Classifying Highly Supersymmetric Solutions

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arXiv:????.????, arXiv:0902.3642, arXiv:0710.1829, hep-th/0610331, hep-th/0606049

15-th European Workshop on String Theory

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- IIB Supergravity and Killing Spinors

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  All *maximally supersymmetric* solutions, i.e. those with 32 linearly independent Killing spinors, are completely classified [Figueroa O'Farrill, Papadopoulos]

  One finds: $\mathbb{R}^{9,1}$, $AdS_5 \times S^5$ and a maximally supersymmetric plane wave solution.

- Conclusions

IIB Supergravity and Killing Spinors

The bosonic fields of IIB supergravity are the spacetime metric g, the axion σ and dilaton ϕ , two three-form field strengths $G^\alpha = dA^\alpha$ $(\alpha = 1, 2)$, and a self-dual five-form field strength F

The axion and dilaton give rise to a complex 1-form P [Schwarz].

The 3-forms are combined to give a complex 3-form G.

To achieve this, introduce a $SU(1,1)$ matrix $U = (V_+^\alpha, V_-^\alpha)$, $\alpha = 1, 2$ such that

$$V_-^\alpha V_+^\beta - V_-^\beta V_+^\alpha = \epsilon^{\alpha\beta} \ , \quad (V_-^1)^* = V_+^2, \qquad (V_-^2)^* = V_+^1$$

$\epsilon^{12} = 1 = \epsilon_{12}$.

The $V_{\pm}^{\alpha}$ are related to the axion and dilaton by

$$\frac{V_{-}^{2}}{V_{-}^{1}} = \frac{1 + i(\sigma + ie^{-\phi})}{1 - i(\sigma + ie^{-\phi})} \ .$$

Then P and G are defined by

$$P_M = -\epsilon_{\alpha\beta} V_{+}^{\alpha} \partial_M V_{+}^{\beta}, \quad G_{MNR} = -\epsilon_{\alpha\beta} V_{+}^{\alpha} G_{MNR}^{\beta}$$

The gravitino Killing spinor equation is

$$\tilde{\nabla}_M \epsilon + \frac{i}{48}\Gamma^{N_1 \ldots N_4}\epsilon F_{N_1 \ldots N_4 M} - \frac{1}{96}(\Gamma_M{}^{N_1 N_2 N_3} G_{N_1 N_2 N_3} \\ -9\Gamma^{N_1 N_2} G_{M N_1 N_2})(C\epsilon)^* = 0$$

where

$$\tilde{\nabla}_M = \partial_M - \frac{i}{2} Q_M + \frac{1}{4}\Omega_{M,AB}\Gamma^{AB}$$

is the standard covariant derivative twisted with $U(1)$ connection Q_M, given in terms of the $SU(1,1)$ scalars by

$$Q_M = -i\epsilon_{\alpha\beta} V_-^\alpha \partial_M V_+^\beta$$

and Ω is the spin connection.

There is also an algebraic constraint

$$P_M \Gamma^M (C\epsilon)^* + \frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0$$

The Killing spinor ϵ is a complex Weyl spinor constructed from two copies of the same Majorana-Weyl representation Δ^+_{16}:

$$\epsilon = \psi_1 + i\psi_2$$

Majorana-Weyl spinors ψ satisfy

$$\psi = C(\psi^*)$$

C is the charge conjugation matrix.

Spinors as Forms

- Let $e_1, \ldots, e_5$ be a locally defined orthonormal basis of $\mathbb{R}^5$.
- Take U to be the span over $\mathbb{R}$ of $e_1, \ldots, e_5$.
- The space of Dirac spinors is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$ (the complexified space of all forms on U).
- Δ_c decomposes into even forms Δ_c^+ and odd forms Δ_c^-, which are the complex Weyl representations of $Spin(9,1)$.

- The gamma matrices are represented on Δ_c as

$$\begin{aligned}
\Gamma_0 \eta &= -e_5 \wedge \eta + e_5 \lrcorner \eta \\
\Gamma_5 \eta &= e_5 \wedge \eta + e_5 \lrcorner \eta \\
\Gamma_i \eta &= e_i \wedge \eta + e_i \lrcorner \eta \qquad\qquad i = 1, \ldots, 4 \\
\Gamma_{5+i} \eta &= i e_i \wedge \eta - i e_i \lrcorner \eta \qquad\quad i = 1, \ldots, 4
\end{aligned}$$

- Γ_j for $j = 1, \ldots, 9$ are hermitian and Γ_0 is anti-hermitian with respect to the inner product

$$< z^a e_a, w^b e_b >= \sum_{a=1}^{5} (z^a)^* w^a \ ,$$

This inner product can be extended from $U \otimes \mathbb{C}$ to Δ_c.

- There is a $Spin(9,1)$ invariant inner product defined on Δ_c defined by

$$B(\epsilon_1, \epsilon_2) = < \Gamma_0 C(\epsilon_1)^*, \epsilon_2 >$$

  B is skew-symmetric in ϵ_1, ϵ_2.

  B vanishes when restricted to Δ_c^+ or Δ_c^-.

- This defines a non-degenerate pairing $\mathcal{B} : \Delta_c^+ \otimes \Delta_c^- \to \mathbb{R}$ given by

$$\mathcal{B}(\epsilon, \xi) = \mathrm{Re}\ B(\epsilon, \xi)$$

Canonical forms of spinors

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$Spin(9,1)$ has one type of orbit with stability subgroup $Spin(7) \ltimes \mathbb{R}^8$ in Δ_{16}^- [Figueroa-O'Farrill, Bryant].

$$\Delta_{16}^- = \mathbb{R} < e_5 + e_{12345} > + \Lambda^1(\mathbb{R}^7) + \Delta_8 \ ,$$

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$\Lambda^1(\mathbb{R}^7)$ is the vector representation of $Spin(7)$ spanned by (j,k=1,...,4) $e_{jk5} - \frac{1}{2}\epsilon_{jkmn}e_{mn5}, i(e_{jk5} + \frac{1}{2}\epsilon_{jkmn}e_{mn5})$ and $i(e_5 - e_{12345})$.

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Δ_8 is the spin representation of $Spin(7)$ spanned by $e_j + \frac{1}{6}\epsilon_{jq_1q_2q_3}e_{q_1q_2q_3}, i(e_j - \frac{1}{6}\epsilon_{jq_1q_2q_3}e_{q_1q_2q_3})$.

$Spin(7)$ acts transitively on the S^7 in Δ_8, with stability subgroup G_2, and G_2 acts transitively on the S^6 in $\Lambda^1(\mathbb{R}^7)$ with stability subgroup $SU(3)$ [Salamon]

Using these transitive actions, any $\nu_1 \in \Delta_{16}^-$ can be written as

$$\nu_1 = a_1(e_5 + e_{12345}) + ia_2(e_5 - e_{12345}) + a_3(e_1 + e_{234})$$

For all possible choices of (real) a_1, a_2, a_3, there exist $Spin(9,1)$ transformations which set $\nu_1 = e_5 + e_{12345}$.

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Having fixed ν_1, it remains to consider ν_2:
By using $Spin(7)$ gauge transformations, which leave ν_1 invariant, one can write

$$\nu_2 = b_1(e_5 + e_{12345}) + ib_2(e_5 - e_{12345}) + b_3(e_1 + e_{234})$$

There are various cases

i) $b_3 \neq 0$. Then using $Spin(7) \ltimes \mathbb{R}^8$ gauge transformations one can take

$$\nu_2 = g(e_1 + e_{234})$$

The stability subgroup of $Spin(9,1)$ which leaves ν_1 and ν_2 invariant is G_2.

ii) If $b_3 = 0$ then

$$\nu_2 = g_1(e_5 + e_{12345}) + ig_2(e_5 - e_{12345})$$

and the stability subgroup is $SU(4) \ltimes \mathbb{R}^8$

iii) If $b_2 = b_3 = 0$ then

$$\nu_2 = g(e_5 + e_{12345})$$

and the stability subgroup is $Spin(7) \ltimes \mathbb{R}^8$.

$N = 31$ Solutions: Algebraic Constraints

Suppose that there exists a solution with exactly (and no more than) 31 linearly independent Killing spinors over $\mathbb{R}$.

Consider the algebraic constraint

$$P_M \Gamma^M (C\epsilon^r)^* + \frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon^r = 0$$

where ϵ^r are Killing spinors for $r = 1, \ldots, 31$.

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where ϵ^r are Killing spinors for $r = 1, \ldots, 31$.

The space of Killing spinors is orthogonal to a single *normal spinor*, $\nu \in \Delta_c^-$ with respect to the $Spin(9,1)$ invariant inner product $\mathcal{B}$. Using $Spin(9,1)$ gauge transformations, this normal spinor can be brought into one of 3 canonical forms:

$$\begin{aligned} Spin(7) \ltimes \mathbb{R}^8 : \quad & \nu = (n + im)(e_5 + e_{12345}) \,, \\ SU(4) \ltimes \mathbb{R}^8 : \quad & \nu = (n - \ell + im) e_5 + (n + \ell + im) e_{12345} \,, \\ G_2 : \quad & \nu = n(e_5 + e_{12345}) + im(e_1 + e_{234}) \,, \end{aligned}$$

In general, one can write

$$\epsilon^r = \sum_{i=1}^{32} f^r{}_i \eta^i$$

where $f^r{}_i$ are real, η^p for $p = 1, \ldots, 16$ is a basis for Δ^+_{16} and $\eta^{16+p} = i\eta^p$.

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The matrix with components $f^r{}_i$ is of rank 31.

The functions $f^r{}_i$ are constrained by the orthogonality condition.

For example, take the case for which $\nu = (n + im)(e_5 + e_{12345})$: set

$$\epsilon^r = f^r{}_1(1 + e_{1234}) + f^r{}_{17} i(1 + e_{1234}) + f^r{}_k \eta^k$$

where η^k are the remaining basis elements orthogonal to $1 + e_{1234}, i(1 + e_{1234})$.

Then the orthogonality relation implies

$$n{f^r}_1 - m{f^r}_{17} = 0$$

and so, taking without loss of generality $n \neq 0$; one finds

$$\epsilon^r = \frac{{f^r}_{17}}{n}(m + in)(1 + e_{1234}) + {f^r}_k \eta^k$$

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Substituting this back into the algebraic Killing spinor equation, one finds

$$P_M \Gamma^M C*[(m+in)(1+e_{1234})] + \frac{1}{24} G_{M_1 M_2 M_3} \Gamma^{M_1 M_2 M_3} (m+in)(1+e_{1234}) = 0$$

and

$$P_M \Gamma^M \eta^p = 0, \qquad G_{M_1 M_2 M_3} \Gamma^{M_1 M_2 M_3} \eta^p = 0, \quad p = 2, \ldots, 16$$

Analogous equations are obtained for $SU(4) \ltimes \mathbb{R}^8$ and G_2 invariant normals.

In all cases, the constraints $P_M \Gamma^M \eta^p = 0$ fix $P = 0$.

This means that the algebraic Killing spinor equation is linear over $\mathbb{C}$, so if there is a background with $N = 31$ linearly independent solutions of the algebraic Killing spinor equation, then this equation must have 32 linearly independent solutions.

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This in turn fixes $G = 0$. However, if $G = 0$ then the gravitino Killing spinor equation also becomes linear over $\mathbb{C}$.

In this case, if the gravitino Killing spinor equation has 31 linearly independent solutions, it must have 32 solutions also. So the background is maximally supersymmetric.

$N = 30$ Solutions: Algebraic Constraints

Having excluded $N = 31$ solutions, consider $N = 30$.

To simplify the analysis, we use a result of Figueroa O'Farrill, Hackett-Jones and Moutsopoulos.

This states that all solutions with $N > 24$ linearly independent Killing spinors are homogeneous, and hence have $P = 0$.

So, for $N = 30$ solutions, the algebraic Killing spinor equation becomes linear over $\mathbb{C}$:

$$\frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0$$

To analyse the case of $N = 30$ solutions, note that the Killing spinors are all orthogonal to a normal spinor $\nu \in \Delta_c^-$ with respect to the inner product B.

This can be brought into canonical form using gauge transformations.

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The solutions to the algebraic Killing spinor equation are

$$\epsilon^r = \sum_{s=1}^{15} z^r{}_s \eta^s \ ,$$

where η^i is a basis normal to ν and z is an invertible 15×15 matrix of spacetime dependent complex functions.

There are three cases to consider, corresponding to the types of normal spinor ν.

In all cases, one can choose the basis (η^i) to have 13 (very simple) common elements, which are orthogonal to ν: $e_{pq}, e_{15pq}, e_{1p}, e_{1q}$ for $p = 2, 3, 4$ and $e_{15} - e_{2345}$.

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The remaining two basis elements are case-dependent

$$\begin{aligned} Spin(7) \ltimes \mathbb{R}^8 : &\qquad 1 - e_{1234}, e_{15} + e_{2345} \, , \\ SU(4) \ltimes \mathbb{R}^8 : &\qquad e_{15} + e_{2345}, (n - \ell + im)1 - (n + \ell + im)e_{1234} \, , \\ G_2 : &\qquad 1 - e_{1234}, m(1 + e_{1234}) + in(e_{15} + e_{2345}) \end{aligned}$$

In all cases, evaluating the algebraic Killing spinor equation on the basis (η^i) produces sufficient constraints to fix $G = 0$.

Integrability Conditions for N=30 Solutions

It remains to consider the integrability conditions of the Killing spinor equations for solutions with $G = P = 0$.

The curvature $\mathcal{R} = [\mathcal{D}, \mathcal{D}]$ of the covariant connection $\mathcal{D}$ of IIB supergravity can be expanded as

$$\mathcal{R}_{MN} = \frac{1}{2}(T^2_{MN})_{PQ}\Gamma^{PQ} + \frac{1}{4!}(T^4_{MN})_{Q_1\dots Q_4}\Gamma^{Q_1\dots Q_4} ,$$

where

$$\begin{aligned}
(T^2_{MN})_{P_1P_2} &= \tfrac{1}{4}R_{MN,P_1P_2} - \tfrac{1}{12}F_{M[P_1}{}^{Q_1Q_2Q_3}F_{|N|P_2]Q_1Q_2Q_3} , \\
(T^4_{MN})_{P_1\dots P_4} &= \tfrac{i}{2}D_{[M}F_{N]P_1\dots P_4} + \tfrac{1}{2}F_{MNQ_1Q_2[P_1}F_{P_2P_3P_4]}{}^{Q_1Q_2}
\end{aligned}$$

The T^2 and T^4 tensors satisfy various algebraic constraints, following from the Bianchi identities and field equations:

$$\begin{aligned}
(T^2_{MN})_{P_1P_2} &= (T^2_{P_1P_2})_{MN} \ , \\
(T^2_{M[P_1})_{P_2P_3]} &= 0 \ , \\
(T^2_{MN})_P{}^N &= 0 \ , \\
(T^4_{[P_1P_2})_{P_3P_4P_5P_6]} &= 0 \\
(T^4_{MN})_{P_1P_2P_3}{}^N &= 0 \ , \\
(T^4_{M[P_1})_{P_2P_3P_4P_5]} &= -\frac{1}{5!}\epsilon_{P_1P_2P_3P_4P_5}{}^{Q_1Q_2Q_3Q_4Q_5}(T^4_{M[Q_1})_{Q_2Q_3Q_4Q_5]} \ .
\end{aligned}$$

And $(T^4{}_{P_1(M})_{N)P_2P_3P_4}$ is totally antisymmetric in P_1, P_2, P_3, P_4.

Analysis of Constraints

The integrability conditions of the gravitino Killing spinor equations

$$\mathcal{R}\epsilon^r = 0$$

One can obtain constraints on the tensors T^2 and T^4 by directly evaluating these constraints on the basis elements η^i and using the constraints and symmetries of T^2, T^4.

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It is more straightforward to note that $\mathcal{R}\epsilon^r = 0$, implies

$$\mathcal{R}_{MN,ab'} = u_{MN,r}\eta^r_a\nu_{b'} + u_{MN}\chi_a\nu_{b'}$$

where u are complex valued, and η^r, χ is a basis for Δ^+_c.

We also have the formula

$$\psi_a \nu_{b'} = -\frac{1}{16} \sum_{k=0}^{2} \frac{1}{(2k)!} B(\psi, \Gamma_{A_1 A_2 \ldots A_{2k}} \nu)(\Gamma^{A_1 A_2 \ldots A_{2k}})_{ab'} \ ,$$

for any positive chirality spinor ψ.

Requiring that the holonomy of the supercovariant connection lie in $SL(16, \mathbb{C})$ implies that

$$u_{MN} B(\chi, \nu) = 0$$

which eliminates the contribution to $\mathcal{R}_{MN,ab'}$ from $u_{MN} \chi_a \nu_{b'}$.

Hence we are left with

$$\begin{aligned}\mathcal{R}_{MN,ab'} &= u_{MN,r}\eta^r_a \nu_{b'} \\ &= -\frac{1}{16} u_{MN,r} \sum_{k=1}^{2} \frac{1}{(2k)!} B(\eta^r, \Gamma_{A_1 A_2 \ldots A_{2k}} \nu)(\Gamma^{A_1 A_2 \ldots A_{2k}})_{ab'}\end{aligned}$$

which in turn relates T^2, T^4 to $u_{MN,r}$ via

$$\begin{aligned}(T^2_{MN})_{A_1 A_2} &= -\frac{1}{16} u_{MN,r} B(\eta^r, \Gamma_{A_1 A_2} \nu) \\ (T^4_{MN})_{A_1 A_2 A_3 A_4} &= -\frac{1}{16} u_{MN,r} B(\eta^r, \Gamma_{A_1 A_2 A_3 A_4} \nu)\end{aligned}$$

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- Translate the T^2 and T^4 constraints into constraints on u
- After some mildly involved computation, one finds that these are sufficient to fix $u_{MN,r} = 0$.
- This then implies that $T^2 = 0$, $T^4 = 0$.
- However these are equivalent (together with $P = 0, G = 0$) to the constraints on maximally supersymmetric backgrounds.

So all $N = 30$ solutions are locally maximally supersymmetric.

There are also no quotients of maximally supersymmetric solutions which preserve 30 supersymmetries.

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- So a background with $N = 29$ linearly independent solutions to the algebraic Killing spinor equation must have at least 30 solutions to this equation.
- By the $N = 30$ analysis, this is sufficient to fix $G = 0$
- As $G = 0$, the gravitino Killing spinor equation is linear over $\mathbb{C}$, and so an exactly $N = 29$ solution is excluded.

Conclusions

There are no solutions of IIB supergravity with exactly $N = 29$, $N = 30$ or $N = 31$ linearly independent Killing spinors

Conclusions

There are no solutions of IIB supergravity with exactly $N = 29$, $N = 30$ or $N = 31$ linearly independent Killing spinors

What about solutions with $N = 28$ supersymmetries? A non-trivial example is known - the plane wave geometry of Bena and Roiban.

In fact in order to have a solution with exactly 28 linearly independent Killing spinors, one is *forced* to take $G \neq 0$.

Analysis of the Killing spinor equation integrability conditions with $G \neq 0$ is *much* more complicated!

The gravitino integrability conditions are

$$\mathcal{S}\epsilon + \mathcal{T}(C\epsilon)^* = 0$$

where

$$\begin{aligned}
\mathcal{T} \quad = \quad & -\frac{\kappa}{96}(\Gamma_{[N}{}^{L_1L_2L_3}D_{M]}G_{L_1L_2L_3} + 9\Gamma^{L_1L_2}D_{[N}G_{M]L_1L_2}) \\
& + \frac{i\kappa^2}{32}(\frac{1}{3}F_{NM}{}^{L_1L_2L_3}G_{L_1L_2L_3} + \Gamma^{L_1L_2}F_{[N|L_1L_2}{}^{Q_1Q_2}G_{|M]Q_1Q_2} \\
& \quad + \frac{1}{3}\Gamma_{[N}{}^{Q}F_{M]Q}{}^{L_1L_2L_3}G_{L_1L_2L_3} - \frac{1}{2}\Gamma^{L_1\ldots L_4}F_{NML_1L_2}{}^{Q}G_{L_3L_4Q} \\
& \quad + \frac{1}{2}\Gamma_{[N}{}^{L_1L_2L_3}F_{M]L_1L_2}{}^{Q_1Q_2}G_{L_3Q_1Q_2} + \frac{1}{4}\Gamma^{L_1\ldots L_4}F_{L_1\ldots L_4}{}^{Q}G_{NMQ} \\
& \quad - \frac{1}{2}\Gamma_{[N|}{}^{L_1L_2L_3}F_{L_1L_2L_3}{}^{Q_1Q_2}G_{|M]Q_1Q_2}) \, .
\end{aligned}$$

$$
\begin{aligned}
\mathcal{S} \quad = \quad & \frac{1}{8} R_{NM}{}^{L_1 L_2} \Gamma_{L_1 L_2} - \frac{1}{2} P_{[N} P^{\star}_{M]} + \frac{i\kappa}{48} \Gamma^{L_1 \ldots L_4} D_{[N} F_{M] L_1 \ldots L_4} \\
& + \frac{\kappa^2}{24} (- \Gamma^{L_1 L_2} F_{[N|L_1}{}^{Q_1 Q_2 Q_3} F_{|M] L_2 Q_1 Q_2 Q_3} + \frac{1}{2} \Gamma^{L_1 \ldots L_4} F_{NML_1}{}^{Q_1 Q_2} F_{L_2 L_3 L_4 Q_1 Q_2} \\
& \qquad\qquad + \frac{1}{2} \Gamma_{[N}{}^{L_1 L_2 L_3} F_{M] L_1}{}^{Q_1 Q_2 Q_3} F_{L_2 L_3 Q_1 Q_2 Q_3}) \\
& + \frac{\kappa^2}{32} (- \frac{1}{2} G_{[N}{}^{L_1 L_2} G^{\star}_{M] L_1 L_2} + \frac{1}{48} \Gamma_{NM} G^{L_1 L_2 L_3} G^{\star}_{L_1 L_2 L_3} \\
& \quad - \frac{1}{4} \Gamma_{[N}{}^{L_1} G_{M]}{}^{L_2 L_3} G^{\star}_{L_1 L_2 L_3} + \frac{1}{8} \Gamma_{[N|}{}^{Q} G_{Q}{}^{L_1 L_2} G^{\star}_{|M] L_1 L_2} \\
& \quad + \frac{3}{16} \Gamma^{L_1 L_2} G_{NM}{}^{L_3} G^{\star}_{L_1 L_2 L_3} - \Gamma^{L_1 L_2} G_{[N|L_1}{}^{Q} G^{\star}_{|M] L_2 Q} \\
& \quad - \frac{3}{16} \Gamma^{L_1 L_2} G_{L_1 L_2}{}^{Q} G^{\star}_{NMQ} + \frac{1}{16} \Gamma_{NM}{}^{L_1 L_2} G_{L_1}{}^{Q_1 Q_2} G^{\star}_{L_2 Q_1 Q_2} \\
& \quad - \frac{1}{16} \Gamma^{L_1 \ldots L_4} G_{L_1 L_2 L_3} G^{\star}_{NML_4} + \frac{1}{8} \Gamma_{[N|}{}^{L_1 L_2 L_3} G_{L_1 L_2}{}^{Q} G^{\star}_{|M] L_3 Q} \\
& \quad + \frac{1}{4} \Gamma^{L_1 \ldots L_4} G_{[N|L_1 L_2} G^{\star}_{|M] L_3 L_4} + \frac{1}{16} \Gamma^{L_1 \ldots L_4} G_{NML_1} G^{\star}_{L_2 L_3 L_4} \\
& \quad + \frac{1}{4} \Gamma_{[N|}{}^{L_1 L_2 L_3} G_{|M] L_1}{}^{Q} G^{\star}_{L_2 L_3 Q} + \frac{1}{24} \Gamma_{[N|}{}^{L_1 \ldots L_5} G_{|M] L_1 L_2} G^{\star}_{L_3 L_4 L_5} \\
& \quad - \frac{1}{48} \Gamma_{[N|}{}^{L_1 \ldots L_5} G_{L_1 L_2 L_3} G^{\star}_{|M] L_4 L_5} - \frac{1}{32} \Gamma_{NM}{}^{L_1 \ldots L_4} G_{L_1 L_2}{}^{Q} G^{\star}_{L_3 L_4 Q} \\
& \quad - \frac{1}{288} \Gamma_{NM}{}^{L_1 \ldots L_6} G_{L_1 L_2 L_3} G^{\star}_{L_4 L_5 L_6})
\end{aligned}
$$

One can show [JG, Gran, Papadopoulos] that the Bena and Roiban plane wave is the unique solution with $N = 28$ supersymmetries:

$$\begin{aligned} ds^2 &= 2dw(dv - (\frac{9}{8} + 2h^2)\delta_{ij}x^i x^j dw) + \delta_{ij}dx^i dx^j \\ G &= -2\sqrt{2}ie^{i\phi}dw \wedge (dx^{15} + dx^{26} + dx^{37} + dx^{48}) \\ F &= 2hdw \wedge (dx^{1256} - dx^{3478}) \end{aligned}$$

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It has also been shown [Gran, JG, Papadopoulos, Roest], that there are no $N = 31$ (and very recently, no $N = 30$) solutions in D=11 supergravity.