Properties of Schrödinger Space-times

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Based on

0904.3304 Matthias Blau, J. H., Blaise Rollier

and work in progress

Introduction & Motivation

- Many systems in nature exhibit critical points with non-relativistic scale invariance $z > 1$:

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- Systems with Schrödinger invariance.
- Holographic approach to the study of such systems: [Son, 2008] [Balasubramanian, McGreevy, 2008]

Contents

- For any $z > 1$:
  - Review of properties of Schrödinger space-times in Poincaré-like coordinates
  - Causal properties
- For $z = 2$:
  - Global coordinates
  - Hilbert space for scalars
  - Comments on Cauchy problem for scalars

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- $\mathtt{sch}_z(d+3) \subset \mathtt{so}(2, d+2)$
- The Schrödinger algebra for $z \neq 1$ consists of:

$SL(2,\mathbb{R})$ (only for $z = 2$)	$\Big\{$	H	time translation
		D	dilatation
		C	special conformal ($\exists$ only for $z = 2$)
Heisenberg	$\Big\{$	N	mass operator (only central for $z = 2$)
		P_a	momenta ($a = 1, \ldots, d$)
		V_a	Galilean boosts
$SO(d)$		M_{ab}	rotations

Geodesic properties

$$ds^2 = -\frac{1}{r^{2z}}dt^2 + \frac{1}{r^2}\left(-2dtd\xi + dr^2 + d\vec{x}^2\right)$$

	Geodesically complete	Tidal forces [Podolsky, 1998]	Bulk to boundary geodesics
$z = 1\ (AdS)$	no	constant	yes
$1 < z < 2$	no	divergent	no
$z = 2$	no	finite (bounded)	no
$z > 2$	no	finite (unbounded)	no

Causality # 1

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  - non-distinguishing: There exist distinct points with identical past and future.

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- Causal future of (t_0, ξ_0, r_0, x_0) contains $\{(t, \xi, r, x) \mid t > t_0\}$.

Causality # 2

Causal Ladder:

-	Globally hyperbolic	->	Minkowski, de Sitter
-	Stably causal	->	Anti-de Sitter, plane waves
-	Strongly causal		
-	Distinguishing		
-	Causal	->	Schrödinger ($z > 1$)
-	Chronological		

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- Only for $z = 2$ does there exist such a Killing vector: $H + \omega^2 C$.
- If there exists a time-independent global coordinate system then only for $z = 2$.
- There is one generator that commutes with $H + \omega^2 C$, namely N.

Global coordinates #2

- To construct a coordinate trafo: $(t, \xi, r, \vec{x}) \to (T, V, R, \vec{X})$ that "diagonalizes" $H + \omega^2 C$ and N: $H + \omega^2 C = \frac{\partial}{\partial T}$ and $N = \frac{\partial}{\partial V}$

$$
\begin{aligned}
t &= \omega^{-1} \tan \omega T \\
r &= \frac{R}{\cos \omega T} \\
\vec{x} &= \frac{\vec{X}}{\cos \omega T} \\
\xi &= V + \frac{\omega}{2} \left(R^2 + \vec{X}^2 \right) \tan \omega T
\end{aligned}
$$

boundary: $r = 0 \to R = 0$

$$
\begin{aligned}
ds^2 &= -\frac{dt^2}{r^4} + \frac{1}{r^2} \left(-2dt d\xi + dr^2 + d\vec{x}^2 \right) \\
&= -\frac{dT^2}{R^4} + \frac{1}{R^2} \left(-2dTdV - \omega^2 \left(R^2 + \vec{X}^2 \right) dT^2 + dR^2 + d\vec{X}^2 \right)
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- NRCFT: primary operators correspond to energy eigenstates of a system in a harmonic potential [Nishida, Son, 2007]
- "Boundary": $R = \text{cst}$ and $V = \text{cst}$

$$ds^2\,|_{R,V=\text{cst}} = -\left(1 + \omega^2\rho^2\right)dT^2 + d\rho^2 + \rho^2 d\Omega_{d-1}^2$$

takes the form of a Newtonian limit with isotropic harmonic oscillator potential.

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- Σ_T:
  - $T = \text{cst}$
  - Induced metric: $ds^2|_{T=\text{cst}} = \frac{1}{R^2}\left(dR^2 + \overbrace{d\rho^2 + \rho^2 d\Omega_{d-1}^2}^{d\vec{X}^2} \right)$
  - Lightlike with normal $\left(\frac{\partial}{\partial V}\right)^\mu = \delta_V^\mu$
  - $d\Sigma^\mu = \delta_V^\mu R^{-(d+1)} \rho^{d-1} dR\, d\rho\, d\Omega_{d-1}$.

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$$\phi_M = e^{-iE_M T} e^{-imV} Y_L(\text{angles})\, \varphi_M(\rho)\, \phi_M(R)\,.$$

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- In $\langle \phi_M \mid \phi_{M'} \rangle$ there is no $\int dV$ integral. The modes ϕ_M will be orthonormal iff $m = m'$ (Bargmann superselection).

A Hilbert space for scalars #3

$$ds^2\,|_{R,V=\text{cst}} = -\left(1+\omega^2\rho^2\right)dT^2 + d\rho^2 + \rho^2 d\Omega_{d-1}^2$$

- The equation for $\varphi_M(\rho)$ is identical to the time-independent Schrödinger equation for a particle in a d-dimensional isotropic harmonic oscillator:

$$\varphi_M(\rho) = e^{-\frac{1}{2}\omega m\rho^2}\rho^L L_n^{L-1+d/2}(\omega m\rho^2)\,.$$

  - $L_n^{L-1+d/2}(\omega m\rho^2)$ are generalized Laguerre polynomials of degree n.

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- $\langle \phi_M \mid \phi_{M'} \rangle \propto e^{i(E_M - E_{M'})T} \delta_{LL'} \delta_{nn'} \int dR R^{-(d+1)} \phi_M(R) \phi_{M'}(R)$

A Hilbert space for scalars #4

- General solution for $\phi_M(R)$:

$$\phi_M(R) = e^{-\frac{1}{2}\omega m R^2} R^{\Delta_+} F(n + \tfrac{L}{2} + \tfrac{d}{4} - \tfrac{E_M}{2\omega}, 1 + \tfrac{\Delta_+ - \Delta_-}{2}, \omega m R^2)$$

$$\Delta_\pm = \tfrac{d+2}{2} \pm \sqrt{\tfrac{(d+2)^2}{4} + m_0^2 + m^2}$$

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- Breitenlohner–Freedman bound: $m_0^2 + m^2 > -\frac{(d+2)^2}{4}$.

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  - F is a confluent hypergeometric function.

- For simplicity consider only modes with $m_0^2 + m^2 > 0$. These modes are normalizable iff $E_M = 2\omega\left(k + n + \frac{L}{2} + \frac{d}{4}\right)$ and given by

$$\phi_M(R) = e^{-\frac{1}{2}\omega m R^2} R^{\Delta_+} L_k^{(\Delta_+ - \Delta_-)/2}(\omega m R^2)\,.$$

A Hilbert space for scalars #5

- Thus for $m_0^2 + m^2 > 0$ with m fixed we have the mode decomposition:

$$
\begin{aligned}
\phi &= \sum_{k,n,L} \left(a_{k,n,L}\phi_{k,n,L} + b^*_{k,n,L}\phi^*_{k,n,L} \right) \\
\phi_{k,n,L} &= A_{k,n,L}\, e^{-iE_{k,n,L}T}\, e^{-imV} Y_L(\text{angles})\, e^{-\frac{1}{2}\omega m(R^2+\rho^2)} R^{\Delta_+}\rho^L \times \\
&\quad \times L_n^{L-1+d/2}(\omega m\rho^2) L_k^{(\Delta_+-\Delta_-)/2}(\omega m R^2) \\
E_{k,n,L} &= 2\omega\left(k + n + \tfrac{L}{2} + \tfrac{d}{4}\right)
\end{aligned}
$$

with coefficients given by

$$
a_{k,n,L} = \langle \phi_{k,n,L} \mid \phi \rangle \qquad\qquad b_{k,n,L} = \langle \phi_{k,n,L} \mid \phi^* \rangle\,.
$$

Causality and initial data #1

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- The surface $T =$ cst is intersected by certain timelike curves more than once. $\rightarrow$ The set $T =$ cst is not achronal.

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Causality and initial data #2

- $T =$ cst is not achronal *but* it is an initial data surface.
- Questions:
  - What kind of curves intersect $T =$ cst more than once?
  - Is there a well-posed initial value formulation?