Matrix-Factorizations and Superpotentials

Marco Baumgartl

ASC-LMU Munich

15th European Workshop on String Theory, Zurich, September 2009

Topics

Motivation

- (phenomenologically) interesting string backgrounds: Calabi-Yau + branes
- open and closed string moduli
- what is their connection? How do brane moduli react on closed string deformations?
- matrix factorization technique via Landau-Ginzburg description (topologically twisted)
- rather explicit connection to worldsheet CFT description

4+6d string theory

- six dimensions may be compactified on an ‘internal’ manifold
- Calabi-Yaus (Kähler, with vanishing Chern class) satisfy the string consistency conditions
- this provides a valid closed string background in 10d supersymmetric string theory
- generally, there are (closed string) moduli

Branes

- for open strings, boundary conditions must be imposed
- these often have a geometric interpretation as hyper-surfaces embedded in the background geometry
- branes often come with (open string) moduli
  - the moduli space can have a rich structure:
    special points, families, webs
- brane-moduli depend crucially on closed string moduli
- what happens to a brane, when the background changes?

From CFT to Calabi-Yau

CY		Landau-Ginzburg		CFT
$W(x_i) = 0$	$\leftrightarrow$	superpotential $W(x_i)$	$\leftrightarrow$	Gepner models $\otimes_i (N=2)_{k_i}$

- e.g. A-type minimal models are realised by $W = x^{k+2}$ with $c = \frac{3k}{k+2}$
- Quintic $W = x_1^5 + \cdots + x_5^5$ is tensor product of five $A_{k=3}$
- complete ADE set known

Landau-Ginzburg description

- The $N = (2, 2)$ LG theory has a Langrangian description

$$S = \int d^2 z d^4 \theta K(x, \bar{x}) + \int d^2 z d^2 \theta W(x) + hc$$

- chiral ring $\mathcal{O}/\partial W$
- boundary conditions for B-branes: W factorizes as

$$W(X) = E(X) \cdot J(X)$$

where $E(X)$ and $J(X)$ are matrices of polynomials

Supersymmetric boundary conditions

- bulk chiral rings extended by Chan-Paton factors

$$\mathcal{R}_\partial \subset \mathrm{Mat}(\mathcal{O})$$

- Q is a graded odd operator with $Q^2 = W$ *(Kontsevich)* (SUSY/BRST)
- In a Clifford representation with grading $\sigma = \mathrm{diag}(1, -1)$, Q has the form

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}$$

with $JE = EJ = W$

Example

- Simple factorization

$$W = x^d = x^n \cdot x^{d-n}$$

$$Q = \begin{pmatrix} 0 & x^n \\ x^{d-n} & 0 \end{pmatrix}$$

- these can be explicitely mapped to boundary states in a single minimal model A_{d-2} [Kapustin; Recknagel et al; Brunner, Gaberdiel]

2-branes on the quintic

$$W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \quad \text{in } \mathbb{CP}^4 \qquad \mathrm{Q} = \mathrm{Q}_1 \odot \mathrm{Q}_2 \odot \mathrm{Q}_3$$

with

$$J_1 = x_1 + x_2 \qquad J_2 = x_4 \qquad J_3 = x_5 + x_3$$

- $J_i = 0$ is a line in $\mathbb{CP}^4 \to$ Nullstellensatz
- this describes a permutation branes *[Recknagel]*
- CFT description known *[Brunner, Gaberdiel]*
- can be generalized to *[MB, Brunner, Gaberdiel]*

$$J_1 = x_1 + x_2 \qquad J_2 = ax_4 - bx_3 \qquad J_3 = ax_5 - cx_3$$

with $a^5 + b^5 + c^5 = 0$ in $\mathbb{CP}^2$

Lines in the quintic

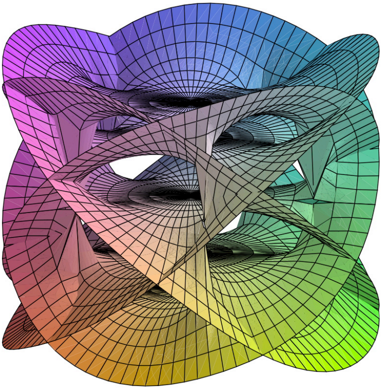
- the common locus of J_i corresponds to a complex line in the quintic
- it can be parametrised as

$$(x_1 : x_2 : x_3 : x_4 : x_5) = (u : -u : av : bv : cv)$$

with $(u : v) \in \mathbb{CP}^1$ and $a^5 + b^5 + c^5 = 0$

- this is a 2-cycle in $W = 0$
- MF has interpretation as D2-brane wrapping this cycle

The moduli space



Im(c) over the b-plane

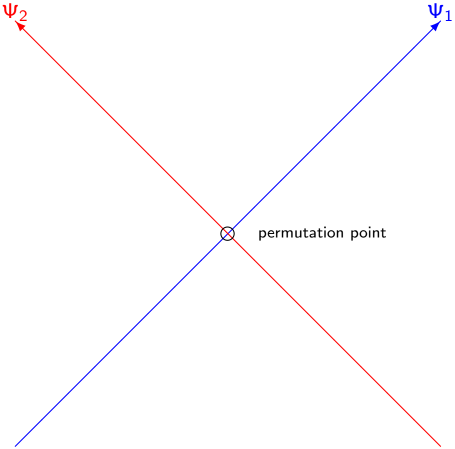
- moduli space known globally
- genus 6 algebraic curve
  $a^5 + b^5 + c^5 = 0$
- cohomology computed!

$$\Psi_1 = \partial_b Q(b)$$

$$\Psi_2 = \frac{x_1}{x_3}\Psi_1$$

- away from the permutation point, Ψ_2 is *obstructed*, due to $\langle \Psi_2 \Psi_2 \Psi_2 \rangle = -\frac{2}{5}\frac{b^4}{c^9}$
- only Ψ_1 is exactly marginal

Directions in moduli space



Red branch: $J_1 \leftrightarrow J_3$

Notation

branch	factorization	intersects with
(α)	(12)(435)	$(\beta), (\zeta), (\rho)$
(β)	(35)(412)	$(\alpha), (\gamma), (\mu)$
(γ)	(14)(325)	$(\beta), (\delta), (\nu)$
(δ)	(23)(415)	$(\gamma), (\epsilon), (\rho)$
(ϵ)	(15)(324)	$(\delta), (\zeta), (\mu)$
(ζ)	(34)(215)	$(\epsilon), (\alpha), (\nu)$
(λ)	(13)(245)	$(\mu), (\nu), (\rho)$
(μ)	(24)(315)	$(\beta), (\lambda), (\epsilon)$
(ν)	(25)(134)	$(\gamma), (\zeta), (\lambda)$
(ρ)	(45)(123)	$(\alpha), (\delta), (\lambda)$

e.g. (12)(435) corresponds to $(u : -u : av : bv : cv)$

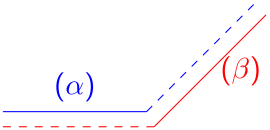
permutation points are given e.g. by $(\alpha\beta), (\mu\lambda)$ *etc*

Transitions

At each permutation point the fermions generating the braches are exchanged

They are related by expressions of the form

$$x_i \Psi_1 = x_j \Psi_2$$

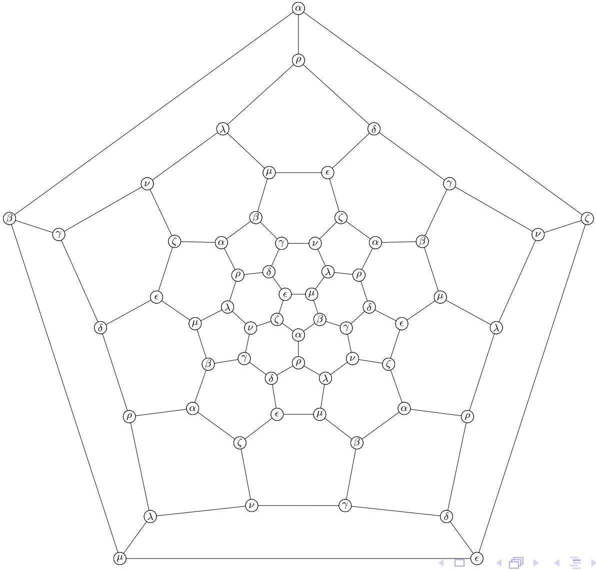


This gives a set of rules how to walk through the moduli space

... it's a truncated icosahedron!

Nodes: moduli branches $(\alpha), (\beta)$ etc

Edges: branch intersections, permutation points $(\alpha\beta), (\beta\gamma)$ etc *[MB, Wood]*



More Calabi-Yaus

$\mathbb{P}_{(1,1,1,1,1)}[5]$ $\quad W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ $\quad a^5 + b^5 + c^5 = 0$

joints with 2 fermions

$\mathbb{P}_{(1,1,1,1,2)}[6]$ $\quad W = x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^3$ $\quad a^6 + b^6 + c^6 = 0$

$a^6 + b^6 + c^3 = 0$

joints with 2 and 3 fermions

$\mathbb{P}_{(1,1,1,1,4)}[8]$ $\quad W = x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2$ $\quad a^8 + b^8 + c^8 = 0$

$a^8 + b^8 + c^2 = 0$

joints with 2 and 5 fermions

$\mathbb{P}_{(1,1,1,2,5)}[10]$ $\quad W = x_1^{10} + x_2^{10} + x_3^{10} + x_4^5 + x_5^2$ $\quad a^{10} + b^{10} + c^{10} = 0$

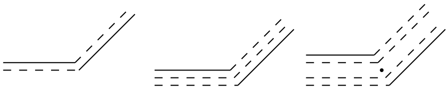
$a^{10} + b^{10} + c^5 = 0$

$a^{10} + b^{10} + c^2 = 0$

$a^{10} + b^5 + c^2 = 0$

joints with 2, 3 and 5 fermions

+ disconnected piece



$$x_j \Psi_1 = x_i \Psi_2 \qquad x_i \Psi_3 = x_j \Psi_2 \qquad \ldots$$

Bulk deformations

- boundary theory 'determined' by bulk

$$W \to W + \lambda G \quad \xrightarrow{\text{if possible}} \quad Q \to Q + u\Psi$$

- branes, cohomology are modified
  - deformations: branes moves along a bulk modulus
  - obstructions: branes cease to exist
- obstructions mean:
  - supersymmetry broken
  - potential for moduli induced
  - renormalization group flow

Bulk deformations

$$W = W_0 + \lambda G \qquad G = x_1^3 s^{(2)}(x_3, x_4, x_5) \qquad s^{(2)} = \sum_{q+r+s=2} s_{qrs} x_3^q x_4^r x_5^s$$

- perturbatively: $Q_0(a, b, c)$ can only be deformed if G is *exact* in $\mathcal{R}_\partial$
- in this case, the factorization extends to finite λ
- $J_1 = J_2 = J_3 = 0$ is a line in $W = W_0 + \lambda G$

[Albano, Katz]

$$s^{(2)}(a, b, c) = 0 \quad \cap \quad a^5 + b^5 + c^5 = 0$$

There are only 10 such points for which branes can be deformed

Renormalization group flow

- for the deforming fermions the conformal weight $h = 1$
- in the patch where $a = 1$ and b is a good coordinate we find for all b

$$\dot{b} = (1-h)b + \frac{\lambda}{2}\langle G\Psi_1\rangle = \frac{\lambda}{50}c^{-4}s^{(2)}(1,b,c)$$

[Fredenhagen, Gaberdiel, Keller; MB, Brunner, Gaberdiel]

- and $\langle G\Psi_2\rangle = 0$, so only Ψ_1 is excited
- the RG fixed points of the CFT are identical to the points where $s^{(2)}(a,b,c) = 0$ obtained from the topological theory

The exact brane potential

- the RG flow equation can be integrated
- the rhs is of the form $\omega_{rs} = b^{r-1}c^{s-5}$ with $1 \leq r, s$ and $r + s \leq 4$
- these are exactly the 6 globally holomorphic functions on the genus-6-curve $1 + b^5 + c^5 = 0$
- thus, $\omega_{rs}\, db$ are the associated differentials

The bulk deformations under which a brane deforms are in one-to-one correspondence to the spectrum of differentials on the moduli space

The exact brane potential

bulk induced effective potential

$$\mathcal{W}(1,b,c) \propto \lambda \sum_{i+j+k=2} s^{(2)}_{ijk} \mathcal{W}_{j+1,k+1}$$

$$\mathcal{W}_{rs} = \frac{b^r}{r}\ {}_2\mathrm{F}_1\left(\tfrac{r}{N}, 1-\tfrac{s}{N}, 1+\tfrac{r}{N}; -b^N\right) \qquad N = 5$$

this can be generalized for the other cases ...

The exact brane potential

[MB, Wood]

CY	moduli curve	bulk deformation	effective superpotential
$\mathbb{P}_{(1,1,1,1,1)}[N=5]$	$a^5+b^5+c^5=0$	$G=\lambda s^{(3)}(x_i,x_j)\cdot s^{(2)}(x_k,x_l,x_m)$	$\mathcal{W}\propto\sum_{i+j+k=2} s^{(2)}_{ijk}\mathcal{W}_{j+1,k+2}$
$\mathbb{P}_{(1,1,1,1,2)}[N=6]$	$a^6+b^6+c^6=0$	$G=\lambda s^{(3)}(x_i,x_5)\cdot s^{(3)}(x_k,x_l,x_m)$	$\mathcal{W}\propto\sum_{i+j+k=3} s^{(3)}_{ijk}\mathcal{W}_{j+1,k+1}$
	$a^6+b^6+c^3=0$	$G=\lambda s^{(4)}(x_i,x_j)\cdot s^{(2)}(x_k,x_l,x_5)$	$\mathcal{W}\propto\sum_{i+j+2k=2} s^{(2)}_{ijk}\mathcal{W}_{j+1,k+1}$
$\mathbb{P}_{(1,1,1,1,4)}[N=8]$	$a^8+b^8+c^8=0$	$G=\lambda s^{(3)}(x_i,x_5)\cdot s^{(5)}(x_k,x_l,x_m)$	$\mathcal{W}\propto\sum_{i+j+k=6} s^{(5)}_{ijk}\mathcal{W}_{j+1,k+1}$
	$a^8+b^8+c^2=0$	$G=\lambda s^{(6)}(x_i,x_j)\cdot s^{(2)}(x_k,x_l,x_5)$	$\mathcal{W}\propto\sum_{i+j+4k=2} s^{(2)}_{ijk}\mathcal{W}_{j+1,4(k+1)}$
$\mathbb{P}_{(1,1,1,2,5)}[N=10]$	$a^{10}+b^{10}+c^5=0$	$G=\lambda s^{(4)}(x_i,x_5)\cdot s^{(6)}(x_l,x_k,x_4)$	$\mathcal{W}\propto\sum_{i+2j+5k=2} s^{(6)}_{ijk}\mathcal{W}_{j+1,2(k+1)}$
	$a^{10}+b^{10}+c^2=0$	$G=\lambda s^{(7)}(x_i,x_4)\cdot s^{(3)}(x_l,x_k,x_5)$	$\mathcal{W}\propto\sum_{i+2j+5k=2} s^{(3)}_{ijk}\mathcal{W}_{j+1,5}$
	$a^{10}+b^5+c^2=0$	$G=\lambda s^{(8)}(x_i,x_j)\cdot s^{(2)}(x_l,x_4,x_5)$	$\mathcal{W}\propto\sum_{i+2j+5k=2} s^{(2)}_{ijk}\mathcal{W}_{2(j+1),5}$

$$\mathcal{W}_{rs}=\frac{b^r}{r}\ {}_2\mathrm{F}_1\left(\tfrac{r}{N},1-\tfrac{s}{N},1+\tfrac{r}{N};-b^N\right)$$

Conclusions

- Matrix factorizations describe B-branes
- and their moduli spaces.
- They provide a new method to investigate bulk induced changes of open moduli spaces,
- in particular the collapse due to RG flow.
- They allow to compute open-closed effective superpotentials on CY exactly
- which are important e.g. for open mirror symmetry

The End