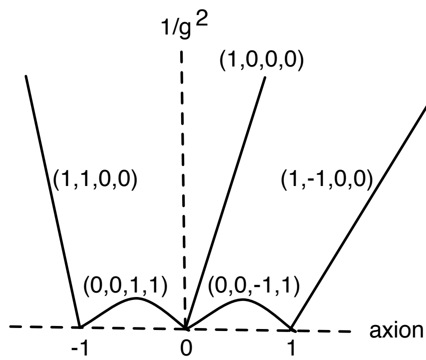
Can be expanded as

$$\sum_{j=1}^{\infty} j\, e^{2\pi ijv} \quad \text{or} \quad \sum_{j=1}^{\infty} j\, e^{-2\pi ijv}$$

It turns out that these two prescription give $d(Q,P)$ in two different domains in the moduli space, both lying inside the weak coupling limit of IIB.

Pope; Gauntlett, Kim, Park, Yi

Two domains in the heterotic axion-dilaton plane



$\rightarrow$ correspond to the contour choice:

$$\Im(\rho), \Im(\sigma) >> \Im(v) > 0$$

and

$$\Im(\rho), \Im(\sigma) >> -\Im(v) > 0$$

Note that $d(Q, P)$ is different on the two sides of the wall but the partition function Ψ is described by the same analytic function of ρ, σ, v.

There are infinite number of other domains but we can determine the form of Ψ in these domains as well as the choice of contour using S-duality invariance.

S-duality invariance $\rightarrow$

$$\Psi(\rho,\sigma,v;\vec{c}) = \Psi(\rho',\sigma',v';\vec{c}\,')$$

$$\rho' \equiv d^2\rho + b^2\sigma + 2bdv, \quad \sigma' \equiv c^2\rho + a^2\sigma + 2acv,$$
$$v' \equiv cd\rho + ab\sigma + (ad+bc)v$$

Choose $\vec{c}$ to be the domain in which we have computed Ψ.

Explicit computation shows that

$$\Psi(\rho,\sigma,v;\vec{c}) = \Psi(\rho',\sigma',v';\vec{c})$$

Thus

$$\Psi(\rho',\sigma',v';\vec{c}\,') = \Psi(\rho',\sigma',v';\vec{c})$$

$\rightarrow$ Ψ is independent of the domain $\vec{c}$.

Comparison with black hole entropy

For this we need to study the behaviour of $d(Q,P)$ for large charges.

Goal: develop a systematic procedure for determining the asymptotic expansion of $d(Q,P)$ in inverse powers of the charges.

$$d(Q,P;\vec{u},\vec{c}) \propto (-1)^{Q\cdot P+1} \int_{\mathcal{C}(\vec{c})} d\rho d\sigma dv\, \Psi(\rho,\sigma,v;\vec{u}) \\ \exp\left[-i\pi(\sigma Q^2 + \rho P^2 + 2vQ\cdot P)\right].$$

To extract the large charge behaviour we deform the contour to the region

$$\Im(\sigma), \Im(\rho), \Im(v) \sim \frac{1}{\text{charge}}$$

The deformed contour does not give contribution growing as exp[constant$\times$charge^2].

Thus the exponentially growing contribution relevant for computation of black hole entropy comes from the poles the contour crosses during this deformation.

We need to identify the pole that contributes to the leading asymptotic expansion.

In all known examples the leading asymptotic growth comes from a pole at

$$\rho\sigma - v^2 + v = 0$$

Result of picking up residue at this pole:

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

for some function $F(\rho,\sigma)$.

Next we do the ρ and σ integral using saddle point approximation.

Define $W(\vec{J})$ through

$$e^{W(\vec{J})} = \int d\rho d\sigma e^{-F(\rho,\sigma)+J_1\rho+J_2\sigma}$$

Then

$$e^{W(\vec{0})} = d(Q,P)$$

Define $\hat{\rho}$, $\hat{\sigma}$, $\Gamma(\hat{\rho},\hat{\sigma})$ through

$$\hat{\rho} = \frac{\partial W(\vec{J})}{\partial J_1}, \qquad \hat{\sigma} = \frac{\partial W(\vec{J})}{\partial J_2}$$

$$\Gamma(\hat{\rho},\hat{\sigma}) = J_1\hat{\rho} + J_2\hat{\sigma} - W(\vec{J})$$

$$\widehat{\rho} = \partial W(\vec{J})/\partial J_1, \quad \widehat{\sigma} = \partial W(\vec{J})/\partial J_2$$

$$\Gamma(\widehat{\rho}, \widehat{\sigma}) = J_1 \widehat{\rho} + J_2 \widehat{\sigma} - W(\vec{J})$$

Then

$$J_1 = \partial \Gamma / \partial \widehat{\rho}, \quad J_2 = \partial \Gamma / \partial \widehat{\sigma}$$

If $\partial \Gamma / \partial \widehat{\rho} = \partial \Gamma / \partial \widehat{\sigma} = 0$ at $(\widehat{\rho}, \widehat{\sigma}) = (\widehat{\rho}_0, \widehat{\sigma}_0)$ then

$$\Gamma(\widehat{\rho}_0, \widehat{\sigma}_0) = -W(\vec{0}) = -\ln d(Q, P)$$

If $\partial\Gamma/\partial\hat{\rho} = \partial\Gamma/\partial\hat{\sigma} = 0$ at $(\hat{\rho}, \hat{\sigma}) = (\hat{\rho}_0, \hat{\sigma}_0)$ then

$$\Gamma(\hat{\rho}_0, \hat{\sigma}_0) = -W(\vec{0}) = -\ln d(Q, P)$$

Thus $\ln d(Q, P)$ is the value of $-\Gamma(\hat{\rho}, \hat{\sigma})$ at its extremum.

$-\Gamma(\hat{\rho}, \hat{\sigma})$ can be called the statistical entropy function.

On the other hand Γ can be calculated by summing over 1PI Feynman diagrams in the 0-dimensional quantum field theory with action $F(\rho, \sigma)$.

Loop expansion parameter: Inverse charge

Example: Heterotic string theory on T^6

Result for Γ after a suitable change of variables from $(\widehat{\rho}, \widehat{\sigma})$ to (a, S):

$$-\Gamma(a, S) = \frac{\pi}{2}\Big[\left(\frac{Q^2}{S} + \frac{P^2}{S}(S^2 + a^2) - 2\,\frac{a}{S}\,Q\cdot P\right) + 128\,\pi\,\phi(a, S)\Big] + \mathcal{O}(Q^{-2}, P^{-2})$$

$$\phi(a, S) = -\frac{3}{16\pi^2}\Big(\ln S + 4\ln|\eta(a + iS)|\Big)$$

Statistical entropy = value of $-\Gamma$ at its extremum with respect to a and S.

How good is the asymptotic formula?

Q^2	P^2	$Q \cdot P$	$d(Q,P)$	S_{stat}	$S_{stat}^{(0)}$	$S_{stat}^{(1)}$
2	2	0	50064	10.82	6.28	10.62
4	4	0	32861184	17.31	12.57	16.90
6	6	0	16193130552	23.51	18.85	23.19
6	6	1	11232685725	23.14	18.59	22.88
6	6	2	4173501828	22.15	17.77	21.94
6	6	3	920577636	20.64	16.32	20.41
6	6	4	110910300	18.52	14.05	18.40

How does this result compare with the entropy of a BPS black hole carrying the same set of charges?

In the presence of higher derivative corrections we must use Wald's formula for black hole entropy.

For extremal black holes this can be implemented via the entropy function formalism.

The supergravity answer for black hole entropy agrees with the leading term in the asymptotic expansion.

If we keep the Gauss-Bonnet term in the effective action besides the leading supergravity action, then the black hole entropy agrees perfectly with S_{stat} up to first non-leading order.

Recall that $d(Q,P)$ changes across walls of marginal stability.

Can we see these changes on the black hole side?

In the large charge limit these changes are exponentially suppressed compared to the leading term.

Thus we would expect that the asymptotic expansion of S_{BH} should not change as we move across the walls of marginal stability.

However there is still an exponentially suppressed change across walls of marginal stability.

Can we see this on the black hole side?

It turns out that this jump in $d(Q,P)$ is associated with 2-centered solutions to the supergravity equations of motion together with higher derivative corrections.

Each of these centers has the near horizon geometry of a small black hole.

(black holes whose entropy is zero at the leading order but is non-zero after taking into account higher derivative corrections.)

Consider such a 2-centered solution with the first center carrying charge (Q_1, P_1) and the second center carrying charges (Q_2, P_2).

If we consider the wall of marginal stability associated with the decay

$$(Q, P) \Rightarrow (Q_1, P_1) + (Q_2, P_2)$$

then the two centered solution exists only on one side of the wall of marginal stability.

As we cross the wall of marginal stability the solution disappears.

Thus from the black hole side the change in the index can be identified as the index associated with the 2-centered solution.

Explicit computation gives

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)$$
$$\left\{\sum_{L_1|(Q_1,P_1)} d_h\left(\frac{Q_1}{L_1},\frac{P_1}{L_1}\right)\right\}\left\{\sum_{L_2|(Q_2,P_2)} d_h\left(\frac{Q_2}{L_2},\frac{P_2}{L_2}\right)\right\}$$

$d_h(q,p)$: index of half-BPS states carrying charges (q,p).

How does this compare with the microscopic result?

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)$$
$$\left\{\sum_{L_1|(Q_1,P_1)} d_h\left(\frac{Q_1}{L_1},\frac{P_1}{L_1}\right)\right\}\left\{\sum_{L_2|(Q_2,P_2)} d_h\left(\frac{Q_2}{L_2},\frac{P_2}{L_2}\right)\right\}$$

In all known cases this agrees exactly with the jump in the index $d(Q,P)$ computed by evaluating the residue of the integrand at the appropriate pole of the partition function $\Psi(\rho,\sigma,v)$.

Thus we see the black holes not only capture the leading asymptotic behaviour of $d(Q,P)$ for large charges, but also capture information about exponentially small corrections to $d(Q,P)$.

So far we have discussed precision counting of extremal black hole microstates in $\mathcal{N} = 4$ supersymmetric string theories.

However in order to systematically compare this with black hole entropy we need to understand corrections to black hole entropy due to

1. Higher derivative corrections

2. Quantum corrections

More generally we need to determine what computation on the black hole side should be compared with the microscopic degeneracy $d_{micro}(q)$.

A general frameork for computing higher derivative corrections to black hole entropy has been developed by Wald.

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

for spherically symmetric black holes in (3+1) dimensions.

In computing $\delta\mathcal{S}/\delta R_{\mu\nu\rho\sigma}$

1. express the action $\mathcal{S}$ in terms of symmetrized covariant derivatives of fields

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

We shall use this to study black hole entropy in the extremal limit.

How do we define extremal black holes in a higher derivative theory?

Take the clue from usual (super-)gravity.

Reissner-Nordstrom solution in $D = 4$:

$$
\begin{aligned}
ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\
&\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\
&\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

$\rho_\pm$: parameters related to mass and charge

Extremal black hole: $\rho_+ = \rho_-$

Instead of studying directly extremal black holes, for which Wald's formula is **not valid**, we shall study the **extremal limit** of regular black holes.

$$ds^2 = -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)}$$
$$+\rho^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Define

$$2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\,\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}$$

and take $\lambda \to 0$ limit.

$$ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$\longrightarrow$ near horizon geometry $AdS_2 \times S^2$

The horizon is at $r = 1$.

The complete near horizon solution:

$$ds^2 = \rho_+^2 \left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right] + \rho_+^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$F_{rt} = \frac{q}{4\pi}, \qquad F_{\theta\phi} = \frac{p}{4\pi} \sin\theta$$

$$\rho_+^2 = G_N \frac{q^2+p^2}{4\pi}$$

q, p: label electric and magnetic charges

The full background has $SO(2,1) \times SO(3)$ isometry.

In general, the near horizon geometry of all known extremal black holes in all dimensions have **time translation symmetry enhanced to** $SO(2,1)$

t and r form an AdS_2 space.

We shall take this as the definition of extremal black holes even in theories with higher derivative terms in the action.

(Partial proof by Kunduri, Lucietti, Reall; Figueras,Kunduri, Lucietti,Rangamani)

Regarding all other directions (including angular coordinates) as compact we can regard the near horizon geometry of an extremal black hole as

$AdS_2 \times$ a compact space (fibered over AdS_2)

Consider classical string theory in such a background containing two dimensional metric $g_{\mu\nu}$ and $U(1)$ gauge fields $A_\mu^{(i)}$ among other fields.

The most general field configuration consistent with $SO(2,1)$ isometry:

$$ds^2 \equiv g^{(2)}_{\mu\nu} dx^\mu dx^\nu = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$

$$F^{(i)}_{rt} = e_i, \quad \cdots\cdots\cdots$$

Let $\mathcal{L}^{(2)}$ be the two dimensional Lagrangian density.

Define

$$\mathcal{E}(\vec{q}, v, \vec{e}, \cdots) \equiv 2\pi \left(e_i \, q_i - v \, \mathcal{L}^{(2)} \right)$$

evaluated on the near horizon geometry.

One finds that for a black hole of charge $\vec{q}$

1. All the near horizon parameters are obtained by extremizing $\mathcal{E}$ with respect to v, e_i and the other near horizon parameters.

2. $S_{BH}(\vec{q}) = \mathcal{E}$ at this extremum.

These results come out of straightforward use of equations of motion and Wald's formula.

We shall now try to generalize this formula taking into account quantum corrections.

Strategy:

1. Find a physical interpretation of the formula for S_{BH} which has a natural quantum generalization.

2. Use this to generalize the definition of S_{BH} into the quantum theory.

Our main tool will be AdS_2/CFT_1 correspondence.

References

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R. Gupta, A.S. arXiv:0806.0053

Related earlier work

Maldacena, Michelson, Strominger, hep-th/9812073

Beasley, Gaiotto, Guica, Huang, Strominger, Yin, hep-th/0608021

Comments

1. We shall not make use of SUSY, although SUSY is undoubtedly useful in ensuring stability of the extremal BPS black holes.

2. Semiclassical part of our analysis will be close to the Euclidean approach to black hole thermodynamics.

However we shall work entirely in the near horizon geometry of the black hole instead of the full black hole solution.

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$

$$F^{(i)}_{rt} = e_i$$

Euclidean continuation:

$$t = -i\theta, \quad r = \cosh\eta, \quad \theta \equiv \theta + 2\pi, \quad 0 \le \eta < \infty$$

This gives

$$\begin{aligned} ds^2 &= v\left(d\eta^2 + \sinh^2\eta\, d\theta^2\right), \\ F^{(i)}_{\theta\eta} &= ie_i \sinh\eta \end{aligned}$$

$$\rightarrow \quad A^{(i)}_\theta = i\, e_i\, (\cosh\eta - 1) = i\, e_i\, (r-1)\,.$$

Classical supergravity partition function:

$$Z_{AdS_2} \simeq e^{-A}, \qquad A = \text{Euclidean action}$$

Since AdS_2 has infinite volume, A would be infinite.

We regularize by putting a cut-off at:

$$\eta = \eta_0 \quad \to \quad r = \cosh \eta_0 = r_0$$

$$\to \quad \int \sqrt{\det g}\, dr\, d\theta = 2\pi\, v\, (r_0 - 1)$$

$$\to A_{bulk} = -\int \sqrt{\det g}\, dr\, d\theta \mathcal{L}^{(2)} = -2\pi v\, (r_0 - 1)\, \mathcal{L}^{(2)}$$

Besides this there may also be boundary contribution proportional to the length of the boundary.

$$A_{boundary} = -K \sinh \eta_0 = -K\, r_0 + \mathcal{O}(r_0^{-1})$$

This gives

$$\begin{aligned} Z_{AdS_2} &\simeq e^{-A_{bulk} - A_{boundary}} \\ &= e^{r_0 (2\pi v\, \mathcal{L}^{(2)} + K) - 2\pi\, v\, \mathcal{L}^{(2)} + \mathcal{O}(r_0^{-1})} \end{aligned}$$

in the classical limit.

Expected form of Z_{AdS_2} in the full quantum theory

$$Z_{AdS_2}(\vec{e}) = e^{Cr_0 - 2\pi v\, \mathcal{L}^{(2)}_{eff}(\vec{e}) + \mathcal{O}(r_0^{-1})}$$

$\mathcal{L}^{(2)}_{eff}$: effective lagrangian density evaluated in the AdS_2 background

e.g. one loop contribution to $\mathcal{L}^{(2)}_{eff}$ is obtained by expressing the determinants of bosonic and fermionic kinetic operators in the AdS_2 background as

$$\exp\left(\mathcal{L}^{(2)}_{eff}\Big|_{\text{one loop}} \times \text{volume of } AdS_2\right)$$

AdS_2/CFT_1 correspondence

By the usual AdS/CFT correspondence we would expect that string theory on AdS_2 should be equivalent to a CFT_1 at the boundary $r = r_0$ of AdS_2.

$$Z_{CFT_1} = Z_{AdS_2}$$

We shall now analyze Z_{CFT_1}.

Conventionally one uses units in which the size of the boundary is fixed but the UV length cut-off is of order $1/r_0$.

We shall use a convention in which the UV cut-off is fixed but the size of the boundary is of order r_0.

Define rescaled angular coordinate near the boundary

$$w \equiv r_0 \theta$$

In this coordinate system the metric and the gauge field near the boundary take the form

$$\begin{aligned} ds^2 &= v\,(d\eta^2 + dw^2) + \mathcal{O}(r_0^{-2}), \\ A_w^{(i)} &= i\,e_i\left(1 - r_0^{-1}\right), \quad w \equiv w + 2\pi r_0\,. \end{aligned}$$

These have finite $r_0 \to \infty$ limit.

$$
\begin{aligned}
ds^2 &= v\,(d\eta^2 + dw^2) + \mathcal{O}(r_0^{-2}), \\
A_w^{(i)} &= i\,e_i\,\left(1 - r_0^{-1}\right), \quad w \equiv w + 2\pi r_0 \,.
\end{aligned}
$$

Define

H: generator of w translation in CFT_1 in the $r_0 \to \infty$ limit

Q_i: Conserved charge dual to $A_\mu^{(i)}$ in CFT_1

Then

$$
Z_{CFT_1} = Tr\left[e^{-2\pi r_0 H - 2\pi e_i Q_i + \mathcal{O}(r_0^{-1})}\right]
$$

$$Z_{CFT_1} = Tr\left[e^{-2\pi r_0 H - 2\pi \vec{e}\cdot\vec{Q} + \mathcal{O}(r_0^{-1})}\right]$$

$$Z_{AdS_2}(\vec{e}) = e^{Cr_0 - 2\pi v\,\mathcal{L}^{(2)}_{eff}(\vec{e}) + \mathcal{O}(r_0^{-1})}$$

Compare the two in the $r_0 \to \infty$ limit:

$\to$ if the ground state energy of H is E_0 and there are $d(\vec{q})$ ground states of charge $\vec{q}$ then

$$e^{-2\pi E_0 r_0} \sum_{\vec{q}} d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}} = e^{Cr_0 - 2\pi v\,\mathcal{L}^{(2)}_{eff}(\vec{e})}$$

$$e^{-2\pi E_0 r_0} \sum_{\vec{q}} d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}} = e^{C r_0 - 2\pi v\, \mathcal{L}^{(2)}_{eff}(\vec{e})}$$

$$\Downarrow$$

$$E_0 = -C/(2\pi), \quad \sum_{\vec{q}} d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}} = e^{-2\pi v\, \mathcal{L}^{(2)}_{eff}(\vec{e})}$$

$$\sum_{\vec{q}} d(\vec{q})\, e^{-2\pi\vec{e}\cdot\vec{q}} = e^{-2\pi v\, \mathcal{L}^{(2)}_{eff}(\vec{e})}$$

In classical limit the l.h.s. is sharply peaked, and in r.h.s. we replace $\mathcal{L}^{(2)}_{eff}$ by $\mathcal{L}^{(2)}$.

$$\ln d(\vec{q}) - 2\pi\vec{e}\cdot\vec{q} = -2\pi v\mathcal{L}^{(2)}$$

at

$$\partial \ln d(\vec{q})/\partial q_i = 2\pi e_i$$

Compare this with Wald entropy

$$S_{BH}(\vec{q}) = 2\pi(\vec{e}\cdot\vec{q} - 2\pi v\mathcal{L}^{(2)})$$

$$\to S_{BH}(\vec{q}) = \ln d(\vec{q})$$

$$S_{BH}(\vec{q}) = \ln d(\vec{q})$$

Thus comparing $S_{BH}(\vec{q})$ to $\ln d_{micro}(\vec{q})$ corresponds to comparing $d(\vec{q})$ with $d_{micro}(\vec{q})$.

This suggests a natural generalization of the $S_{BH} \leftrightarrow S_{micro}$ correspondence in the full quantum theory.

$$d(q) \leftrightarrow d_{micro}(\vec{q})$$

Up to overall factors of the form e^{Cr_0} which can be absorbed by a shift in the ground state energy,

$$Z_{AdS_2}(\vec{e}) = \sum_{\vec{q}} d(\vec{q}) e^{-2\pi \vec{q}\cdot\vec{e}}$$

as a consequence of AdS_2/CFT_1 correspondence.

Define

$$Z_{micro}(\vec{e}) \equiv \sum_{\vec{q}} d_{micro}(\vec{q}) e^{-2\pi \vec{q}\cdot\vec{e}}$$

Thus $d(\vec{q}) \leftrightarrow d_{micro}(\vec{q})$ corresponds to

$$Z_{AdS_2}(\vec{e}) \leftrightarrow Z_{micro}(\vec{e})$$

Special case: Type IIA on CY_3

In this case Z_{AdS_2} may be computable due to SUSY.

Recall:

$$Z_{AdS_2} \simeq e^{-2\pi v \mathcal{L}^{(2)}_{eff}}$$

after removing cut-off dependent terms.

If we evaluate $v\mathcal{L}^{(2)}_{eff}$ using only the F-type terms in the effective action then

$$Z_{AdS_2} \simeq e^{-2\pi v \mathcal{L}^{(2)}_{eff}} = |Z_{top}|^2$$

Ooguri, Strominger, Vafa
Beasley, Gaiotto, Guica, Huang, Strominger, Yin

Quantum corrections should be strongly constrained due to SUSY.

Expect

$$Z_{AdS_2} = |Z_{top}|^2 \times \text{simple measure factor}$$

It may not be impossible to calculate this completely.

We can then compare this with $Z_{micro}(\vec{e})$ when the latter is known.

One subtle issue

$d_{micro}(\vec{q})$ depends on the point in the moduli space where we are computing it.

$\longrightarrow$ can jump across walls of marginal stability.

A natural choice: Attractor point corresponding to $\vec{e}$.

At the saddle point it is also the attractor point corresponding to $\vec{q}$.

$\longrightarrow$ $d_{micro}(\vec{q})$ counts only the degeneracies of single centered black holes and does not suffer from entropy enigma. Denef, Moore