Holographic superconductors with Lifshitz scaling

Erling J. Brynjólfsson, University of Iceland

0908.2611 EJB, Ulf Danielsson, Lárus Thorlacius, Tobias Zingg

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Overview

- Introduction
- Lifshitz black holes
- Charged Lifshitz black holes
- Coupling to charged matter
- Conclusions
- Outlook

Lifshitz scaling

$$t \to \lambda^z t, \qquad \mathbf{x} \to \lambda \mathbf{x}, \qquad z > 1.$$

Arises naturally at quantum critical points.

- $z = 1$: Spin systems.
- $z = 2$: The Schrödinger equation, antiferromagnetism in heavy fermion metals.
- $z = 3$: Ferromagnetism in heavy fermion metals
- Noninteger z.

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Question: Can we give a gravity dual description of a strongly coupled system which exhibits Lifshitz scaling? Yes!

Gravity theory with background fluxes

Kachru, Liu & Mulligan '08

$$S = \int \mathrm{d}^4x\sqrt{-g}\ (R - 2\Lambda)$$
$$- \frac{1}{2}\int *F_{(2)} \wedge F_{(2)} - \frac{1}{2}\int *H_{(3)} \wedge H_{(3)} - c\int B_{(2)} \wedge F_{(2)},$$

with $H_{(3)} = dB_{(2)}$.

Field equations:

$$d * F_{(2)} = -cH_{(3)}, \qquad d * H_{(3)} = -cF_{(2)},$$

$$G_{\mu\nu}+\Lambda g_{\mu\nu} = \frac{1}{2}(F_{\mu\lambda}F_{\nu}^{\ \lambda}-\frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma})+\frac{1}{4}(H_{\mu\lambda\sigma}H_{\nu}^{\ \lambda\sigma}-\frac{1}{6}g_{\mu\nu}H_{\lambda\sigma\rho}H^{\lambda\sigma\rho}).$$

The metric

$$ds^2 = L^2 \left(-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\mathbf{x}^2 \right),$$

is a solution to the gravity theory with $\Lambda = -\frac{z^2+z+4}{2L^2}$, $c = \frac{\sqrt{2z}}{L}$.
It's invariant under

$$t \to \lambda^z t, \qquad r \to r/\lambda, \qquad \mathbf{x} \to \lambda \mathbf{x}.$$

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Finite temperature $\to$ Look for black hole solutions

Black hole ansatz

$$ds^2 = L^2\left(-r^{2z}f(r)^2dt^2 + \frac{g(r)^2}{r^2}dr^2 + r^2(d\theta^2 + \chi(\theta)^2 d\phi^2)\right)$$

$$\begin{aligned} r \to r_0: \quad & f(r) = \sqrt{r - r_0}\,\left(f_0 + f_1(r - r_0) + \cdots\right) \\ & g(r) = \tfrac{1}{\sqrt{r-r_0}}\,\left(g_0 + g_1(r - r_0) + \cdots\right) \\ r \to \infty: \quad & f(r) \to 1, \qquad g(r) \to 1 \end{aligned}$$

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$$\chi(\theta) = \begin{cases} \sin\theta & \text{if} \quad k = 1, \quad \text{spherical horizon} \\ \theta & \text{if} \quad k = 0, \quad \text{flat horizon} \\ \sinh\theta & \text{if} \quad k = -1, \quad \text{hyperbolic horizon} \end{cases}$$

And for the form fields:

$$\begin{aligned} F_{(2)} &= \sqrt{2z}\, L\, h(r)\, \frac{g(r)}{r}\, dr \wedge r^z f(r)\, dt \\ H_{(3)} &= 2\, L^2\, j(r)\, \frac{g(r)}{r}\, dr \wedge r\, d\theta \wedge r\chi(\theta)\, d\phi \end{aligned}$$

$$r \to \infty : \quad h(r) \to \sqrt{z-1}, \quad j(r) \to \sqrt{z-1}$$

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$$

$$
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$$

Reduced system:

$$
\begin{aligned}
rf' &= -\frac{1+2z}{2} f + \frac{fg^2}{2}\left[\frac{k}{r^2} + \frac{z^2+z+4}{2} - \frac{z}{2}h^2 + j^2\right] \\
rg' &= \frac{3g}{2} - \frac{g^3}{2}\left[\frac{k}{r^2} + \frac{z^2+z+4}{2} - \frac{z}{2}h^2 - j^2\right] \\
rh' &= -2h + 2gj \\
rj' &= \frac{j}{2} + zgh - \frac{jg^2}{2}\left[\frac{k}{r^2} + \frac{z^2+z+4}{2} - \frac{z}{2}h^2 + j^2\right]
\end{aligned}
$$

Linearizing around the Lifshitz point

$$g = 1 + \delta g \qquad h = \sqrt{z-1}(1+\delta h) \qquad j = \sqrt{z-1}(1+\delta j)$$

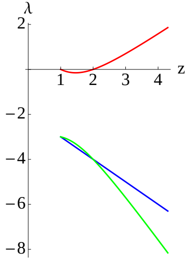
$$r\frac{d}{dr}\begin{pmatrix}\delta g\\ \delta h\\ \delta j\end{pmatrix} = \begin{pmatrix} -3 & \frac{z(z-1)}{2} & z-1 \\ 2 & -2 & 2 \\ -(z+1) & \frac{z(z+1)}{2} & 1-2z \end{pmatrix}\begin{pmatrix}\delta g\\ \delta h\\ \delta j\end{pmatrix} - \frac{k}{2r^2}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

- Expanding f, g, h, j close to the horizon $\Longrightarrow$ solutions depend on r_0 and $h_0 = h(r)|_{r=r_0}$
- Finite energy $\Longrightarrow$ leading mode is absent $\Longrightarrow$ fine-tuning of r_0 and h_0

  Danielson & Thorlacius '08

  Ross & Saremi '09

  Bertoldi, Burrington & Peet '09



- Effectively one-parameter family of solutions similar to Schwarzschild black holes.
- h and j vanish as $z \to 1$ and we recover the AdS gravity.
- The form fields $F_{(2)}$ and $H_{(3)}$ are auxilliary fields, included to obtain asymptotically Lifshitz geometry.

Known exact black hole solutions

$z = 2$: topological black hole with hyperbolic horizon

Mann '09

$$f(r) = \frac{1}{g(r)} = \sqrt{1 - \frac{1}{2r^2}}$$

$z = 4$: black hole with spherical horizon

Bertoldi, Burrington & Peet '09

$$f(r) = \frac{1}{g(r)} = \sqrt{1 + \frac{1}{10r^2} - \frac{3}{400r^4}}$$

and hyperbolic horizon

$$f(r) = \frac{1}{g(r)} = \sqrt{1 - \frac{1}{10r^2} - \frac{3}{400r^4}}$$

Isolated points of the one-parameter families for given z.

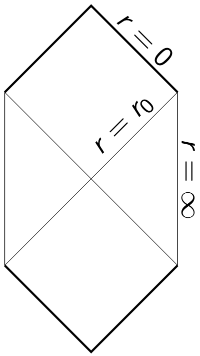
Global geometry

Use exact solutions to study the black hole interior.

The Ricci scalar is

$$z = 2: \qquad R = \frac{1}{L^2}\left(\frac{1}{r^2} - 22\right),$$
$$z = 4: \qquad R = \frac{1}{L^2}\left(\frac{9k^2}{200r^4} - \frac{3k}{5r^2} - 54\right).$$

Null singularity at $r \to 0$.



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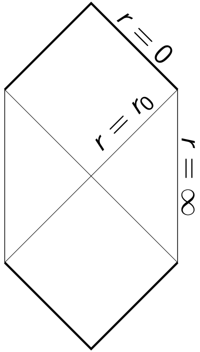

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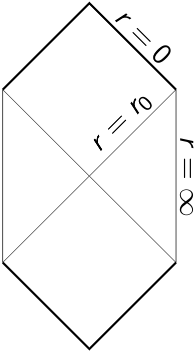


- This is true for generic asymptotically Lifshitz black holes with a single non-degenerate horizon.
- Two categories of black holes characterized by $R \propto \frac{1}{r^2}$ and $R \propto \frac{1}{r^4}$.

Charged Lifshitz black holes

- Dual field theory applications involve $k = 0$.
- All non-zero temperatures are equivalent: Can be mapped into each other by coordinate transformations.
- No phase transitions possible.
- We need to introduce another length scale.
- We add a Maxwell field and consider charged black holes.

Add new term to the action

$$S_{\mathcal{F}} = -\frac{1}{2}\int *\mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)}.$$

The black hole charge, Q, provides a new length scale.

$$T_H = \frac{r_0^{z-1} f_0}{4\pi g_0}$$

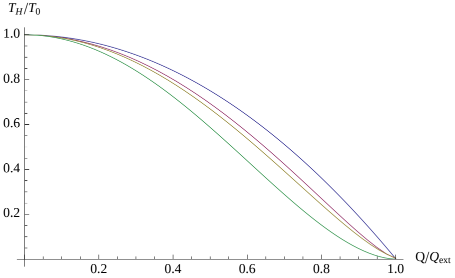


Figure: From top to bottom: $z = 1$, $z = 3/2$, $z = 2$, $z = 4$. Fixed $r_0 = 1$

Exact charged black hole solutions

z=1: AdS-Reissner-Nordström: A check of the formalism.

z=4: New family of solutions

$$f(r) = \frac{1}{g(r)} = \sqrt{1 + \frac{k}{10r^2} - \frac{3k^2}{400r^4} - \frac{Q^2}{2r^4}},$$

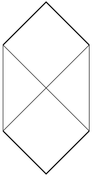
reduces to the previous $z = 4$ solutions for $Q = 0$.

Global geometry of the exact charged $z = 4$ solution.

- The Ricci scalar is

$$R = \frac{1}{L^2}\left(\frac{3Q^2}{r^4} + \frac{9k^2}{200r^4} - \frac{3k}{5r^2} - 54\right).$$

- The singularity is null.
- The conformal diagram remains the same.
- There is no inner horizon!



Coupling to scalar matter

- We add scalar ψ to the system.
- Charged under the new gauge field $\mathcal{F}_{(2)}$ but neutral under the Lifshitz form fields $F_{(2)}$ and $H_{(3)}$.
- Allows our black holes to grow scalar hair.
- Charged plane symmetric black hole with hair is a gravity dual of a holographic superconductor.
- Under certain conditions, the hair corresponds to the condensation of a charged operator, at low temperatures, in the dual field theory.

Add charged scalar field

$$S_\psi = -\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}(\partial_\mu\psi^* + iq\mathcal{A}_\mu\psi^*)(\partial_\nu\psi - iq\mathcal{A}_\nu\psi) + m^2\psi^*\psi\right).$$

For $r \to \infty$ the scalar solution is

$$\psi(r) = \frac{c_-}{r^{\Delta_-}} + \frac{c_+}{r^{\Delta_+}} + \cdots$$

with

$$\Delta_\pm = \frac{z+2}{2} \pm \sqrt{\left(\frac{z+2}{2}\right)^2 + m^2L^2}.$$

Choice of m^2

The requirement of having finite Euclidian action restricts the behaviour of ψ as $r \to \infty$. For

$$1 - \left(\frac{z+2}{2}\right)^2 < L^2 m^2$$

we must have $c_- = 0$. For

$$-\left(\frac{z+2}{2}\right)^2 < L^2 m^2 < 1 - \left(\frac{z+2}{2}\right)^2$$

we can either have

$$c_- = 0 \qquad \psi \quad \text{dual to an operator of dimension} \quad \Delta_+$$

or

$$c_+ = 0 \qquad \psi \quad \text{dual to an operator of dimension} \quad \Delta_- .$$

- For numerical calculations we pick

$$L^2 m^2 = \frac{1}{4} - \left(\frac{z+2}{2} \right)^2 .$$

- Within the range that gives a choice of two boundary theories.
- Convenient values of the operator dimensions, $\Delta_\pm = \frac{z+2}{2} \pm \frac{1}{2}$.
- Nonlinear descendants of $\frac{c_-}{r^{\Delta_-}}$ fall off faster than $\frac{c_+}{r^{\Delta_+}}$.
- For given values of z, m and q there is a three-parameter family of initial values of the dynamical fields, Q, $\psi_0 = \psi(r)|_{r=r_0}$ and h_0.
- As before, the initial data must be fine-tuned to obtain asymptotically Lifshitz solution with finite energy.

Signal of superconducting condensate

$$c_+ = 0 \qquad \text{and} \qquad \langle \mathcal{O}_- \rangle = c_- \neq 0,$$

or

$$c_- = 0 \qquad \text{and} \qquad \langle \mathcal{O}_+ \rangle = c_+ \neq 0.$$

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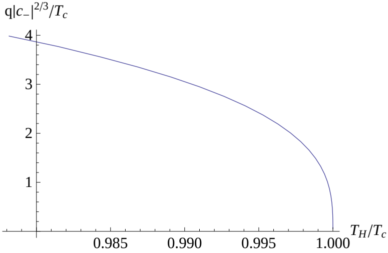

Figure: The onset of condensation in a $z = 2$ system.

Conclusions

- Lifshitz black holes provide a window onto finite temperature in strongly coupled Lifshitz models.
- The global extensions of Lifshitz black holes have null-singularities inside the horizon.
- The exact charged Lifshitz black hole has no smooth inner horizon.
- Charged Lifshitz black holes with hair are dual to the superconducting phase of holographic superconductors at $z > 1$.

Outlook

- Improved understanding of holographic renormalization in $z > 1$ models.
- Calculation of correlation functions in $z > 1$ holographic superconductors: Conductivity, thermal conductivity.
- Comparison to known CM results: E. g. ultralocality in $z = 2$ Lifshitz models.